Exercise, show $\lim_{x\to 0} g(x)$ does not exist.

return to prop.

Def: Let $f: U \to \mathbb{R}$ w/ $U \subseteq \mathbb{R}$ open. If $f'(x_0)$ exists for all $x_0 \in U$, we say f is differentiable (on U). Consequently, $x_0 \mapsto f'(x_0)$ defines a function
$$f': U \to \mathbb{R}$$
we call f' the derivative of f. Also denote $f' = \frac{d}{dx}f = \frac{df}{dx}$

Rules of Differentiation V.2

Prop: Let $f, g: U \to \mathbb{R}$ for $U \subseteq \mathbb{R}$ open. If f and g are diff'ble at $x_0 \in U$, then so are $f \pm g$, $f \cdot g$, and, if $g(x_0) \neq 0$, f/g. Their derivatives are given by:
$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$
$$(f \cdot g)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$$
$$(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Pf: Simply use limit arithmetic.
$$(f \cdot g)'(x_0) = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}$$
$$= \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}\right) + \left(g(x_0) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right)$$
using cty of f at x_0 $= f(x_0) \cdot g'(x_0) + g(x_0)f'(x_0)$

Check rest as hw. $\square$

11/15/24

Cor: (i) If $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}$ open, diff'ble at $x_0 \in U$ then $\forall a \in \mathbb{R}$ $(a \cdot f)'(x_0) = a f'(x_0)$

(ii) $\forall n \in \mathbb{N}$, $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$.

Prop (Chain Rule)

Let $U, V \subseteq \mathbb{R}$ be open. Let $f: U \to V$, $g: V \to \mathbb{R}$. For $x_0 \in U$, assume f diff'ble at x_0, and that g is diff'ble at $f(x_0)$. Then $g \circ f$ is diff'ble at x_0 with

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Pf: Define $Q: V \to \mathbb{R}$ by

$$Q(y) = \begin{cases} \frac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } y \in V \setminus \{f(x_0)\} \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$$

By def. of $g'(f(x_0))$, Q is cts at $f(x_0)$. Since the comp. of cts functions is cts, we have that $Q \circ f$ is cts at x_0.

Thus

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{Q(f(x)) (f(x) - f(x_0))}{x - x_0}$$

$$= Q(f(x_0)) \cdot f'(x_0)$$

$$= g'(f(x_0)) \cdot f'(x_0). \quad \square$$

The Mean Value Theorem V.3

Prop: Let $f: U \to \mathbb{R}$ with $U \subseteq \mathbb{R}$ open. Assume f attains a minimum or maximum value at $x_0 \in U$. If f is diff'ble at x_0, then $f'(x_0) = 0$.

Pf: Suppose, TAC, $|f'(x_0)| \neq 0$. Then letting $\varepsilon = \frac{|f'(x_0)|}{2}$, $\exists \delta > 0$ s.t. if $x \neq x_0$ satisfies $|x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \frac{|f'(x_0)|}{2}$$

$$\iff \quad f'(x_0) - \frac{|f'(x_0)|}{2} < \frac{f(x) - f(x_0)}{x - x_0} < f'(x_0) + \frac{|f'(x_0)|}{2}$$

$$\left.\begin{matrix} \frac{f'(x_0)}{2} & \text{if } f'(x_0) > 0 \\ \frac{3f'(x_0)}{2} & \text{if } f'(x_0) < 0 \end{matrix}\right\} = \qquad \begin{cases} \frac{3f'(x_0)}{2} & \text{if } f'(x_0) > 0 \\ \frac{f'(x_0)}{2} & \text{if } f'(x_0) < 0 \end{cases}$$