Since $\varepsilon>0$ was arbitrary, we must have

$$\int_a^b f(x)\,dx \geq 0. \qquad \square$$

<u>Cor</u> If $f,g:[a,b]\to\mathbb{R}$ are integrable and $f(x)\leq g(x)$ $\forall x\in[a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

<u>Pf:</u> Apply the previous prop to $h=g-f$:

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g-f)(x)\,dx \geq 0. \qquad \square$$

<u>Cor</u> If $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$ and $\exists\, m,M\in\mathbb{R}$ s.t.

$$m\leq f(x)\leq M \qquad \forall x\in[a,b]$$

then

$$m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a)$$

<u>Pf:</u> Apply the previous cor. and Ex 1:

$$m(b-a)=\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a) \qquad \square$$

Existence of the Integral VI.3

(Cauchy-criterion for Riemann sums)

<u>Lemma 1:</u> $f:[a,b]\to\mathbb{R}$ is int'ble iff $\forall\varepsilon>0$ $\exists\delta>0$ s.t. $|S_1-S_2|<\varepsilon$ whenever S_1 and S_2 are Riemann sums for f corresponding to partitions of $[a,b]$ of width less than δ.

<u>Pf:</u> ($\Longrightarrow$) Suppose f is int'ble. Let $\varepsilon>0$, and let $\delta>0$ be s.t.

$$\left|S-\int_a^b f(x)\,dx\right|<\frac{\varepsilon}{2}$$

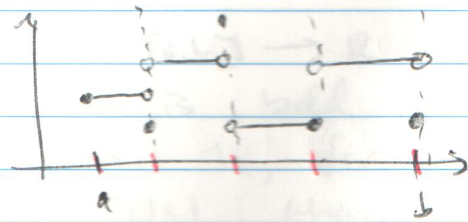
whenever S is a Riemann sum for f corresponding to a partition of width $<\delta$.

Then if S_1, S_2 are such Riemann sums, we have
$$|S_1 - S_2| \leq |S_1 - \int_a^b f(x)dx| + |\int_a^b f(x)dx - S_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$) Suppose f satisfies this Cauchy criterion on Riemann sums. For each $n \in \mathbb{N}$, consider a partition
$$a = x_0 < x_1 < \cdots < x_N = b$$
of width $< \frac{1}{n}$. Pick arbitrary rep's $x_i' \in [x_{i-1}, x_i]$, $i = 1, \ldots, N$, and denote
$$S^{(n)} := \sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}).$$
By our hypotheses, $(S^{(n)})_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy sequence and hence converges to some $A \in \mathbb{R}$. We claim A is the integral of f.

Let $\varepsilon > 0$, and let $\delta > 0$ be s.t.
$$|S_1 - S_2| < \frac{\varepsilon}{2}$$
whenever S_1 and S_2 are R.S. corresponding to partitions of width $< \delta$. Let $n \in \mathbb{N}$ be s.t. $\frac{1}{n} < \delta$ and s.t.
$$|S^{(n)} - A| < \frac{\varepsilon}{2}.$$
Then for any R.S. S for f corresponding to a partition of width $< \delta$, we have:
$$|S - A| \leq |S - S^{(n)}| + |S^{(n)} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Thus f is int'ble with $\int_a^b f(x)dx = A$. $\square$

<u>Def</u>: A function $f: [a,b] \to \mathbb{R}$ is called a <u>step function</u> if $\exists$ a partition
$$a = x_0 < x_1 < \cdots < x_N = b$$
s.t. f is constant in the open intervals (x_0, x_1), $(x_1, x_2), \ldots, (x_{N-1}, x_N)$ [and arbitrary on $\{x_0, x_1, \ldots, x_N\}$]



graph of a step function

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Lemma 2 Step functions are int'ble.

