is a (the) open covering of $P_i(a)$ with

$$\sum_{k=1}^{\infty} |B_k| = \sum_{k=1}^{\infty} k^{n-1} \cdot \frac{2\varepsilon}{2^{k+1} k^{n-1}} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Letting $\varepsilon \to 0$ yields $m^*(P_i(a)) = 0$. $\square$

4/6/2018

## 6.2 Measurability

Our definition of the outer measure yields a map(s)

$$m^* : 2^{(\mathbb{R}^n)} \longrightarrow [0, +\infty].$$

which is certainly more robust than Riemann measurability. However, for ~~[illegible]~~ set theoretic reasons (and to avoid things like the Banach Tarski paradox) we will want to restrict $m^*$ to a subcollection of subsets of $\mathbb{R}^n$. ~~[illegible]~~

<u>Def</u>: A set $E \subset \mathbb{R}^n$ is said to be <u>Lebesgue measurable</u> (or just <u>measurable</u>) if $\forall X \subseteq \mathbb{R}^n$ we have

$$(*) \quad m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$$

The set of Lebesgue measurable subsets is denoted $\mathcal{M}$ ($= \mathcal{M}(\mathbb{R}^n)$). For $E \subseteq \mathcal{M}$, the <u>Lebesgue measure</u> of $E$ is the quantity:

$$m(E) := m^*(E)$$

(That is, the $m(E)$ is just the outer measure, but we drop the '$*$' only if $E$ is measurable).
((*) is also called, more generally the <u>Carathéodory condition</u>)
The reason we use (*) as the criterion for measurability is the following: ~~[illegible]~~ lemma:

~~are disjoint, then~~

~~$m(A \cup B) = m(A) + m(B)$~~

~~Indeed, using $X = A$ in (*), we have~~

<u>Lemma</u>: For $A, B \in \mathcal{M}$, $A \cup B \in \mathcal{M}$. Moreover, if $A$ and $B$ are disjoint then

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

<u>Pf</u> Let $X \subseteq \mathbb{R}^n$, then since $A$ and $B$ are measurable we have:

$$m^*(X) = m^*(X \cap A) + m^*(X \cap A^c)$$
$$= m^*(X \cap A \cap B) + m^*(X \cap A \cap B^c) + m^*(X \cap A^c \cap B) + m^*(X \cap A^c \cap B^c)$$

Note that $(A \cup B)^c = A^c \cap B^c$ while

$$(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) = A \cup (A^c \cap B) = A \cup B.$$

So by subadditivity we have:

$$m^*(X) \geq m^*(X \cap (A \cup B)) + m^*(X \cap (A \cup B)^c).$$

The reverse inequality always holds (again by subadditivity), thus $A \cup B \in \mathcal{M}$.

Now, suppose $A \cap B = \emptyset$. Then since $A \in \mathcal{M}$

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^c)$$
$$= m^*(A) + m^*(B) = m(A) + m(B) \qquad \square$$

<u>Thm</u> $\mathcal{M}$ satisfies

(i) $\emptyset \in \mathcal{M}$

(ii) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$

(iii) If $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$

Moreover, $m$ is countably additive on $\mathcal{M}$: whenever $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ is a collection of pairwise disjoint sets ($E_n \cap E_m = \emptyset \ \forall n, m \in \mathbb{N}$)

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

<u>Def</u>: A collection $\mathcal{A}$ of subsets of $\mathbb{R}^n$ satisfying (i), (ii), and (iii) above is called a <u>$\sigma$-algebra</u>.

Pf (Thm):

(i) For $X \subseteq \mathbb{R}^n$, we have $X \cap \phi = \phi$ and $X \cap (\phi)^c = X \cap \mathbb{R}^n = X$.

Thus
$$m^*(X \cap \phi) + m^*(X \cap \phi^c) = m^*(\phi) + m^*(X) = m^*(X),$$
since $\phi$ is a zero set. Thus $\phi \in \mathcal{M}$.

(ii) Since the def. of measurable is symmetric with respect to $E$ and $E^c$, this is clear.

