Measurable Functions

We want to use this improved notion of measurability to define an improved integral (the "Lebesgue Integral"). We must first discuss the types of functions that we will attempt to integrate.

<u>Def</u> A function $f: \mathbb{R}^n \to \mathbb{R}$ is <u>Lebesgue</u> ~~integ~~ <u>measurable</u> (or just <u>measurable</u>) if for all open sets $U \subseteq \mathbb{R}$, we have $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^n)$.

Since open sets are measurable, we immediately see that $f: \mathbb{R}^2 \to \mathbb{R}$ cts is Lebesgue measurable. We can say more:

<u>Prop:</u> If $f: \mathbb{R}^n \to \mathbb{R}$ is Riemann ~~integrable~~ integrable, then it is Lebesgue measurable.

<u>Pf</u>: Let $D \subseteq \mathbb{R}^n$ be the discty. set of f. By the Riemann-Lebesgue Theorem, D is a zero set. Hence $D \in \mathcal{M}$ with $m(D) = 0$. Define cts $g: D^c \to \mathbb{R}$ by $g(x) = f(x)$. Then for any $S \subseteq \mathbb{R}$, it is easy to see that

$$g^{-1}(S) \subseteq f^{-1}(S) \subseteq g^{-1}(S) \cup D.$$

In particular, if S is open then g cts implies *$g^{-1}(S)$ is open and hence measurable. But then

$$f^{-1}(S) = \underset{\mathcal{M}}{\underset{\uparrow}{g^{-1}(S)}} \cup \underset{\mathcal{M} \text{ as a zero set.}}{\underset{\uparrow}{(D \setminus f^{-1}(S))}}$$

So $f^{-1}(S) \in \mathcal{M}$ and so f is Lebesgue measurable. $\square$

* technically, $g^{-1}(S)$ is open relative to D^c, but then $\exists U \subseteq \mathbb{R}^n$ open s.t. $g^{-1}(S) = U \cap D^c$ which is still in $\mathcal{M}$.

The proof of the following prop is straightforward and so we leave it as an exercise.

Prop: Let $f: \mathbb{R}^n \to \mathbb{R}$. TFAE

(i) f is Lebesgue measurable.

(ii) For all closed subsets $V \subseteq \mathbb{R}$, $f^{-1}(V) \in \mathcal{M}$.

(iii) For all $a \in \mathbb{R}$, $f^{-1}((a,\infty)) \in \mathcal{M}$

(iv) For all $a \in \mathbb{R}$, $f^{-1}((-\infty,a]) \in \mathcal{M}$

(v) For all $a \in \mathbb{R}$, $f^{-1}([a,\infty)) \in \mathcal{M}$

(vi) For all $a \in \mathbb{R}$, $f^{-1}((-\infty,a)) \in \mathcal{M}$.

Prop Let $f, g: \mathbb{R}^n \to \mathbb{R}$ be measurable. Then $f+g$, and $f \cdot g$, are measurable.

Pf. Let $a \in \mathbb{R}$. By the previous prop, it suffices to show

$$(f+g)^{-1}([a,\infty)) \in \mathcal{M}$$

Observe that

$$(f+g)(x) \geq a \iff g(x) \geq a - f(x).$$

We claim

$$(f+g)^{-1}([a,\infty)) = \bigcup_{n \in \mathbb{Z}} \underbrace{f^{-1}((-\infty,n])}_{\mathcal{M}} \cap \underbrace{g^{-1}([a-n,\infty))}_{\mathcal{M}}$$

in which case we are done since the right-hand side is a c-ble union of measurable sets. Let $x \in (f+g)^{-1}([a,\infty))$. Then $\exists\, n \in \mathbb{Z}$ s.t. $f(x) \leq n$. Consequently,

$$(f+g)(x) \geq a \Rightarrow g(x) \geq a - f(x) \geq a - n.$$

Hence x is in the left-hand-side set. Conversely if x — in ——, then $\exists\, n \in \mathbb{Z}$ s.t.

$$f(x) \leq n \text{ and } g(x) \geq a - n.$$

Hence

We claim

$$(f+g)^{-1}([a,\infty)) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} f^{-1}((-\infty,\tfrac{k}{n}]) \cap g^{-1}([a-\tfrac{k}{n},\infty))$$

For x in RHS, $f(x)+g(x) \geq a \Rightarrow g(x) \geq a - f(x)$.