In particular, for a partition
$$a = x_0 < x_1 < \cdots < x_N = b$$
and $c_1, \dots, c_N \in \mathbb{R}$, if $f(x) = c_i$ for $x \in (x_{i-1}, x_i)$, $i = 1, \dots, N$, then
$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} c_i (x_i - x_{i-1})$$
[Note: we don't care about $f(x_0), f(x_1), \dots, f(x_N)$]

These functions are finite linear combinations of Ex (2) and Ex (3)

Step

Pf: For each $i = 1, \dots, N$, define $\varphi_i : [a,b] \to \mathbb{R}$ by
$$\varphi_i(x) = \begin{cases} c_i & \text{if } x_{i-1} \le x \le x_i \\ 0 & \text{otherwise} \end{cases}$$
Then
$$f - \sum_{i=1}^{N} \varphi_i$$
is zero except at finitely many points (i.e. $\{x_0, x_1, \dots, x_N\}$). By the same proof as in Ex (2)
$$\int_a^b f(x) - \sum_{i=1}^{N} \varphi_i(x)\,dx = 0$$
Note
$$\int_a^b \varphi_i(x)\,dx = c_i(x_i - x_{i-1})$$
by Ex (3). Thus $f = (f - \sum_i \varphi_i) + \sum_i \varphi_i$ is integrable with
$$\int_a^b f(x)\,dx = \int_a^b \Big(f - \sum_{i=1}^{N} \varphi_i\Big)(x)\,dx + \sum_{i=1}^{N} \int_a^b \varphi_i(x)\,dx$$
$$= \sum_{i=1}^{N} c_i (x_i - x_{i-1})$$
$\square$

Lemma 3 If $f:[a,b]\to\mathbb{R}$ is int'ble on $[a,b]$, then it is bdd on $[a,b]$.

Pf: By lemma 1, for $\varepsilon=1$, $\exists\,\delta>0$ s.t. $|S_1-S_2|<1$ whenever S_1 and S_2 are Riemann sums for f corresponding to partitions of width $<\delta$. Fix a partition $a=x_0<x_1<\cdots<x_N=b$ of width $<\delta$. Then for any $x_i', x_i''\in[x_{i-1},x_i]$, $i=1,\dots,N$ we have:

$$\left|\sum_{i=1}^{N} f(x_i')(x_i-x_{i-1}) - \sum_{i=1}^{N} f(x_i'')(x_i-x_{i-1})\right|<1$$

$$\left|\sum_{i=1}^{N}\big(f(x_i')-f(x_i'')\big)(x_i-x_{i-1})\right|$$

Fix $j\in\{1,\dots,N\}$ Choose rep's $x_i'=x_i''$ whenever $i\neq j$, and for $i=j$ choose $x_j'\in[x_{j-1},x_j]$ arbitrary and $x_j''=x_j$. Then the above estimate yields

$$|f(x_j')-f(x_j)|\,(x_j-x_{j-1})<1$$

so

$$|f(x_j')| \le |f(x_j)| + \frac{1}{x_j-x_{j-1}}$$

Thus f is bdd on $[x_{j-1},x_j]$. Repeating this argument for each $j=1,\dots,N$ shows f is bounded on all on of $[a,b]$ $\square$

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Prop $f:[a,b]\to\mathbb{R}$ is integrable if and only if $\forall\,\varepsilon>0$ $\exists$ step functions $f_1,f_2:[a,b]\to\mathbb{R}$ s.t.

$$f_1(x)\le f(x)\le f_2(x)\qquad \forall x\in[a,b]$$

and

$$\int_a^b f_2(x)-f_1(x)\,dx<\varepsilon.$$

Pf: $(\Longrightarrow)$ Suppose f is int'ble. Let $\varepsilon>0$ and let $\delta>0$ be as lemma 1 for $\frac{\varepsilon}{2}$. Fix a partition $a=x_0<x_1<\cdots<x_N=b$ of width $<\delta$

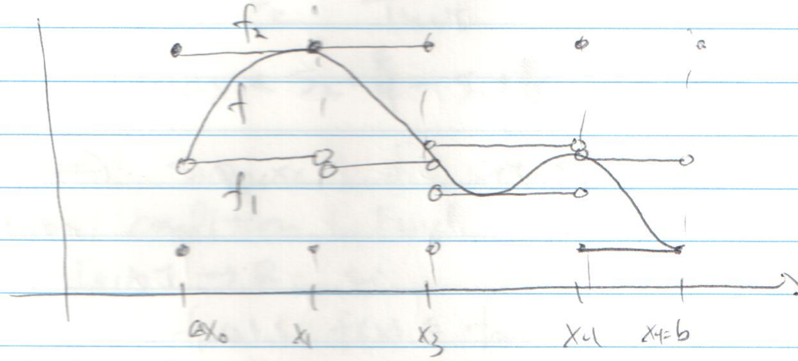
By lemma 3, f is bdd, so for each $i=1,\dots,N$ we can define:

