$y = f(x)$ for some $x \in (a,b)$. Applying the IVT to $f|_{[a,x]}$ we obtain $c \in (a,x)$ s.t. $f(c) = f(b)$. Since $c < x < b$, this contradicts f being 1-1. Thus $f([a,b]) \subseteq [f(a), f(b)]$ (QED).

Sequences of Functions IV.6

Let (E,d) and (E',d') be metric spaces. Given a collection of functions $\{f_n : n \in \mathbb{N}\}$ s.t.

$$f_n : E \to E' \quad \forall n \in \mathbb{N}$$

~~[illegible]~~ for any $x \in E$, observe that

$$(f_n(x))_{n \in \mathbb{N}} \subseteq E'$$

is a sequence. We might wonder about convergence of this sequence and whether it holds for all $x \in E$ or only some $x \in E$. We will think of the collection of functions as a sequence itself:

$$(f_n)_{n \in \mathbb{N}}.$$

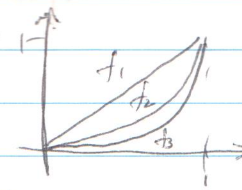
Def: Let (E,d) and (E',d') be metric spaces, and for each $n \in \mathbb{N}$ let $f_n : E \to E'$ be a function. For $x \in E$, we say $(f_n)_{n \in \mathbb{N}}$ converges at x if the sequence $(f_n(x))_{n \in \mathbb{N}} \subseteq E'$ converges. ~~[illegible] converges [illegible]~~ For $S \subseteq E$, we say $(f_n)_{n \in \mathbb{N}}$ converges pointwise on S if it converges at every $x \in S$. If $S = E$, we simply say $(f_n)_{n \in \mathbb{N}}$ converges pointwise, or is pointwise convergent. In this case the function $f : E \to E'$ defined by

$$f(x) = \lim_{n \to \infty} f_n(x)$$

is called the <u>limit function</u> of $(f_n)_{n\in\mathbb{N}}$.

Ex: ① For each $n\in\mathbb{N}$, define $f_n: [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$.

Then $(f_n)_{n\in\mathbb{N}}$ converges pointwise (on $[-1,1]$) to:



$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Indeed, if $-1 < x < 1$, then $|x| < 1$ and so by an earlier result

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} |x|^n = 0.$$

That is, $(f_n)_{n\in\mathbb{N}}$ converges at x to 0.

If $x = 1$, then $f_n(1) = 1^n = 1$ $\forall n\in\mathbb{N}$ and so

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1.$$

Thus $(f_n)_{n\in\mathbb{N}}$ converges at 1 to 0.

~~Remark: If we extend the domain~~

If $x = -1$, then $f_n(-1) = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases}$

Since $-1, 1, -1, 1, \ldots$ doesn't converge, f_n doesn't converge pointwise at -1.

If $x \in \mathbb{R} \setminus [-1,1]$, then $|x|^n > 1$. Consequently

$$(f_n(x))_{n\in\mathbb{N}} = (x^n)_{n\in\mathbb{N}}$$

is unbdd and thus non-convergent.

11/6/2017

② For each $n\in\mathbb{N}$, define $f_n: \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{x}{n}$.

Then $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$ with limit function: $f(x) = 0$.

Indeed, for any $x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{x}{n} = 0$.

Suppose a sequence $(f_n)_{n\in\mathbb{N}}$ of functions ~~on~~ $f_n: E\to E'$ converges pointwise to a $f: E\to E'$. Formally this means:

$$\forall x\in E\ \forall \varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } \forall n\geq N\ d'(f_n(x), f(x))<\varepsilon.$$

Consequently, N may depend on $x\in E$.

Indeed, in Ex ① above, the closer x is to -1 or 1, the larger N must be.

In Ex ② N also depends on x: need

~~Def~~ $\quad |\frac{x}{n}-0|<\varepsilon \iff \frac{|x|}{|n|}<\varepsilon \iff N>\frac{|x|}{\varepsilon}$.

