by Taylor's theorem:

$$|f(x) - f_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

for some c between x and a. By assumption

$$\leq \frac{R^{n+1} \cdot r^{n+1}}{(n+1)!} = \frac{(R \cdot r)^{n+1}}{(n+1)!}$$

Since this doesn't depend on x, we have:

$$\sup\{|f(x) - f_n(x)| : x \in U\} \leq \frac{(R\cdot r)^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$

by Lemma. Thus $(f_n)_{n\in\omega}$ conv. unif. to f on U. ☐

Riemann Integration VI

Def: Let $a, b \in \mathbb{R}$ with $a < b$. A partition of $[a,b]$ is a finite sequence $x_0, x_1, x_2, \ldots, x_N$ s.t.

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

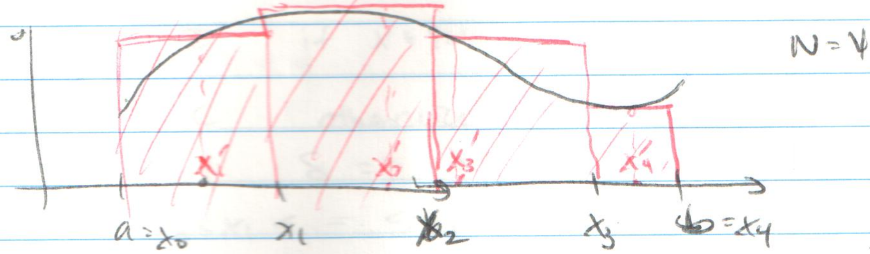
The width of the partition $\{x_0, x_1, \ldots, x_N\}$ is $\max\{x_i - x_{i-1} : i = 1, \ldots, N\}$

Def If $f : [a,b] \to \mathbb{R}$, a Riemann sum for f (corresponding to the partition ~~$\{x_0, x_1, \ldots, x_N\}$~~ $a = x_0 < x_1 < \cdots < x_N = b$) is the quantity:

$$f(x_1')\cdot(x_1 - x_0) + f(x_2')\cdot(x_2 - x_1) + \cdots + f(x_N')(x_N - x_{N-1}) =: \sum_{i=1}^{N} f(x_i')(x_i - x_{i-1})$$

where $x_i' \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, N$. We call $x_1', \ldots, x_N'$ representatives.

Note that each choice of $x_1', \ldots, x_N'$ gives a different Riemann sum for f.



Riemann sum S is the sum of the area of the red rectangles.

11/20/2017 **Def:** Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a,b] \to \mathbb{R}$. We say f is Riemann integrable (on $[a,b]$) if $\exists A \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. whenever S is a Riemann sum corresponding to a partition of width less than δ, we have $|S - A| < \varepsilon$.

In this case A is called the Riemann integral of f (from a to b) and is denoted

$$A = \int_a^b f(x)\,dx.$$

Exercise: The integral, if it exists, is unique.

Ex: (1) $f : [a,b] \to \mathbb{R}$, $f(x) = c \in \mathbb{R}$ $\forall x \in [a,b]$.

Then f is Riemann integrable with

$$\int_a^b f(x)\,dx = c(b-a)$$

Indeed, for any partition $a = x_0 < x_1 < \dots < x_N = b$ and any representative $x_1', \dots, x_N'$

$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{N} c \cdot (x_i - x_{i-1})$$

$$= c \cdot [(x_1 - x_0) + (x_2 - x_1) + \dots + (x_N - x_{N-1})]$$

$$= c \cdot (-x_0 + x_N) = c \cdot (b-a)$$

So $\int_a^b f(x)\,dx = c \cdot (b-a)$

(2) Let $c \in \mathbb{R}$, $w_1, \dots, w_d \in [a,b]$.

Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \{w_1, \dots, w_d\} \\ c & \text{otherwise} \end{cases}$$

<u>Claim</u>: $\int_a^b f(x) = 0$.

Let $\varepsilon > 0$. Set $\delta = \boxed{\min\left\{ \frac{b-a}{2d}, \frac{\varepsilon}{2d|c|} \right\}}$

Suppose $a = x_0 < x_1 < \dots < x_N = b$ is a partition of width less than δ, and let $x_1', \dots, x_N'$ be rep's. Then

$$\left| \sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) \right| \le \sum_{i=1}^{N} |f(x_i')| \cdot |x_i - x_{i-1}|$$

$$\le \sum_{i=1}^{N} |f(x_i')| \cdot \delta$$

Observe that $f(x_i') \neq 0$ only when $x_i' \in \{w_1, \dots, w_d\}$. Thus at most $2d$ of those terms is non-zero:

$$\le \min\{N, 2d\} \cdot |c| \cdot \delta$$

~~However,~~ $N \cdot \delta \ge b - a \implies N \ge \frac{b-a}{\delta}$

Thus if $\frac{b-a}{\delta} > 2d \iff \delta < \frac{b-a}{2d}$ then $N > 2d$.