$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$
$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}.$$

Consider the step functions:

$$f_1(x) = \begin{cases} m_i & \text{if } x_{i-1} < x < x_i \\ \min\{m_1,\dots,m_N\} & \text{if } x \in \{x_0, x_1,\dots,x_N\} \end{cases}$$

$$f_2(x) = \begin{cases} M_i & \text{if } x_{i-1} < x < x_i \\ \max\{M_1,\dots,M_N\} & \text{if } x \in \{x_0, x_1,\dots,x_N\} \end{cases}$$



By def of f_1, f_2 we have

$$f_1(x) \le f(x) \le f_2(x) \quad \forall x \in [a,b]$$

so it remains to show the integrals of f_1 and f_2 are close. By lemma 2:

$$\int_a^b f_1(x)\,dx = \sum_{i=1}^{N} m_i (x_i - x_{i-1})$$

$$\int_a^b f_2(x)\,dx = \sum_{i=1}^{N} M_i (x_i - x_{i-1})$$

These are <u>almost</u> Riemann sums, since f doesn't necessarily attain its inf. or sup. on each subinterval. But, it does get close. ~~Fix~~

~~Fix~~ $\varepsilon > 0$. For each $i=1,\dots,N$ find $x_i', x_i'' \in [x_{i-1}, x_i]$ s.t.

$$m_i \le f(x_i') \le m_i + \frac{\varepsilon}{4N(x_i - x_{i-1})}$$

$$M_i - \frac{\varepsilon}{4N(x_i - x_{i-1})} \le f(x_i'') \le M_i$$

Then

$$\int_a^b f_1(x)dx = \sum_{i=1}^N m_i(x_i - x_i) \le \underbrace{\sum_{i=1}^N f(x_i')\cdot(x_i - x_{i-1})}_{=: S_1} < \int_a^b f_1(x)\,dx + \frac{\varepsilon}{4}$$

Similarly

$$\int_a^b f_2(x)dx - \frac{\varepsilon}{4} < \underbrace{\sum_{i=1}^N f(x_i'')(x_i - x_{i-1})}_{=: S_2} \le \sum_{i=1}^N M_i(x_i - x_{i-1}) = \int_a^b f_2(x)dx$$

By our choice of δ, $|S_1 - S_2| < \varepsilon/2$.

$\Rightarrow S_2 - S_1 < \frac{\varepsilon}{2}$ Thus

$$\int_a^b f_2(x) - f_1(x)dx \le S_2 + \frac{\varepsilon}{4} - S_1 + \frac{\varepsilon}{4} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

~~Lett~~ ($\Leftarrow$). Suppose f satisfies the step function condition. Find step functions $f_1, f_2 : [a,b] \to \mathbb{R}$ s.t.

$$f_1(x) \le f(x) \le f_2(x) \qquad \forall x \in [a,b]$$

and s.t.

$$\int_a^b f_2(x) - f_1(x)dx < \frac{\varepsilon}{3}.$$

As int'ble functions (Lemma 2), $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$|S_i - \int_a^b f_i(x)dx| < \frac{\varepsilon}{3}$$

whenever S_i is a R.S. for f_i, ~~each~~ corresponding to a partition of width $< \delta_i$, $i = 1, 2$.