(iii) By the previous lemma and an easy induction argument, if $E_1, \ldots, E_N \in \mathcal{M}$ then
$$\bigcup_{n=1}^{N} E_n \in \mathcal{M}.$$
Moreover, if they are pairwise disjoint then
$$m\left(\bigcup_{n=1}^{N} E_n\right) = m(E_1) + \cdots + m(E_N).$$

Now, let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$. For each $n \in \mathbb{N}$, define
$$F_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1}).$$
$$= E_n \cap (E_1 \cup \cdots \cup E_{n-1})^c$$
$$= (E_n^c \cup (E_1 \cup \cdots \cup E_{n-1}))^c$$

Then $F_n \in \mathcal{M}$ since $\mathcal{M}$ is closed under complements and finite unions. Also the $F_n$ are pairwise disjoint (by def) and
$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Thus it suffices to show $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$. Set
$$G_N := \bigcup_{n=1}^{N} F_n \qquad N \in \mathbb{N}.$$

Then using $F_N \in \mathcal{M}$ for any $X$ we have:
$$m^*(X \cap G_N) = m^*(X \cap G_N \cap F_N) + m^*(X \cap G_N \cap F_N^c)$$
$$= m^*(X \cap F_N) + m^*(X \cap G_{N-1})$$
induction:
$$= m^*(X \cap F_N) + \cdots + m^*(X \cap F_1)$$

Thus, using $G_N \in \mathcal{M}$ we have

$$\mu^*(X) = \mu^*(X \cap G_N) + \mu^*(X \cap G_N^c)$$
$$= \sum_{n=1}^{N} \mu^*(X \cap F_n) + \mu^*(X \cap G_N^c)$$
$$\text{(monotonicity)} \geq \sum_{n=1}^{N} \mu^*(X \cap F_n) + \mu^*\left(X \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right)$$

Letting $N \to \infty$ yields

$$\mu^*(X) \geq \sum_{n=1}^{\infty} \mu^*(X \cap F_n) + \mu^*\left(X \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right)$$
$$\text{(subadditivity)} \geq \mu^*\left(X \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right) + \mu^*\left(X \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c\right)$$
$$\text{(subadd.)} \geq \mu^*(X)$$

Thus we have equality in the above. This implies $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$ as desired. Moreover, taking $X = \bigcup_{n=1}^{\infty} F_n$, the equalities above yield

$$m\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu^*\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu^*\left(\left(\bigcup_{n=1}^{\infty} F_n\right) \cap F_n\right) + \mu^*\left(X \cap \left(\bigcup F_n\right)^c\right)$$
$$= \sum_{n=1}^{\infty} \mu^*(F_n) + 0$$
$$= \sum_{n=1}^{\infty} m(F_n)$$

So that $m$ is countably additive on $\mathcal{M}$. $\square$

<u>Remark</u>: Since

$$\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c$$

it follows that $\mathcal{M}$ is closed under countable intersections.

<u>Cor</u> (a) (<u>Continuity from below</u>) If $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ satisfy

$$E_1 \subset E_2 \subset \cdots$$

then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n)$$

(b) (Continuity from above) If $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ satisfy $m(E_1) < \infty$ and

$$E_1 \supset E_2 \supset \cdots$$

then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

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Pf: (a) Define for each $n \in \mathbb{N}$

$$F_n := E_n \setminus E_{n-1} \; (= E_n \setminus (E_1 \cup \cdots \cup E_{n-1}))$$

Then $\{F_n\}_{n\in\mathbb{N}}$ are pairwise disjoint with the same union as $\{E_n\}_{n\in\mathbb{N}}$. Hence by countable additivity we have:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} m(F_n)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} m(F_n)$$

$$= \lim_{N\to\infty} m(F_1 \cup \cdots \cup F_N)$$

$$= \lim_{N\to\infty} m(E_N).$$

~~(b) Define for each $n \in \mathbb{N}$~~

~~$F_n := E_n \setminus E_{n+1} \; (= E_n \setminus (\bigcap_{k=n+1}^{\infty} E_k))$~~

~~Then $\{F_n\}_{n\in\mathbb{N}}$ are pairwise disjoint and~~

~~$E_1 = \left(\bigcup_{n=1}^{\infty} F_n\right) \cup \left(\bigcap_{n=1}^{\infty} E_n\right)$~~

~~and the above is a disjoint union. Hence by part (a) countable additivity and part (a) we have~~

~~$m(E_1) = \sum_{n=1}^{\infty}$~~

(b) Define for each $n \in \mathbb{N}$

$$F_n := E_1 \setminus E_n.$$

Then

$$F_1 \subset F_2 \subset \cdots$$

and $m(E_1) = m(F_n) + m(E_n)$.

