The Lebesgue Integral

Let us denote by L^+ $(=L^+(\mathbb{R}^d))$ the space of all Lebesgue measurable functions $f:\mathbb{R}^d \to [0,\infty]$

~~For~~

Def: For a simple function $\phi = \sum_{j=1}^{n} a_j \chi_{E_j} \in L^+$ in std. rep we define the Lebesgue integral of ϕ as the quantity:

$$\int \phi \, dm := \sum_{j=1}^{n} a_j \, m(E_j)$$

Remark: If $m(E_j) = +\infty$ and $a_j > 0$, then $\int \phi \, dm = +\infty$. Otherwise, if each $m(E_1), \ldots, m(E_n) < \infty$, then $\int \phi \, dm < \infty$.

Notation: If we wish to make the argument (input) of ϕ explicit, we will write

$$\int \phi(x) \, dm(x) := \int \phi \, dm$$

If $A \in \mathcal{M}$, then $\phi \cdot \chi_A = \sum_{j=1}^{n} a_j \chi_{E_j \cap A} \in L^+$, so its Lebesgue integral is defined as above. We write

$$\int_A \phi \, dm := \int \phi \, \chi_A \, dm.$$

Prop: Let ϕ and ψ be simple functions in L^+.

(a) If $c \geq 0$, $\int c\phi \, dm = c \int \phi \, dm$

(b) $\int (\phi + \psi) dm = \int \phi \, dm + \int \psi \, dm$

(c) If $\phi \leq \psi$, then $\int \phi \, dm \leq \int \psi \, dm$

Pf: (a) This is immediate.

(b) & (c) Exercise: If $\phi = \sum a_j \chi_{E_j}$, $\psi = \sum b_k \chi_{F_k}$ (in std. rep) consider the sets

$$E_j \cap F_k \in \mathcal{M} \quad \forall j, k$$

Def: For $f \in L^+$, the Lebesgue integral (or just integral) of f is the quantity

$$\int f\,dm := \sup\left\{\int \phi\,dm : 0 \le \phi \le f,\ \phi \text{ simple}\right\}$$

Rem If f above is simple, the part (c) of the previous Prop. implies this new def'n agrees with the old one.

Thm: Let $B \subseteq \mathbb{R}^d$ be a box and suppose $f : \mathbb{R}^d \longrightarrow [0,\infty]$ is Riemann int'ble over B. Then

$$\underbrace{\int_B f}_{\text{Riemann integral}} = \underbrace{\int_B f\,dm}_{\text{Lebesgue integral}}$$

Pf HW Exercise. ☐

Prop (a) For $f, g \in L^+$, if $f \le g$ then $\int f\,dm \le \int g\,dm$

(b) For $f \in L^+$ and $c > 0$, $\int cf\,dm = c\int f\,dm$

Pf: (a) For any simple ϕ satisfying $0 \le \phi \le f$, we have $0 \le \phi \le g$. Thus $\int \phi\,dm \le \int g\,dm \Rightarrow \int f\,dm \le \int g\,dm$.

(b) Since this holds for simple functions, and since a factor of c can be extracted from the supremum, we have $\int cf\,dm = c\int f\,dm$. ☐

Thm (The Monotone Convergence Thm)

Let $(f_n)_{n\in\mathbb{N}} \subseteq L^+$ be a sequence satisfying

$$f_1 \le f_2 \le \cdots$$

Define $f = \lim_{n\to\infty} f_n \left(= \sup_n f_n\right)$. Then

$$\int f\,dm = \lim_{n\to\infty} \int f_n\,dm$$

~~[Note: It may be that $f(x) = \infty$ for some points x.~~

~~this is ok, all of our theory holds for such functions provided we accept the convention that $0\cdot\infty = 0$~~

4/18/2018

Pf: ~~Since~~ By the previous prop $(\int f_n \, dm)_{n\in\mathbb{N}} \subset \mathbb{R}$ is an increasing sequence, so its limit exists, though it may be infinite. Furthermore, $f_n \le f \ \forall n \in \mathbb{N}$ implies

$$\lim_{n\to\infty} \int f_n \, dm = \sup_n \int f_n \, dm \le \int f \, dm.$$

So it suffices to show the reverse inequality. Fix $\alpha \in (0,1)$, and let $0 \le \phi \le f$ be a simple function. Consider

$$E_n := \{x : f_n(x) \ge \alpha \phi(x)\}.$$

Then

$$E_1 \subseteq E_2 \subseteq \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d$$

Indeed, $\uparrow$ is clear, and for any $x \in \mathbb{R}^d$, ~~if $f(x) = \infty$, then $(f_n(x))$ is unbounded so $\exists n \in \mathbb{N}$ s.t. $f_n(x) \ge \alpha\phi(x)$. If $f(x) < \infty$,~~ $f(x) > \alpha \phi(x)$ (the strict inequality is important here). Thus by def'n of the supremum, $\exists n \in \mathbb{N}$ st $f(x) \ge f_n(x) > \alpha\phi(x)$. Hence $x \in E_n$.

