104 Review 12/6

Basic Objects:

① Metric Spaces

(E,d) where E is a set & $d: E \times E \to \mathbb{R}$

s.t.

(i) $d(x,y) \geq 0 \quad \forall x,y \in E$

(ii) $d(x,y) = 0 \iff x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

d represents a distance within E

② Continuous functions

Two metric spaces (E,d), (E',d')

$f: E \to E'$

f is continuous at $x_0 \in E$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $d(x,x_0) < \delta$ for $x \in E$, $d'(f(x), f(x_0)) < \varepsilon$.

f is continuous on E if it is cts at every $x_0 \in E$.

f is uniformly cts if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x, x_0 \in E$ satisfy $d(x,x_0) < \delta$, $d'(f(x_0), f(x)) < \varepsilon$.

Difference: in regular cty, δ can depend on x_0. In unif cty, same δ works $\forall x,y \in E$.

③ Limits of Sequences

(E,d) a metric space, a sequence $(x_n)_{n \in \mathbb{N}} \subset E$

A sequence converges to $x \in E$ if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x) < \varepsilon$.

Philosophy for choosing N: balance of <u>simplicity</u> & <u>efficacy</u>.

Convergent Sequences:
- have unique limit
- are bounded. ($\exists y \in E$ & $R > 0$ s.t. $x_n \in B(y, R)$ $\forall n \in \mathbb{N}$)
- every subsequence is also convergent w/ same limit

Link btw cty & convergence gives us alternative defn for cty. → $f: E \to E'$ is cts at x_0 $\iff$ $\forall (x_n)_{n \in \mathbb{N}} \subset E$ converging to $x_0 \in E$, $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x_0)$.

Utility: to check cty at a pt, can check limits of sequences (good for fxns of $\mathbb{R}$).

④ Open & Closed Balls

Open ball: For (E, d), a metric space, $x \in E$ & $r > 0$, $B(x, r) = \{y \in E : d(x,y) < r\}$

Closed ball: Same as above except $B[x, r] = \{y \in E : d(x,y) \leq r\}$

Another alt defn for cty using connection to open balls → $f: E \to E'$ cts at $x_0 \in E$ $\iff$ $\forall$ open balls $B(f(x_0), \varepsilon) \subseteq E'$, $\exists$ $B(x_0, \delta) \subseteq E$ s.t. $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$.

alt defn for conv using connection to open balls → $(x_n)_{n\in\mathbb{N}} \subseteq E$ converges to $x \in E$ $\iff$ for every open ball $B(x, \varepsilon)$ centered at x, only finitely many indices $n \in \mathbb{N}$ satisfy $x_n \notin B(x, \varepsilon)$.

Open sets

$S \subseteq E$ is open if $\forall\, x \in S$, $\exists\, r > 0$ s.t. $B(x, r) \subseteq S$.

(every pt in S has room to wiggle w/out leaving S)

ex of open sets)
- $B(x, r)$ open balls
- $\emptyset$, E
- in $\mathbb{R}$, (a, b)
- arbitrary unions of open sets
- finite intersections of open sets

(not true for infinite intersections

Ex.) $S_n = (-\frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$

↳ open

But, $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ ← singleton set not open.)

In general, openness correspond to strict inequalities

this defn more useful for working w/ abstract metric spaces & functions → $f: E \to E'$ cts on E $\iff$ $\forall\, U \subseteq E'$ open,

$f^{-1}(U) = \{x \in E : f(x) \in U\} \subseteq E$ is open

Closed sets: $S \subseteq E$ is closed if either/both of following hold:

which defn you use depends on whether you "know more" about S or S^c

(i) $E \setminus S = S^c$ is open

(ii) $\forall$ conv sequences $(x_n)_{n\in\mathbb{N}} \subseteq S$, you also have $\lim_{n\to\infty} x_n \in S$.

Non-ex: $S = [0,1)$

$(1-\frac{1}{n})_{n\in\mathbb{N}} \subseteq S$, but $\lim_{n\to\infty}(1-\frac{1}{n}) = 1 \notin S$

So S isn't closed

Ex of closed sets

- $B[x,r]$ closed balls
- $\emptyset$, E
- $[a,b] \subseteq \mathbb{R}$
- arbitrary intersections of closed sets
- finite unions of closed sets

(Ex) $S_n = [\frac{1}{n}, 1-\frac{1}{n}]$ $n \geq 2$

$\bigcup_{n=2}^{\infty} S_n = (0,1) \leftarrow$ open set.

So only finite unions work w/ prev result

analogous to our open ball defn

$f: E \to E'$ is ctg on $E \iff \forall V \subseteq E'$ closed, $f^{-1}(V)$ closed.

Warning: sets can be
(a) open but not closed
(b) closed but not open
(c) both open & closed
(d) neither open nor closed

sequence clustering up, but we might not know which pt they cluster about.

Cauchy Sequences

$(x_n)_{n\in\mathbb{N}} \subseteq E$ is Cauchy if

$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t. $\forall n, m \geq N$,

$d(x_n, x_m) < \varepsilon$.

Convergent sequences are always Cauchy, but reverse not true.

Ex) $E = [0,1)$ $\quad (1-\frac{1}{n})_{n \in \mathbb{N}}$ is Cauchy but not convergent in (E,d).
(desired limit aka 1 not in metric space).

$f: E \to E'$ is uniformly cts $\Rightarrow$ for $(x_n)_{n \in \mathbb{N}} \subseteq E$ Cauchy, then $(f(x_n))_{n \in \mathbb{N}} \subseteq E'$ is Cauchy.

Note, cts fxns take conv sequences to conv sequences. Need uniform cty to take Cauchy → Cauchy

Completeness
A metric space (E,d) is complete if every Cauchy sequence converges.

ex) $\mathbb{R}$, $\mathbb{R}^n$ $\quad$ Non-ex: $\mathbb{Q}$, $[0,1)$
d: usual $\quad$ n-dim Euc

If E is complete, $S \subseteq E$, a closed subset of E, is complete.