Taylor's Theorem Application

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Lemma. *For any $x \in \mathbb{R}$, $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$.*

Proof. Let $\epsilon > 0$. Let $k_1, k_2 \in \mathbb{N}$ be such that

$$k_1 < |x| \leq k_1 + 1 \qquad \text{and} \qquad k_2 - 1 \leq 2|x| < k_2.$$

Then for any $k \in \{k_1+1, k_1+2, \dots, k_2-1\}$ we have $\frac{|x|}{k} \leq 1$, and for any $k \geq k_2$ we have $\frac{|x|}{k} < \frac{1}{2}$. Define

$$M := \frac{|x|^{k_1}}{k_1!}.$$

Let $K \in \mathbb{N}$ be such that for any $k \geq K$, $\left(\frac{1}{2}\right)^k < \frac{\epsilon}{M}$. Then for $N := K + k_2$ if $n \geq N$ we have $n - k_2 \geq K$ so that:

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} = \frac{|x|}{n}\frac{|x|}{n-1}\cdots\frac{|x|}{k_2}\frac{|x|}{k_2-1}\cdots\frac{|x|}{k_1+1}M \leq \left(\frac{1}{2}\right)^{n-k_2+1}\cdot 1\cdot M \leq \left(\frac{1}{2}\right)^{K+1} M < \frac{\epsilon}{M}M = \epsilon. \qquad \square$$

Proposition. *Let $U \subset \mathbb{R}$ be an open interval of finite length. Suppose $f\colon U \to \mathbb{R}$ is n-times differentiable for each $n \in \mathbb{N}$, and that there exists $R > 0$ so that*

$$\sup\{|f^{(n)}(x)|\colon x \in U\} \leq R^n \qquad \forall n \in \mathbb{N}.$$

Fix any $a \in U$. For each $n \in \mathbb{N}$, define a degree n polynomial f_n by

$$f_n(x) := f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on U.

Proof. Let $\epsilon > 0$ and let r be the length of the interval U. Then for any $x \in U$ we have $|x - a| < r$. Now, by Taylor's theorem

$$f(x) - f_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between a and x. Thus we have

$$|f(x) - f_n(x)| \leq \frac{R^{n+1}}{(n+1)!}r^{n+1} = \frac{(Rr)^{n+1}}{(n+1)!}.$$

Since this upper bound holds for all $x \in U$, we have

$$\sup\{|f(x) - f_n(x)|\colon x \in U\} \leq \frac{(Rr)^{n+1}}{(n+1)!}.$$

By the Lemma, the right-hand side tends to zero as n tends to infinity and therefore $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on U. $\square$