Exercises:

1. For each of the following, decide whether the objects and operations described form a vector space. If they do, show that they satisfy the axioms. If not, show that they fail to satisfy at least one axiom.

   (a) The set $\mathbb{R}^3$ of 3 dimensional columns of real numbers, with addition defined by

   $$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 4v_1 + 4w_1 \\ 4v_2 + 4w_2 \\ 4v_3 + 4w_3 \end{pmatrix},$$

   and the usual scalar multiplication.

   (b) The set of real polynomials of degree **exactly** n:

   $$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

   with $a_n \neq 0$, and with addition and scalar multiplication defined the same way we did in class for polynomials of degree at most n.

   (c) The subset of $\mathbb{R}^3$ given by:

   $$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + 2y - z = 0 \right\},$$

   with the usual addition and scalar multiplication.

   (d) The subset of $\mathbb{R}^3$ given by:

   $$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + 2y - z = 3 \right\},$$

   with the usual addition and scalar multiplication.

   (e) The subset of $\mathbb{R}^3$ given by:

   $$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^6 + y^2 + z^4 = 0 \right\},$$

   with the usual addition and scalar multiplication.

   (f) The set of functions

   $$V = \{f : \mathbb{R} \to \mathbb{R} \mid f(5) = 0\},$$

   with addition given by $(f+g)(x) = f(x) + g(x)$ and scalar multiplication by $(\alpha f)(x) = \alpha(f(x))$.

2. Let V be a general vector space.

   (a) For $\mathbf{v} \in V$, prove that $\mathbf{v}$ has a unique additive inverse.

   (b) For $\mathbf{v} \in V$ and a scalar α, prove that $(-\alpha)\mathbf{v}$ is the additive inverse of $\alpha\mathbf{v}$.

   (c) Prove that the additive inverse of $\mathbf{0}$ is $\mathbf{0}$.

3. Let V be a general vector space with zero vector $\mathbf{0}$. Prove that $\alpha\mathbf{0} = \mathbf{0}$ for any scalar α.
   [**Hint:** treat the cases $\alpha = 0$ and $\alpha \neq 0$ separately.]

Solutions:

1. (a) This is **not** a vector space because it fails the zero element axiom. Suppose, towards a contradiction, that $\mathbf{w} = (w_1, w_2, w_3)^T \in \mathbb{R}^3$ is the zero vector. Then for any $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ we should have $\mathbf{v} + \mathbf{w} = \mathbf{v}$. The way addition is defined for this set, this implies

$$\begin{pmatrix} 4v_1 + 4w_1 \\ 4v_2 + 4w_2 \\ 4v_3 + 4w_4 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

or $4v_j + 4w_j = v_j$ for each $j = 1, 2, 3$. Solving for w_j, we obtain $w_j = \frac{3}{4}v_j$ for each $j = 1, 2, 3$. This is a contradiction because each w_j is fixed, while we can take v_j to be any real number. Thus zero vector does not exist and this is not a vector space. □

[**Note:** it is not sufficient to argue that the usual zero vector $(0, 0, 0)^T$ fails to be a zero vector. This only shows that that particular vector is not the zero vector for this addition operation, but it *could* be that there is some other strange vector that plays the role of the zero vector here. That is why in the proof above we consider a vector $\mathbf{w}$ with no other assumptions on it.]

(b) This is **not** a vector space for a number of reasons. One is that it is not closed under scalar multiplication: the polynomial $p(x) = x^n$ is in this set, but the scalar multiple $0p(x) = 0$ is not because it does not have degree n.

You can also argue that it is not closed under addition: $p(x) = x^n$ and $q(x) = -x^n$ are both in the set but $p(x) + q(x) = 0$ is not.

Finally, you can argue that it fails the zero element axiom. Indeed, suppose towards a contradiction that $q(x) = b_n x^n + \cdots + b_1 x + b_0$ is a zero vector. □

(c) This is a vector space. First note that if $\mathbf{v} = (x, y, z)^T, \mathbf{w} = (a, b, c)^T \in V$ then $\mathbf{v} + \mathbf{w} = (x + a, y + b, z + c)^T$ and

$$(x + a) + 2(y + b) - (z + c) = (x + 2y - z) + (a + 2b - c) = 0 + 0 = 0.$$

So $\mathbf{v} + \mathbf{w} \in V$. Also, for a scalar α we have $\alpha\mathbf{v} = (\alpha x, \alpha y, \alpha z)^T$ and

$$\alpha x + 2(\alpha y) - \alpha z = \alpha(x + 2y - z) = \alpha(0) = 0,$$

So $\alpha\mathbf{v} \in V$. That is, V is closed under addition and scalar multiplication.

Regarding the axioms, commutativity, additivity, multiplicative identity, multiplicative associativity, and the distributive laws all follow since they hold in $\mathbb{R}^3$. So it remains to show that there is a zero vector and that every element has an additive inverse. Observe that the usual zero vector $\mathbf{0} = (0, 0, 0)^T$ in V since $0 + 2(0) - 0 = 0$. This vector then does indeed play the role of the zero vector. Finally, if $\mathbf{v} = (x, y, z)^T \in V$ then its usual additive inverse $-\mathbf{v} = (-x, -y, -z)^T$ also in V:

$$(-x) + 2(-y) - (-z) = -(x + 2y - z) = -0 = 0.$$

Hence all the axioms are satisfied and so this is indeed a vector space. □

(d) This is **not** a vector space for a number of reasons. First it is not closed under scalar multiplication: if $\mathbf{v} = (x, y, z)^T \in V$ then $0v = (0, 0, 0)^T$ and $0 + 2(0) - 0 \neq 3$.