Then $\forall\, n \in \mathbb{N}$, $\exists\, k \in \mathbb{Z}$ s.t. $f(x) \leq \frac{k}{n}$. Hence $g(x) \geq a - \frac{k}{n}$.

So $x \in \bigcup_{k \in \mathbb{Z}} f^{-1}((-\infty,\frac{k}{n}]) \cap g^{-1}([a-\frac{k}{n},\infty))$

Since this holds $\forall n \in \mathbb{N}$, x is in LHS.
Conversely, let $x \in$ LHS. Then $\forall n \in \mathbb{N}$, $\exists k \in \mathbb{Z}$ s.t.

$$\frac{k-1}{n} \le f(x) \le \frac{k}{n}.$$

Since $f(x) \le \frac{k}{n} \Rightarrow g(x) \ge a - \frac{k}{n}$. Thus

$$f(x)+g(x) \ge \frac{k-1}{n} + a - \frac{k}{n} = a - \frac{1}{n}.$$

Letting $n \to \infty$ yields $f(x)+g(x) \ge a$. So $x \in$ RHS. Now, since f and g are meas, the prop implies

$$f^{-1}((-\infty, \tfrac{k}{n}]) \cap g^{-1}([a-\tfrac{k}{n}, \infty))$$

is meas. Since $\mathcal{M}$ is closed under ctble unions and ctble intersections, we obtain ~~f~~ $(f+g)^{-1}([a,\infty)) \in \mathcal{M}$.

The proof for fg is similar, but now one must be careful about the signs of $a, \frac{n}{k}$. $\square$

<u>Prep</u>: Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of uniformly bdd measurable functions from $\mathbb{R}^n \to \mathbb{R}$. Then

$$g_1(x) := \sup_{k \in \mathbb{N}} f_k(x) \qquad g_2(x) := \inf_k f_k(x)$$

$$g_3(x) := \limsup_{k \to \infty} f_k(x) \qquad g_4(x) := \liminf_{k \to \infty} f_k(x)$$

are all measurable. If the sequence converges pointwise, then $\lim_k f_k$ is measurable.

<u>Pf</u>: Observe that $\forall a \in \mathbb{R}$

$$g_1^{-1}((a,\infty]) = \bigcup_{k=1}^{\infty} f_k^{-1}((a,\infty]) \in \mathcal{M}$$

$$g_2^{-1}((-\infty,a)) = \bigcup_{k=1}^{\infty} f_k^{-1}((-\infty,a)) \in \mathcal{M}$$

so that g_1 and g_2 are measurable. But then
$$g_3(x) = \inf_N \left(\sup_{k \geq N} f_k(x) \right)$$
is measurable and similarly for g_4. Finally, $\lim_{k\to\infty} f_k$ exists, it equals $g_3 = g_4$, so is meas. $\square$

<u>Cor</u>: If $f, g: \mathbb{R}^n \to \mathbb{R}$ are measurable, then so are $\max(f,g)$ and $\min(f,g)$.

Recall that for the Riemann ~~measuring~~ integral, the fundamental functions that all Riemann integrable functions were approximated by were step functions. The key difference between the Riemann integral and the "Lebesgue integral" will come from swapping out step functions for "simple functions."

<u>Def</u>: A function $f: \mathbb{R}^n \to \mathbb{R}$ is called a <u>simple function</u> if $\exists\, c_1, \dots, c_d \in \mathbb{R}$ and $E_1, \dots, E_d \in \mathcal{M}$ s.t.
$$f = \sum_{j=1}^{d} c_j \chi_{E_j}$$

That is, a simple function is a finite, linear combination of characteristic functions of measurable subsets.

Observe that for $a \in \mathbb{R}$, and f a simple function as above
$$f^{-1}([a,\infty)) = \begin{cases} \bigcup_{c_j \geq a} E_j & \text{if } a > 0 \\ \bigcup_{c_j \geq a} E_j \cup (E_1 \cup \cdots \cup E_j)^c & \text{if } a \leq 0 \end{cases} \in \mathcal{M}$$

So f is measurable.

Remark An equivalent definition of a simple function is a measurable function $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $f(\mathbb{R}^n)$ is a finite set. Indeed, let $\{c_1, \dots, c_d\} = f(\mathbb{R}^n)$, then

$$f = \sum c_j \chi_{f^{-1}(\{c_j\})}$$

Thm: Let $f: \mathbb{R}^n \to [0,\infty)$ be Lebesgue measurable. Then $\exists$ a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of simple functions s.t.