Def: Let (E,d) and (E',d') be metric spaces, and for each $n\in\mathbb{N}$ let $f_n: E\to E'$ be a function. Let $S\subseteq E$ and let $f: S\to E'$ be another function. We say $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on S if

$$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } \forall n\geq N\ \forall x\in S\ d'(f_n(x), f(x))<\varepsilon ~~\forall x\in E~~.$$

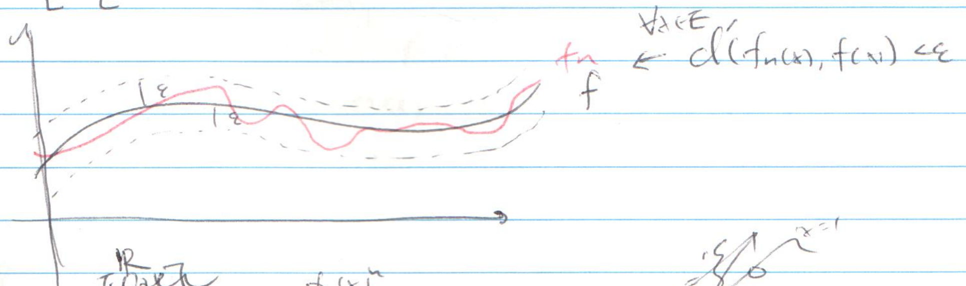
We also say $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on S.

If $S=E$, we simply say $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f ~~on E~~. ~~if (f_n) converges uniform~~ ~~$\forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $\forall n\geq N$, $\forall x\in S$ $d'(f_n(x), f(x))<\varepsilon$~~ call f limit function still.

Remark: Unif conv. on $S \Rightarrow$ unif. conv. on $T\subseteq S$. In particular $\Rightarrow$ pointwise converg.

- ~~Note that if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, it also converges pointwise to f, so we still call f the limit function.~~

Picture: $E=E'$

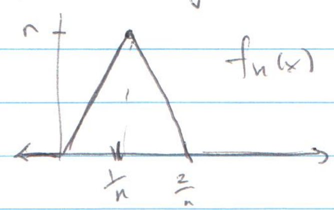


Ex ① $\quad f_n: ~~[0,1]~~ \to \mathbb{R}$, $f(x)^n$ ~~converges pointwise to f, but~~ does not converge uniformly to $f = \begin{cases} 0 \\ 1 \end{cases}$ on $[0,1]$

Indeed set ε to be anything in $(0,1)$.
Let $N \in \mathbb{N}$. We need to show $\exists n \geq N$ and $x \in [0,1]$
s.t. $$|f_n(x) - f(x)| \geq \varepsilon.$$
Note that for $x \in [0,1)$ we have for any $n \in \mathbb{N}$
~~Let $n \in \mathbb{N}$, then for $x \in$~~
$$|f_n(x) - f(x)| = |x^n - 0| = x^n.$$
This is $\geq \varepsilon \iff x \geq \varepsilon^{1/n}$. Since $\varepsilon \in (0,1)$
we can ~~always find set~~ take $x = \varepsilon^{1/n}$ ~~an x~~. So simply
set $n = N$ and let ~~$x \in [\varepsilon^{1/N}, 1)$~~ $x = \varepsilon^{1/N}$.

(2) Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. Then f_n converges uniformly to $f(x) = 0$ on any bounded subset.
Let $S \subseteq \mathbb{R}$ be bdd: $S \subseteq B[0,R]$ for some $R > 0$.
Let $\varepsilon > 0$, then set $N \in \mathbb{N}$ s.t. $N > \frac{R}{\varepsilon}$.
For $n \geq N$ and $x \in S$ we have
$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} \leq \frac{R}{n} < \varepsilon.$$

(3) For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by
$$f_n(x) = \begin{cases} 0 & x < 0 \\ n^2 x & 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \frac{2}{n} < x \end{cases}$$



Note that ~~max~~ $\sup\{f_n(x) : x \in \mathbb{R}\} = f_n(\frac{1}{n}) = n$
~~Hence~~ f_n converges pointwise ~~but~~ not uniformly,
with limit function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$.
To see pointwise convergence, fix ~~$x \in \mathbb{R}$ and let $\varepsilon > 0$~~.
If $x = 0$, then $f_n(x) = 0$ $\forall n \in \mathbb{N}$ so ~~$f(x)$~~ $\lim_{n\to\infty} f_n(x) = 0$.
If $x \neq 0$, let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{2}{N} < |x|$.
Consequently $x \notin [0, \frac{2}{n}]$ $\forall n \geq N \Rightarrow f_n(x) = 0$ $\forall n \geq N$.