You can also argue that it is not closed under addition: we have that $(3, 0, 0)^T$ and $(0, 0, -3)^T$ are in V but their sum $(3, 0, -3)^T$ is not since $3 + 2(0) - (-3) = 6 \neq 0$.

You can also argue that it also fails to have a zero vector. Since addition is defined the same way as for $\mathbb{R}^3$, it must be that the zero vector is of the form $\mathbf{0} = (0, 0, 0)^T$ but this vector fails to be in V.

Finally, you can argue that it fails the additive inverse axiom. Since addition is defined the same way as for $\mathbb{R}^3$, for $\mathbf{v} = (x, y, z)^T \in V$ its additive inverse must be the usual one $-\mathbf{v} = (-x, -y, -z)^T$. But $-x + 2(-y) - (-z) = -(x + 2y - z) = -(3) = -3 \neq 3$. □

(e) This is a vector space. The only way to satisfy the condition $x^6 + y^2 + z^4 = 0$ (since the degrees are all even) is if $x = y = z = 0$. Thus V consists of exactly one vector: the zero vector $\mathbf{0} = (0, 0, 0)^T$. Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha\mathbf{0} = \mathbf{0}$ for all scalars α, V is closed under addition and scalar multiplication.

Moreover, since these are inherited from $\mathbb{R}^3$, they automatically satisfy commutativity, additivity, multiplicative identity, multiplicative associativity, and the distributive laws. Finally, V has a zero vector and additive inverses:

$$-\mathbf{0} = (-0, -0, -0)^T = (0, 0, 0)^T = \mathbf{0}.$$

Thus V is a vector space.

(f) This is a vector space. We first note that it is closed under addition since for $f, g \in V$ we have $(f+g)(5) = f(5)+g(5) = 0+0 = 0$, so that $f+g \in V$. It is also closed under scalar multiplication: for $f \in V$ and a scalar α we have $(\alpha f)(5) = \alpha(f(5)) = \alpha(0) = 0$ so that $\alpha f \in V$.

2. (a) Fix $\mathbf{v} \in V$. Suppose that there are $\mathbf{w}_1, \mathbf{w}_2 \in V$ such that $\mathbf{v} + \mathbf{w}_j = \mathbf{0}$ for $j = 1, 2$; that is, $\mathbf{w}_1$ and $\mathbf{w}_2$ are both additive inverses of $\mathbf{v}$. We must show $\mathbf{w}_1 = \mathbf{w}_2$. Using the zero vector and associativity of addition we have

$$\mathbf{w}_1 = \mathbf{w}_1 + \mathbf{0} = \mathbf{w}_1 + (\mathbf{v} + \mathbf{w}_2) = (\mathbf{w}_1 + \mathbf{v}) + \mathbf{w}_2 = \mathbf{0} + \mathbf{w}_2 = \mathbf{w}_2.$$

Thus $\mathbf{w}_1 = \mathbf{w}_2$. □

(b) Fix $\mathbf{v} \in V$ and a scalar α. Then using the distributive law we have

$$\alpha\mathbf{v} + (-\alpha)\mathbf{v} = (\alpha + -\alpha)\mathbf{v} = 0\mathbf{v}.$$

From class, we know that $0\mathbf{v} = \mathbf{0}$. Thus $\alpha\mathbf{v} + (-\alpha)\mathbf{v} = \mathbf{0}$. That means $(-\alpha)\mathbf{v}$ is an additive inverse for $\alpha\mathbf{v}$, and so by the previous part of the exercise it must be that $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$. □

[**Note:** observe that, in particular, this implies that $(-1)\mathbf{v} = -(1\mathbf{v}) = -\mathbf{v}$ by the multiplicative identity axiom.]

(c) By the zero element axiom, we know $\mathbf{0} + \mathbf{0} = \mathbf{0}$. But this also means that $\mathbf{0}$ is its own additive inverse. □

3. Fix a scalar α. We will consider two cases: $\alpha = 0$ and $\alpha \neq 0$.

First suppose $\alpha = 0$. We showed in class that $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. So in particular $0\mathbf{0} = \mathbf{0}$, which finishes this case.

Next, suppose that $\alpha \neq 0$. We have seen in class that the zero vector in a vector space is unique. So if we can show $\mathbf{v} + \alpha\mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$, then $\alpha\mathbf{0}$ is a zero vector and therefore must equal *the* zero vector $\mathbf{0}$. Note that since $\alpha \neq 0$, $1/\alpha$ exists. So for any $\mathbf{v} \in V$ we use the multiplicative identity, distributive law, and zero vector axioms to compute:

$$\mathbf{v} + \alpha\mathbf{0} = \left(\alpha\frac{1}{\alpha}\right)\mathbf{v} + \alpha\mathbf{0} = \alpha\left(\frac{1}{\alpha}\mathbf{v} + \mathbf{0}\right) = \alpha\left(\frac{1}{\alpha}\mathbf{v}\right) = \mathbf{v}.$$

So $\alpha\mathbf{0}$ is indeed an additive identity and must therefore equal $\mathbf{0}$. This completes the second case and the proof. □