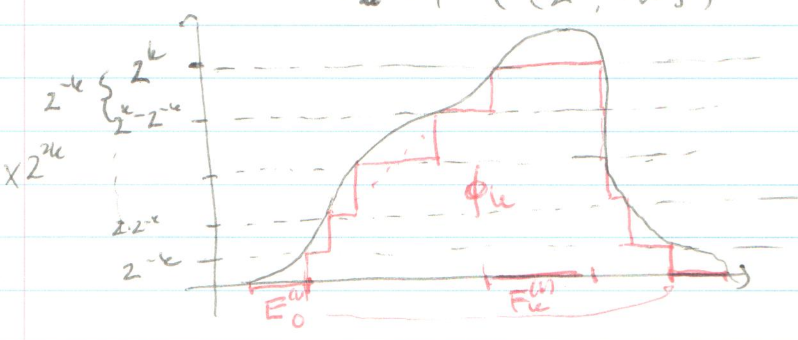
$$0 \le \phi_1 \le \phi_2 \le \dots \le f$$

and $\lim_{k\to\infty} \phi_k(x) = f(x)$ $\forall x \in \mathbb{R}^n$. Moreover, if f is bdd on $E\in\mathcal{M}$, then $(\phi_k)_{k\in\mathbb{N}}$ converges uniformly to f on E.

Pf: Fix $k\in\mathbb{N}$, then for each integer $0 \le j \le 2^{2k}-1$ let

$$E_j^{(k)} := f^{-1}((j2^{-k}, (j+1)2^{-k}])$$

$$F^{(k)} := f^{-1}((2^k, +\infty])$$



Set

$$\phi_k := \sum_{j=0}^{2^{2k}-1} j 2^{-k} \chi_{E_j^{(k)}} + 2^k \chi_{F^{(k)}}$$

Then $0 \le \phi_k \le f$. Also, since

$$E_j^{(k)} \subseteq E_{2j}^{(k+1)}$$

we have $j2^{-k}\chi_{E_j^{(k)}} = 2j\, 2^{-(k+1)} \chi_{E_j^{(k)}} \le 2j\, 2^{-(k+1)} E_{2j}^{(k+1)}$

Cor: For $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ measurable, $\exists (\phi_k)_{k\in\mathbb{N}}$ simple functions s.t.

$$0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$$

$\phi_k \to f$ ptwise, and $\phi_k \to f$ unif. on any set where f is bdd.

and so $\phi_k \le \phi_{k+1}$. Also, on ~~$F^{(k)}$~~

$$(F^{(k)})^c = f^{-1}([0, 2^n])$$

we have

$$0 \le f(x) - \phi_k(x) \le 2^{-k} \quad \forall x \in (F^{(k)})^c$$

Thus $(\phi_k)_{k\in\mathbb{N}}$ converges uniformly to f on sets its bdd on. This, in particular implies that $(\phi_k)_{k\in\mathbb{N}}$ converges to f ptwise. $\square$

Pf (Cor) ~~Apply~~ Thm to $f_+ = f\cdot\chi_{f^{-1}([0,\infty])}$, $f_- = -f\cdot\chi_{f^{-1}([-\infty,0])}$. $\square$

Def: ~~We say a property~~ Given $f, g: \mathbb{R}^n \to \overline{\mathbb{R}}$ we say $f = g$ almost everywhere (a.e.) if

$$\{x \in \mathbb{R}^n : f(x) \ne g(x)\}$$

is a zero set.

Prop: (i) If $f: \mathbb{R}^n \to \overline{\mathbb{R}}$ is measurable and $g: \mathbb{R}^n \to \overline{\mathbb{R}}$ ~~[illegible]~~ satisfies $f = g$ a.e., then g is meas.

(ii) If $(f_k)_{k\in\mathbb{N}}$ is a sequence of measurable functions on $\mathbb{R}^n$, and $g: \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfies

$$\lim_{k\to\infty} f_k = g \quad \text{a.e.}$$

then g is measurable.

Pf: Exercise. $\square$

~~The point is that as the less ... on our Lebesgue integral~~

We will see that $f = g$ a.e. implies their Lebesgue integrals agree. Recall we already proved this for Riemann integrable functions on the homework.