Now,

$$\int f_n \, dm \ge \int_{E_n} f_n \, dm \ge \int_{E_n} \alpha \phi \, dm = \alpha \int_{E_n} \phi \, dm$$

Suppose $\phi = \sum a_j \chi_{B_j}$. Then by cty from below we have:

$$\lim_{n\to\infty} \alpha \int_{E_n} \phi \, dm = \lim_{n\to\infty} \alpha \sum_j a_j \, m(B_j \cap E_n)$$

$$= \alpha \sum a_j \, m(B_j) = \alpha \int \phi \, dm.$$

So $\lim_{n\to\infty} \int f_n \, dm \ge \alpha \int \phi \, dm$. Since $0 \le \phi \le f$ was arbitrary, we have

$$\lim_{n\to\infty} \int f_n \, dm \ge \alpha \int f \, dm$$

Letting $\alpha \to 1$ completes the proof. $\square$

Rem: The def'n of $\int f\,dm$ involves the sup over a (most likely) unctable set, so it is in practice hard to compute. The MCT allows us to reduce this to a simple limit. In particular, we know $\forall f \in L^+$ $\exists (\phi_n)_{n\in\mathbb{N}}$ simple functions s.t. $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ s.t. $\lim_{n\to\infty} \phi_n = f_n$.

Thm For $(f_n)_{n\in\mathbb{N}} \subseteq L^+$,

$$\int \sum_{n=1}^{\infty} f_n\,dm = \sum_{n=1}^{\infty} \int f_n\,dm.$$

Pf: First, we claim $\int f_1 + f_2\,dm = \int f_1\,dm + \int f_2\,dm$. By the above remark and the MCT, $\exists$ $(\phi_k)_{k\in\mathbb{N}}$ $(\psi_k)_{k\in\mathbb{N}}$ sequences of $\uparrow$ simple functions whose limit is f_1 and f_2, respectively. Note that $0 \le \phi_k + \psi_k \le f_1 + f_2$ for each $k\in\mathbb{N}$ and $\lim_{k\to\infty}(\phi_k + \psi_k) = (f_1 + f_2)$. Hence by the MCT we have:

$$\int (f_1 + f_2)\,dm = \lim_{k\to\infty} \int \phi_k + \psi_k\,dm = \lim_{k\to\infty}\left(\int \phi_k\,dm + \int \psi_k\,dm\right)$$

$$= \int f_1\,dm + \int f_2\,dm.$$

By an induction argument, we then have

$$\int \sum_{n=1}^{N} f_n\,dm = \sum_{n=1}^{N} \int f_n\,dm$$

for all $N \in \mathbb{N}$. Then applying the MCT again yields:

$$\sum_{n=1}^{\infty} \int f_n\,dm = \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n\,dm = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n\,dm = \int \sum_{n=1}^{\infty} f_n\,dm. \quad \square$$

Prop If $f \in L^+$, then $\int f\,dm = 0$ iff $f = 0$ a.e.

Pf: If f is simple, say $f = \sum a_j \chi_{E_j}$, then

$$\int f\,dm = 0 \iff \sum a_j m(E_j) = 0 \iff \text{for each } j \text{ either } a_j = 0 \text{ or } m(E_j) = 0$$

$$\iff f = 0 \text{ a.e.}$$

More generally, suppose $f = 0$ a.e. If $0 \le \phi \le f$ is simple, then $\phi = 0$ a.e. ~~Hence if $\phi = \sum a_j \chi_{E_j}$ then for every j either $a_j = 0$ or $m(E_j) = 0$ $\Rightarrow \int \phi \, dm = 0$. Hence~~ Thus $\int \phi \, dm = 0$ and so

$$\int f \, dm = \sup \{ \int \phi \, dm : 0 \le \phi \le f \ \phi \text{ simple} \} = \sup \{ 0 : \text{---} " \text{---} \} = 0.$$

Conversely, suppose $\int f \, dm = 0$. Let

$$E_n = \{ x : f(x) > \tfrac{1}{n} \}.$$

Then $E_1 \subseteq E_2 \subseteq \cdots$

$$E := \{ x : f(x) > 0 \} = \bigcup_{n=1}^{\infty} E_n$$

if $f \neq 0$ a.e., then $m(E) > 0$. By cty from below, we have

$$m(E) = \lim_{n \to \infty} m(E_n),$$

and so $\exists n \in \mathbb{N}$ s.t. $m(E_n) > 0$. But then $\phi := \frac{1}{n} \cdot \chi_{E_n} \le f$ and so,

$$\int f \, dm \ge \int \phi \, dm = \tfrac{1}{n} m(E_n) > 0.$$

a contradiction. □

Cor: If $f, g \in L^+$ satisfy $f = g$ a.e., then $\int f \, dm = \int g \, dm$.

Cor If $(f_n)_{n \in \mathbb{N}} \subseteq L^+$ and $f \in L^+$ are s.t.

$$f_1 \le f_2 \le \cdots \le f$$

and $f(x) = \sup f_n(x)$ for a.e. x, then $\int f \, dm = \lim_{n \to \infty} \int f_n \, dm$.