Also, note that

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (E_1 \cap E_n^c) = E_1 \cap \left(\bigcup_{n=1}^{\infty} E_n^c\right)$$
$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)^c$$
$$= E_1 \setminus \left(\bigcap_{n=1}^{\infty} E_n\right).$$

So by part (a) we have:

$$m(E_1) = m\left(\bigcap_{n=1}^{\infty} E_n\right) + m\left(\bigcup_{n=1}^{\infty} F_n\right)$$
$$= m\left(\bigcap_{n=1}^{\infty} E_n\right) + \lim_{n\to\infty} m(F_n)$$
$$= m\left(\bigcap_{n=1}^{\infty} E_n\right) + \lim_{n\to\infty} \left(m(E_1) - m(E_n)\right)$$

Since $m(E_1) < \infty$, we can subtract this from each side to obtain the desired equality. □

We have shown so far that $m$ on $\mathcal{M}$ has some very nice properties, we would like to show that $\mathcal{M}$ contains lots of sets, so that we apply those nice properties of $m$ as broadly as possible.

Prop: Every zero set $Z$ is measurable: $Z \in \mathcal{M}$. Moreover, for any set $E \in \mathcal{M}$ and any zero set $Z$, we have $E \cup Z$, $E \setminus Z \in \mathcal{M}$ with
$$m^*(E \cup Z) = m^*(E \setminus Z) = m(E)$$
That is, zero sets do not affect measurability or the value of the measure itself.

Pf: Let $Z$ be a zero set. Then for any $X$, $X \cap Z \subseteq Z$ is a zero set hence by subadd. and monotonicity we have:
$$m^*(X) \leq m^*(X \cap Z) + m^*(X \cap Z^c)$$
$$= 0 + m^*(X \cap Z^c) \leq m^*(X)$$

~~Pf~~ $E \setminus Z = E \cap Z^c \in \mathcal{M}$, so
$m(E) = m(E \setminus Z) + m(Z) = m(E \setminus Z)$

Thus $Z \in \mathcal{M}$.
For $E \in \mathcal{M}$ and $Z$ a zero set, $Z \in \mathcal{M}$ implies $E \cup Z$. Then by subadd. and monotonicity we have:
$$m(E) \le m(E \cup Z) \le m(E) + m(Z) = m(E).$$
Thus these are all equalities. $\square$

<u>Prop</u>: For $a \in \mathbb{R}$, $i = 1, \dots, n$ the half-spaces:
$$H_i(a) := \{(x_1, \dots, x_n) \in \mathbb{R}^n : a \le x_i\}$$
are measurable. Moreover,
$$A = I_1 \times \cdots \times I_n \in \mathbb{R}^n$$
where each $I_i$ is of the form
$$(a_i, b_i),\ [a_i, b_i),\ (a_i, b_i],\ \text{or}\ [a_i, b_i]$$
is measurable.

<u>Pf</u>: Let $X \subseteq \mathbb{R}^n$ be arbitrary. We must show:
$$m^*(X) = m^*(X \cap H_i(a)) + m^*(X \cap H_i(a)^c)$$
Since the plane
$$P_i(a) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = a\}$$
is a zero set, for any subset $Y \subset \mathbb{R}^n$
$$m^*(Y) \le m^*(Y \setminus P_i(a)) + m^*(P_i(a) \cap Y) = m^*(Y \setminus P_i(a)) \le m^*(Y)$$
That is, $m^*(Y \setminus P_i(a)) = m^*(Y)$. Thus it suffices to assume $X \cap P_i(a) = \emptyset$ (by replacing $X$ with $X \setminus P_i(a)$).
Consequently:
$$X_+ := X \cap H_i(a) = \{(x_1, \dots, x_n) \in X : x_i > a\} \text{ and } X = X_+ \cup X_-$$
$$X_- := X \cap H_i(a)^c = \{(x_1, \dots, x_n) \in X : x_i < a\}$$
Let $\varepsilon > 0$, ∃ $\{B_k\}_{k \in \mathbb{N}}$ be a c'tble covering of $X$ by open boxes s.t.
$$\sum |B_k| \le m^*(X) + \varepsilon$$
Define for each $k \in \mathbb{N}$,

$B_k^+ := \{(x_1,\dots,x_n)\in B_k : x_i > a\}$ boxes

$B_k^- := \{(x_1,\dots,x_n)\in B_k : x_i < a\}$.