So

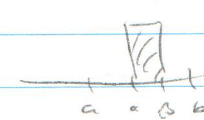
$$\le 2d \cdot |c| \cdot \delta$$

If $\delta < \frac{\varepsilon}{2d|c|}$, we have $\dots < \varepsilon$.

Thus f is Riemann integrable w/ $\int_a^b f(x)dx = 0$.

(3) Let $a < \alpha < \beta < b$, $c \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, ~~such~~

$$f(x) = \begin{cases} c & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$



<u>Claim</u>: $\int_a^b f(x)dx = c(\beta - \alpha)$

Let $\varepsilon > 0$. Set $\delta = \min\left\{ \frac{\beta - \alpha}{2}, \frac{\varepsilon}{2c} \right\}$

Suppose $a = x_0 < x_1 < \dots < x_N = b$ is a partition of width less δ. Let $p, q \in \{0, \dots, N\}$ be s.t.

$$x_{p-1} \le \alpha < x_p \qquad x_{q-1} < \beta \le x_q$$

Note that $\delta < \frac{\beta - \alpha}{2} \implies \exists\, x_j \in (\alpha, \beta)$ $x_j, x_{j+1} \in (\alpha, \beta)$

and so $p \le j \cancel{\le q} + 1 \le q-1$

$\Rightarrow \quad p+1 \le q-1$.

Now, ~~fix~~ rep's $x_i' \in [x_{i-1}, x_i]$ ~~and~~ denote the corresponding Riemann sum

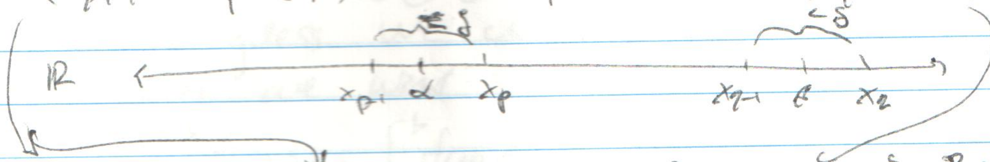
$$S = \sum_{i=1}^{N} f(x_i')(x_i - x_{i-1})$$

We have

$$\underbrace{\sum_{i=p+1}^{q-1} c\cdot(x_i - x_{i-1})}_{= c\cdot(x_{q-1} - x_p)} \le S \le \underbrace{\sum_{i=p}^{q} c\cdot(x_i - x_{i-1})}_{= c\cdot(x_q - x_{p-1})}$$

Subtracting $c\cdot(\beta-\alpha)$ yields

$$c\cdot((x_{q-1}-\beta) - \underbrace{(x_p - \alpha)}_{\le \delta}) \le S - c\cdot(\beta-\alpha) \le c(\underbrace{(x_q-\beta)}_{<\delta} - (x_{p-1}-\alpha))$$



$$-2c\cdot\delta \le \quad \le S - c(\beta-\alpha) \le \quad \le 2c\cdot\delta$$

So $\delta < \frac{\varepsilon}{2c}$ yields: $-\varepsilon \le S - c(\beta-\alpha) \le \varepsilon$

$\iff |S - c\cdot(\beta-\alpha)| < \varepsilon$.

Thus f is int'ble w/ claimed integral.

(4) Let $f:[a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Then f is not Riemann integrable.

We'll show that for arbitrarily ~~small~~ narrow partitions, we can find rep's yielding Riemann sums ~~as~~ of $b-a$ and 0.

Let $\delta > 0$ and suppose $a = x_0 < x_1 < \dots < x_N = b$ is a partition of width $< \delta$.

(i) Since $\mathbb{Q}$ is dense, can pick ~~[illegible]~~
$x_i' \in \mathbb{Q} \cap [x_{i-1}, x_i]$ $i = 1, \dots, N$.

Then

$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{N} x_i - x_{i-1} = b-a.$$

(ii) Since $\mathbb{R}\setminus\mathbb{Q}$ is dense, can also pick
$x_i' \in (\mathbb{R}\setminus\mathbb{Q}) \cap [x_{i-1}, x_i]$ $i=1,..,N$
Then
$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{N} 0 = 0.$$
Thus there is no $A \in \mathbb{R}$ s.t. $|S-A|$ small.

~~Prop: If f is Riemann integrable on [a,b], then its integral is unique.~~

Remark: The ~~role of~~ 'x' in $\int_a^b f(x)\,dx$ is meaningless. We can replace it by any character we want:
$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(☺)\,d☺$$

11/15/2017

Linearity and order Properties of the Integral VI.2

Prop: (i) If $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ then so is $f+g$ with
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(ii) If $f : [a,b] \to \mathbb{R}$ is int'ble on $[a,b]$ ~~and $c \in \mathbb{R}$~~ then so is $c \cdot f$ for any $c \in \mathbb{R}$ with
$$\int_a^b (c \cdot f)(x)\,dx = c \int_a^b f(x)\,dx.$$

Pf (i) Let $\varepsilon > 0$. Then f, g int'ble means $\exists\ \delta_1, \delta_2 > 0$ s.t. if S_1 and S_2 are Riemann sums of f and g corresponding to partitions of width