To see it does not converge unif.

Set $\varepsilon = 1$ and let $N \in \mathbb{N}$. For $n = N$, and $x = \frac{1}{n}$ we have

$$|f_n(\tfrac{1}{n}) - f(\tfrac{1}{n})| = |n - 0| = n \geq 1 = \varepsilon.$$

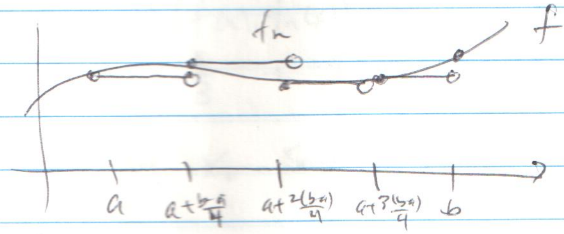
Thus the conv. is not unif.

Ex ④ Let $f : [a,b] \to \mathbb{R}$ be a cts function. Since $[a,b]$ is compact, f is unif cts. For each $n \in \mathbb{N}$, define $f_n : [a,b] \to \mathbb{R}$ piecewise on subintervals:

$$[a, a+\tfrac{b-a}{n}),\ [a+\tfrac{b-a}{n}, a+\tfrac{2(b-a)}{n}),\ \dots,\ [a+\tfrac{(n-1)(b-a)}{n}, b)\ \{b\}$$

as follows: for $x \in [a + \frac{k(b-a)}{n}, \frac{(k+1)(b-a)}{n})$ $k = 0, \dots, n-1$

$$f_n(x) = f(a + \tfrac{k(b-a)}{n}) \quad \text{and } f_n(b) = f(b).$$



Then $(f_n)_{n \in \mathbb{N}}$ converges unif. to f on $[a,b]$.
Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. if $x, y \in [a,b]$ satisfy $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.
Let $N \in \mathbb{N}$ be s.t. $\frac{b-a}{N} < \delta$. For $n \geq N$, if $x \in [a,b]$, we have either $x = b$ in which case

$$|f_n(b) - f(b)| = |f(b) - f(b)| = 0 < \varepsilon$$

or $\exists\, k \in \{0, 1, 2, \dots, n-1\}$ s.t.

$$x \in [a + \tfrac{k(b-a)}{n}, a + \tfrac{(k+1)(b-a)}{n})$$

Since

$$|x - a + \tfrac{k(b-a)}{n}| \leq \tfrac{b-a}{n} < \delta$$

we know

~~$|f(x)| - f(a+$~~

$$|f_n(x) - f_n(x)| = |f(a + \tfrac{k(b-a)}{n}) - f(x)| < \varepsilon.$$

Thus f converges unif. to f.

<u>Thm</u> Let (E,d) and (E',d') be metric spaces, and let $(f_n)_{n\in\mathbb{N}}$ be a sequences of cts functions $f_n: E\to E'$. ~~Sup~~ If $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f: E\to E'$, then f is cts.

<u>Pf</u> ($\frac{\varepsilon}{3}$ proof ~ famous)

Fix $x_0\in E$ and let $\varepsilon>0$. Let $N\in\mathbb{N}$ be s.t. $\forall n\geq N$

$$d'(f_n(x), f(x)) < \frac{\varepsilon}{3} \qquad \forall x\in E$$

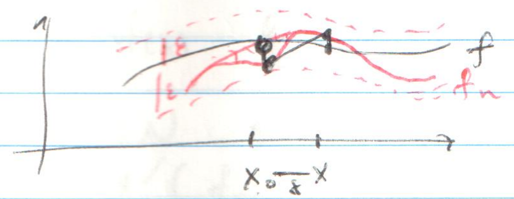
Fix $n\geq N$. Since f_n is cts at x_0, $\exists\,\delta>0$ s.t. if $x\in E$ satisfies $d(x,x_0)<\delta$ then

$$d'(f_n(x), f_n(x_0)) < \varepsilon/3.$$

Suppose $x\in E$ satisfies $d(x,x_0)<\delta$. Then

$$d'(f(x), f(x_0)) \leq d'(f(x), f_n(x)) + d'(f_n(x), f_n(x_0)) + d'(f_n(x_0), f(x_0))$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus f is cts at x_0. Since $x_0\in E$ was arbitrary, f is cts on E. $\square$