Let $a = x_0 < x_1 < \dots < x_N = b$ be a partition of width less than $\delta = \min\{\delta_1, \delta_2\}$, and let $x_i' \in [x_{i-1}, x_i]$, $i = 1, \dots N$. Set

$$S := \sum_{i=1}^N f(x_i')(x_i - x_{i-1}).$$

Then

$$\int_a^b f_1(x)dx - \frac{\varepsilon}{3} < \sum_{i=1}^N f_1(x_i')(x_i - x_{i-1})$$

$$\le S \le$$

$$\sum_{i=1}^N f_2(x_i')(x_i - x_{i-1}) < \int_a^b f_2(x)dx + \frac{\varepsilon}{3}$$

Thus S is contained in the interval
$$\left(\int_a^b f_1(x)dx - \frac{\varepsilon}{3}, \int_a^b f_2(x) + \frac{\varepsilon}{3}\right)$$
which has length
$$\int_a^b f_2(x)dx + \frac{\varepsilon}{3} - \int_a^b f_1(x)dx + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus if S_1 and S_2 are R.S.'s for f corresponding to partitions of width $< \delta$, then they are contained in the above interval and hence $|S_1 - S_2| < \varepsilon$. By Lemma 1, we have that f is int'ble. $\square$

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<u>Thm</u> If $f: [a,b] \to \mathbb{R}$ is cts, then it is int'ble on $[a,b]$.

<u>Pf</u>: Since $[a,b]$ is compact, f is unif cts. Let $\varepsilon > 0$ and find $\delta > 0$ st
$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$$
whenever $x', x'' \in [a,b]$ satisfy $|x' - x''| < \delta$.
Fix a partition $a = x_0 < x_1 < \cdots < x_N = b$ of width $< \delta$. Since f is cts, it attains its min & max on each subinterval $[x_{i-1}, x_i]$. Define $x_i', x_i'' \in [x_{i-1}, x_i]$, $i = 1, \ldots, N$ by:
$$f(x_i') = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$
$$f(x_i'') = \sup\{f(x) : x_{i-1} \le x \le x_i\}.$$
Define step functions:
$$f_1(x) = \begin{cases} f(x_i') & \text{if } x_{i-1} < x < x_i \\ f(x) & \text{if } x \in \{x_0, x_1, \ldots, x_N\} \end{cases}$$
$$f_2(x) = \begin{cases} f(x_i'') & \text{if } x_{i-1} < x < x_i \\ f(x) & \text{if } x \in \{x_0, x_1, \ldots, x_N\} \end{cases}$$
Then
$$f_1(x) \le f(x) \le f_2(x) \quad \forall x \in [a,b]$$
and

$$\int_a^b f_2(x) - f_1(x)\,dx = \sum_{i=1}^{N} (f(x_i'') - f(x_i')) \cdot (x_i - x_{i-1})$$
$$< \sum_{i=1}^{N} \frac{\varepsilon}{b-a} \cdot (x_i - x_{i-1}) = \varepsilon.$$

Thus the prop $\Rightarrow$ f is int'ble. □

The Fundamental Theorem of Calculus VI.4

<u>Prop</u> Let $a, b, c \in \mathbb{R}$, $a < b < c$, and let $f: [a,c] \to \mathbb{R}$. Then f is int'ble on $[a,c]$ if and only if f is int'ble on $[a,b]$ and $[b,c]$, in which case:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

<u>Pf</u> ($\Rightarrow$): suppose f is int'ble on $[a,c]$. Let $\varepsilon > 0$ and find step functions $f_1, f_2 : [a,c] \to \mathbb{R}$ s.t.

$$f_1(x) \le f(x) \le f_2(x) \quad \forall x \in [a,c]$$

and s.t.

$$\int_a^c f_2(x) - f_1(x)\,dx < \varepsilon.$$

Then restricting these functions to $[a,b]$ and $[b,c]$ (and using the formula for step functions which is easily checked), we see that f is integrable on $[a,b]$ and $[b,c]$. Using the formula for step functions, we can see that it holds for f as well.

($\Leftarrow$): Suppose f is int'ble on $[a,b]$ and $[b,c]$. Let $\varepsilon > 0$. Then $\exists\, h_1, h_2 : [a,b] \to \mathbb{R}$, $k_1, k_2 : [b,c] \to \mathbb{R}$ s.t.

$$h_1(x) \le f(x) \le h_2(x) \quad \forall x \in [a,b]$$
$$k_1(x) \le f(x) \le k_2(x) \quad \forall x \in [b,c]$$

and s.t.

$$\int_a^b h_2(x) - h_1(x)\,dx < \varepsilon/2$$
$$\int_b^c k_2(x) - k_1(x)\,dx < \varepsilon/2$$

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