Pf: Let

$$E = \{ x : f(x) = \sup_n f_n(x) \}$$

Then $m(E^c) = 0$ and hence ~~$f \chi_{E^c} = 0$ a.e.~~ ~~so they are positive and~~

$$f = f \chi_E \quad \text{~~+ } f \chi_{E^c}\text{~~}$$
$$f_n = f_n \chi_E \quad \text{~~+ } f_n \chi_{E^c}\text{~~}$$

~~and equal the zeroes~~ almost everywhere. By the previous corollary and the MCT:

$$\int f \, dm = \int_E f \, dm = \lim_{n \to \infty} \int_E f_n \, dm = \lim_{n \to \infty} \int f_n \, dm$$ □

done

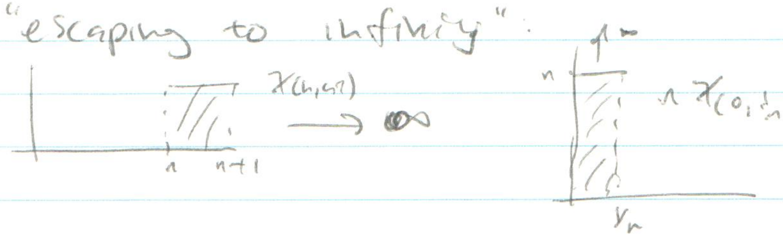
Remark: The hypothesis "$f_1 \le f_2 \le \dots$" in the MCT cannot be readily discarded:

$$\chi_{(n,n+1)} \to 0 \quad \text{yet} \quad \int \chi_{(n,n+1)}\,dm = 1$$

also

$$n \cdot \chi_{(0,\frac{1}{n})} \to 0 \quad \text{yet} \quad \int n\,\chi_{(0,\frac{1}{n})}\,dm = 1$$

These examples demonstrate the area under the graph "escaping to infinity":



Thm (Fatou's Lemma) For any $(f_n)_{n\in\mathbb{N}} \subseteq L^+$,

$$\int \liminf_{n\to\infty} f_n\,dm \le \liminf_{n\to\infty} \int f_n\,dm$$

Pf: For $N \in \mathbb{N}$, $\inf_{n\ge N} f_n \le f_k$ for all $k \ge N$. Thus by monotonicity of the integral we have

$$\int \inf_{n\ge N} f_n\,dm \le \int f_k\,dm \quad \forall k \ge N$$

$$\Longrightarrow \quad \int \inf_{n\ge N} f_n\,dm \le \inf_{k\ge N} \int f_k\,dm.$$

The MCT then implies

$$\lim_{N\to\infty} \int \inf_{n\ge N} f_n\,dm = \int \lim_{N\to\infty} \inf_{n\ge N} f_n\,dm$$

swap

$$\left(= \int \liminf_{n\to\infty} f_n\,dm \right) \le \liminf_{n\to\infty} \int f_n\,dm \quad \square$$

4/20/2018

Cor If $(f_n)_{n\in\mathbb{N}} \subseteq L^+$, $f \in L^+$, and $f_n \to f$ a.e. then

$$\int f\,dm \le \liminf_{n\to\infty} \int f_n\,dm$$

Pf: If $f_n \to f$ everywhere, then by Fatou's Lemma we are done. We can achieve this by modifying the f_n's and f on a measure zero set, which, as we have seen, does not affect the integrals. □

Exercise: ~~Apply~~ Check this corollary on (a) $\chi_{(n,n+1)}$ and (b) $n\chi_{(0,1/n)}$.

Exercise: Suppose $f \in L^+$ satisfies $\int f\,dm < \infty$. Show that $f < \infty$ a.e.

Integrating $\overline{\mathbb{R}}$-valued Functions

Suppose $f: \mathbb{R}^d \to \overline{\mathbb{R}}$ is Lebesgue measurable. Then if

$$E = \{x \in \mathbb{R}^d : f(x) \geq 0\} = f^{-1}([0,\infty]),$$

we have $E \in \mathcal{M}$ and so

$$f_+ := f\chi_E \quad \text{and} \quad f_- := -f\chi_{E^c}$$

are elements of L^+ with

$$f = f_+ - f_-.$$

We want to define the Lebesgue integral of f as

$$\int f_+\,dm - \int f_-\,dm,$$

but if $\int f_\pm\,dm = \infty$, we cannot make sense of "$\infty - \infty$". However, note that

$$|f| = f_+ + f_- \in L^+$$

If $\int |f|\,dm < \infty$, then $\int f_\pm\,dm < \infty$ and so we can reconcile $\int f_+\,dm - \int f_-\,dm$.

Def: We say a ~~function~~ Lebesgue measurable function $f: \mathbb{R}^d \to \overline{\mathbb{R}}$ is ~~integrable~~ Lebesgue integrable if $\int |f|\,dm < \infty$.
(Equivalently, if $\int f_+\,dm, \int f_-\,dm < \infty$.)
In this case we define the Lebesgue integral