11/8/2017

<u>Prop</u> Let (E,d) and (E',d') be metric spaces, and assume (E',d') is complete. For each $n\in\mathbb{N}$, let $f_n: E\to E'$ be a function. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if

$$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } \forall n,m\geq N \text{ and } \forall x\in E$$
$$d'(f_n(x), f_m(x)) < \varepsilon$$

(Cauchy-type condition)

<u>Pf</u> ($\Longrightarrow$) Suppose $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some $f: E\to E'$. Then $\forall\varepsilon>0\ \exists N\in\mathbb{N}$

s.t. $\forall n \geq N$ and $\forall x \in E$

$$d'(f_n(x), f(x)) < \frac{\varepsilon}{2}$$

Thus $\forall n, m \geq N$ and $\forall x \in E$

$$d'(f_n(x), f_m(x)) \leq d'(f_n(x), f(x)) + d'(f(x), f_m(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Longleftarrow$) Suppose $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ and $\forall x \in E$

$$d'(f_n(x), f_m(x)) < \varepsilon$$

Fix $x \in E$, then this implies $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence. Since (E', d') is complete, it converges. This happens for each $x \in E$, so we can define $f: E \to E'$ by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

We need to show $(f_n)_{n \in \mathbb{N}}$ converges unif. to f.
Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be s.t. $\forall n, m \geq N$ and $\forall x \in E$

$$d'(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$$

Now, ~~then~~ fix $x \in E$. Since

$$f(x) = \lim_{m \to \infty} f_m(x)$$

we can find $m_x \geq N$ s.t.

$$d'(f_{m_x}(x), f(x)) < \frac{\varepsilon}{2}$$

Now, ~~for arbitrary let take then take~~ for $n \geq N$ and any $x \in E$

$$d'(f_n(x), f(x)) \leq d'(f_n(x), f_{m_x}(x)) + d'(f_{m_x}(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(f_n)_{n \in \mathbb{N}}$ converges unif. to f. $\square$

We want to ~~know~~ construct a metric space where the "points" are cts functions between two fixed metric spaces.

<u>Lemma</u> Let (E, d) and (E', d') be metric spaces,

and let $f, g: E \to E'$ be cts functions.
Then $h: E \to \mathbb{R}$ defined by

$$h(x) := d'(f(x), g(x))$$

is cts.

Pf: Fix $x_0 \in E$ and let $\varepsilon > 0$. Let $\delta_1, \delta_2 > 0$ be s.t. for $x \in E$

if $d(x_0, x) < \delta_1$ then $d'(f(x_0), f(x)) < \frac{\varepsilon}{2}$

if $d(x_0, x) < \delta_2$ then $d'(g(x_0), g(x)) < \frac{\varepsilon}{2}$

(by cty of f and g). Let $\delta = \min\{\delta_1, \delta_2\}$, and suppose $x \in E$ satisfies $d(x_0, x) < \delta$.
Then

$$|h(x_0) - h(x)| = |d'(f(x_0), g(x_0)) - d'(f(x), g(x))|$$
$$= |d'(f(x_0), g(x_0)) - d'(f(x), g(x_0)) + d'(f(x), g(x_0)) - d'(f(x), g(x))|$$
$$\overset{\Delta\text{-ineq } |\cdot|}{\leq} |d'(f(x_0), g(x_0)) - d'(f(x), g(x_0))| + |d'(f(x), g(x_0)) - d'(f(x), g(x))|$$
$$\overset{\text{rev. } \Delta\text{-ineq } d'}{\leq} d'(f(x_0), f(x)) + d'(g(x_0), g(x))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus h is cts at x_0. Since $x_0 \in E$ was arbitrary, we see h is cts on E. □

Fix now metric spaces (E, d) and (E', d').
Assume E is compact. Define

$$C(E, E') = \{ f: E \to E' \mid f \text{ cts} \}$$

For $f, g \in C(E, E')$, define

$$D(f, g) := \sup\{ d'(f(x), g(x)) : x \in E \}$$

By the previous lemma, $E \ni x \mapsto d'(f(x), g(x)) \in \mathbb{R}$ is cts. Since E is compact, the sup is attained, and hence

$$D(f, g) = \max\{ d'(f(x), g(x)) : x \in E \} < \infty.$$

Note also that $D(f, g) \geq 0$ $\forall x \in E$.