Then $\{B_k^\pm\}_{k\in\mathbb{N}}$ is an ctble covering of $X_\pm$ by open boxes. Hence

$$m^*(X\cap H_i(a)) + m^*(X\cap H_i(a)^c) = m^*(X_+) + m^*(X_-)$$
$$\le \sum_{k=1}^\infty |B_k^+| + \sum_{k=1}^\infty |B_k^-|$$
$$\le \sum_{k=1}^\infty |B_k| \le m^*(X)+\varepsilon.$$

Letting $\varepsilon \to 0$ yields

$$m^*(X\cap H_i(a)) + m^*(X\cap H_i(a)^c) \le m^*(X).$$

Since the other inequality always holds, we see that $H_i(a)$ is measurable.

Now, towards seeing that $A$ as above is measurable, note that

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n : a<x_i\} = H_i(a)\cap P_i(a)^c$$
$$\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_i<b\} = H_i(b)^c$$
$$\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_i\le b\} = H_i(b)^c \cup P_i(b)$$

are all measurable (since measurability is closed under finite unions and intersections and complements, and $P_i(a), P_i(b)$ are measurable as zero sets). But then $A$ is simply a (finite) intersection of such sets

(e.g. $[a_1,b_1)\times\cdots\times[a_n,b_n) = (H_1(a_1)\cap H_1(b_1)^c)\cap\cdots\cap(H_n(a_n)\cap H_n(b_n)^c)$)

Thus $A$ is measurable. $\square$

__Thm__ Every open and closed set in $\mathbb{R}^n$ is measurable.

__Pf__: Since a closed set is the complement of an open set, it suffices to show closed sets are measurable. But every open set is a countable union of open boxes and hence is measurable.

~~[illegible]~~ - Exercise $\square$

Since we can take unions and intersections ad nauseam, this isn't the end of the story.

Def: A $G_\delta$-set in $\mathbb{R}^n$ is a ctble intersection of open sets. An $F_\sigma$-set in $\mathbb{R}^n$ is a countable union of closed sets.

$\delta$ = durchschnitt
$\sigma$ = sum

11/1/2018 Exercise: Show open and closed sets are both $G_\delta$ and $F_\sigma$.

We immediately see that all $G_\delta$ and $F_\sigma$ sets are measurable.

Thm: For $E \in \mathcal{M}$, if and only if $\exists$ an $F_\sigma$-set $F \subseteq E$ and a $G_\delta$-set $G \supseteq E$ s.t. $m(G \setminus F) = 0$.

Pf: ($\Rightarrow$) First, suppose $E$ is bounded, and let $R \subseteq \mathbb{R}^n$ be a large open rectangle containing $E$.
For each $m \in \mathbb{N}$, let $\{B_k^{(m)}\}_{k \in \mathbb{N}}$ be a ctble covering of $E$ by open boxes s.t.

$$\sum |B_k^{(m)}|_{box} \le m(E) + \tfrac{1}{m}$$

Define

$$B^{(m)} := \bigcup_{k=1}^{\infty} B_k^{(m)} \leftarrow \text{open}$$

So that $E \subseteq B^{(m)}$ and by ctble subadd:

$$m(B^{(m)}) \le \sum_{k=1}^{\infty} m(B_k^{(m)}) \le m(E) + \tfrac{1}{m}.$$

Then define

$$G = \bigcap_{m=1}^{\infty} B^{(m)} \cap R$$

so that $G$ is a $G_\delta$ set containing $E$, and by cty from above

$$m(G) = \lim_{m \to \infty} m(B^{(1)} \cap \cdots \cap B^{(m)})$$
$$\le \lim_{m \to \infty} m(B^{(m)}) \le \lim_{m \to \infty} m(E) + \tfrac{1}{m} = m(E).$$

So $m(G) \le m(E)$, and by monotonicity we have $m(G) = m(E)$.
Applying the above argument $E^c \cap R$ yields a $G_\delta$ set $E^c \cap R \subseteq \tilde{G}$ s.t.

$$m(G') = m(E^c \cap \bar{R}) = m(\bar{R}) - m(E)$$

(by measurability of $E$). Define $F = \bar{R} \setminus G'$, then

$$\begin{aligned} m(F) &= m(\bar{R}) - m(G') \\ &= m(\bar{R}) - (m(\bar{R}) - m(E)) \\ &= m(E) \end{aligned}$$

Moreover, $F$ is an $F_\sigma$ set (exercise: check this), ~~Thus~~ and

$$\begin{aligned} F = \bar{R} \cap (G')^c &\subseteq \bar{R} \cap (E^c \cap \bar{R})^c \\ &= \bar{R} \cap (E \cup (\bar{R})^c) \\ &= E. \end{aligned}$$

So $F \subseteq E \subseteq G$ and $m(G \setminus F) = m(G) - m(F) = m(E) - m(E) = 0$.