- We claim that $(C(E,E'), D)$ is a metric space:
  - (1) $D(f,g) \geq 0$ for all $f,g \in C(E,E')$ ✓
  - (2) $D(f,g) = 0$ iff $f=g$:
    Indeed, clearly $D(f,f)=0$. If $D(f,g)=0$, then $0 \leq d'(f(x), g(x)) \leq D(f,g) = 0 \quad \forall x \in E$
    $\Rightarrow d'(f(x),g(x)) = 0 \quad \forall x \in E \Rightarrow f(x)=g(x) \quad \forall x \in E$
    $\Rightarrow f=g$. ✓
  - (3) $D(f,g) = D(g,f) \quad \forall f,g \in C(E,E')$:
    Indeed, since $d'(f(x),g(x)) = d'(g(x), f(x))$. ✓
  - (4) $D(f,g) \leq D(f,h) + D(h,g) \quad \forall f,g,h \in C(E,E')$
    Indeed, fix $x_0 \in E$. Then
    $d'(f(x_0), g(x_0)) \leq d'(f(x_0), h(x_0)) + d'(h(x_0), g(x_0))$
    $\leq D(f,h) + D(h,g)$
    Since this holds for all $x_0 \in E$, we get
    $D(f,g) \leq D(f,h) + D(h,g)$. ✓

  Thus $(C(E,E'), D)$ is a metric space.

- Consider a sequence in this metric space: $(f_n)_{n\in\mathbb{N}} \subseteq C(E,E')$.
  Then for each $n \in \mathbb{N}$
  $f_n : E \to E'$ is a function.
  Suppose $(f_n)_{n\in\mathbb{N}}$ converges to some $f \in C(E,E')$ w.r.t. D. Then
  $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \quad D(f_n, f) < \varepsilon$
  where $D(f_n,f) = \sup\{d'(f_n(x), f(x)) : x \in E\}$

  Thus this is equiv. to $(f_n)_{n\in\mathbb{N}}$ converging unif. to f on E

- Suppose (E', d') is complete. Then we claim $(C(E,E'), D)$ is complete. Indeed, suppose $(f_n)_{n\in\mathbb{N}} \subseteq C(E,E')$ is a Cauchy seq. That is,
  $\forall \varepsilon > 0, \ \exists N \in \mathbb{N}$ s.t. $\forall n,m \geq N$

$$D(f_n, f_m) < \varepsilon$$

$\Leftrightarrow$ $\sup\{d'(f_n(x), f_m(x)) : x \in E\} < \varepsilon$

$\Rightarrow$ $\forall x \in E$ $d'(f_n(x), f_m(x)) < \varepsilon$.

So by an earlier prop, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some $f : E \to E'$. Since each f_n is cts, the unif. conv. $\Rightarrow$ f is cts and therefore $f \in C(E,E')$. Finally, as we saw above, $(f_n)_{n\in\mathbb{N}}$ conv unif. to f $\Leftrightarrow$ $(f_n)_{n\in\mathbb{N}}$ conv to f wrt D. Thus the Cauchy seq. $(f_n)_{n\in\mathbb{N}}$ conv. to f, and so $(C(E,E'), D)$ is complete. ∎

11/13/2017

~~Thus~~ We summarize the above as the following thm:

Thm Let (E,d) be a compact metric space, and let (E',d') be a complete metric space. If

$$C(E,E') = \{f : E \to E' \mid f \text{ cts}\}$$

then

$$D(f,g) = \sup\{d'(f(x),g(x)) : x \in E\} \qquad f,g \in C(E,E')$$

defines a metric on $C(E,E')$ s.t. $(C(E,E'), D)$ is complete. Moreover, $(f_n)_{n\in\mathbb{N}} \subseteq C(E,E')$ conv. ~~conv.~~ to $f \in C(E,E')$ w.r.t. D iff $(f_n)_{n\in\mathbb{N}}$ conv unif to f on E.

Differentiation V

- We ~~turn~~ focus now on functions

$$f : U \to \mathbb{R} \quad \text{with } U \subseteq \mathbb{R} \text{ open.}$$

We wish to formalize the notion of a derivative. This will require limits of functions.