If $E$ is unbounded, ~~let~~ find $G_\delta$ sets $G_n$, $F_\sigma$-sets $F_n$

$$F_n \subseteq E \cap B(\bar{0}, n) \subseteq G_n$$

Then

$$F = \bigcup F_n \qquad \text{and} \qquad G = \bigcup G_n$$

are $F_\sigma$ and $G_\delta$-sets, respectively (exercise).

Moreover, by monotonicity and subadd we have:

$$\begin{aligned} m\left(\bigcup G_n \setminus \bigcup F_n\right) &\leq m\left(\bigcup (G_n \setminus F_n)\right) \\ &\leq \sum_{n=1}^{\infty} m(G_n \setminus F_n) = \sum_{n=1}^{\infty} 0 = 0. \end{aligned}$$

($\Longleftarrow$) Suppose for $E \subseteq \mathbb{R}^n$, $\exists F \subseteq E \subseteq G$, $F_\sigma$ and $G_\delta$ sets s.t. $m(G \setminus F)$. Then

$$E = F \cup (E \setminus F)$$

~~Since~~ Since $E \setminus F \subseteq G \setminus F$, it is measurable as a zero set, and $F \in \mathcal{M}$. Thus $E \in \mathcal{M}$. $\square$

Exercise/Cor: Every Riemann measurable set is Lebesgue measurable.

Cor: Every $E \in \mathcal{M}$ is a $F_\sigma$ set union a zero set, and is a $G_\delta$ set take away a zero set.

Pf: Let $F \subset E \subset G$ be $F_\sigma$ and $G_\delta$ sets with $m(G \setminus F) = 0$. Then $Z = E \setminus F$ and $Z' = G \setminus E$ are zero sets and $E = F \cup Z = G \setminus Z'$. $\square$

<u>Example</u> (A non-measurable set)

Define an equivalence relation on $[0,1]$

$$x \sim y \iff x - y \in \mathbb{Q}$$

(Exercise: check this is an equiv. relation). Let $\mathcal{E}$ denote the collection of equivalence classes $[x]$, $x \in [0,1]$. Using the axiom of choice, let $N$ be a set consisting of exactly one representative from each equiv. class. We claim $N$ is non-measurable.

Pf: Suppose not. Since $m^*$ is trans. inv., $N + x$ is meas. for any $x \in \mathbb{R}$. In particular, let $\{r_k\}_{k \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [-1,1]$, and define

$$N_k := N + r_k$$

Then $m(N_k) = m(N)$ $\forall k \in \mathbb{N}$. Observe that $N_k \cap N_\ell = \emptyset$ for $k \neq \ell$. Indeed, if $x \in N_k \cap N_\ell$, then

$$x = y + r_k = z + r_\ell$$

for some $y, z \in N$. Since $k \neq \ell$, $r_k \neq r_\ell \Rightarrow y \neq z$. However, $y - z = r_\ell - r_k \in \mathbb{Q}$, so $y \sim z$. But distinct elements of $N$ belong to distinct equiv. classes, contradiction. So $N_k \cap N_\ell = \emptyset$.

Now, we next claim

$$[0,1] \subseteq \bigcup_{k=1}^{\infty} N_k \subseteq [-1,2].$$

The second inclusion follows from $N \subseteq [0,1]$ and $r_k \in [-1,1]$. For the first inclusion, given any $x \in [0,1]$, $\exists\, y \in N$ s.t. $x \sim y$. Hence $x - y \in \mathbb{Q}$. Moreover, $x - y \in \mathbb{Q} \cap [-1,1]$. Thus $\exists k$ s.t. $x - y = r_k \Rightarrow x \in N_k$.

Finally, by ctble additivity and monotonicity, we have

$$1 = m([0,1]) \leq m\Big(\bigcup_{k=1}^{\infty} N_k\Big) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N)$$

so $m(N) > 0$. But then $\hookrightarrow = +\infty$, contradicting

$$m\Big(\bigcup_{k=1}^{\infty} N_k\Big) \leq m([-1,2]) = 3. \qquad \square$$

Thus $N$ is non-measurable.