is cts at x_0 for each $j = 1, \ldots, n$.

Pf ($\Longrightarrow$) If f is cts at x_0, then $\pi_j \circ f$ is cts at x_0 for each $j = 1, \ldots, n$ as the composition of two cts functions.

($\Longleftarrow$) Suppose each $\pi_1 \circ f, \ldots, \pi_n \circ f$ are cts at x_0. Let $\varepsilon > 0$. Then for each $j = 1, \ldots, n$, $\exists \delta_j > 0$ s.t.

$$* \qquad |\pi_j \circ f(x) - \pi_j \circ f(x_0)| < \frac{\varepsilon}{\sqrt{n}}$$

whenever $x \in E$ satisfies $d(x, x_0) < \delta_j$.

Set $\delta = \min\{\delta_1, \ldots, \delta_n\}$. If $x \in E$ satisfies $d(x, x_0) < \delta$, then ($*$) holds for each coordinate $j = 1, \ldots, n$. Hence

$$d_n(f(x), f(x_0)) = \sqrt{(\pi_1 \circ f(x) - \pi_1 \circ f(x_0))^2 + \cdots + (\pi_n \circ f(x) - \pi_n \circ f(x_0))^2}$$

$$< \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon.$$

Thus f is cts at x_0. $\square$

Continuous Functions on Compact Sets, IV.4

Thm: Let (E, d) and (E', d') be metric spaces, and let $f: E \to E'$ be a cts function. If $S \subseteq E$ is compact, then so is $f(S)$.

Pf: Let $\{U_i\}_{i \in I}$ be an open cover of $f(S) \subseteq E'$. Since f is cts,

$$f^{-1}(U_i) \subseteq E$$

is open for each $i \in I$. Moreover,

$$S \subseteq \bigcup_{i \in I} f^{-1}(U_i)$$

Indeed, if $x \in S$, then $f(x) \in f(S)$ and so $\exists i_0 \in I$ s.t. $f(x) \in U_{i_0}$. But then $x \in f^{-1}(U_{i_0})$. So the claimed inclusion holds.

We have shown that $\{f^{-1}(U_i)\}_{i \in I}$

is an open cover for S. Since S is compact, $\exists$ a finite subcover: $i_1, \dots, i_n \in I$ s.t.

$$S \subseteq f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}).$$

We claim that

$$f(S) \subseteq U_{i_1} \cup \dots \cup U_{i_n},$$

i.e. that $\{U_{i_1}, \dots, U_{i_n}\}$ is the desired finite subcover.

Indeed, if $y \in f(S)$ then $\exists\, x \in S$ s.t. $f(x) = y$. Since $x \in f^{-1}(U_{i_j})$ for some $j \in \{1, \dots, n\}$, we have $y = f(x) \in U_{i_j}$. Since $y \in f(S)$ was arbitrary, we obtain the desired inclusion. $\square$

- Recall that compactness was related to several other metric space concepts, and had a nice characterization in $\mathbb{R}^n$. We now examine these connections under the lens of cty.

<u>Def</u>: For $f: E \to E'$ a function, we say f is ~~bdd~~ <u>bounded</u> if $f(E)$ is a bounded subset of E'. That is, if $\exists\, y \in E'$ and $R > 0$ s.t. $f(E) \subseteq B(y, R)$.

In particular, if $E' = \mathbb{R}$, f is bdd iff $\exists\, R > 0$ s.t. $|f(x)| \leq R \quad \forall x \in E$.

<u>Cor</u>: ~~[illegible]~~ Let (E, d) and (E', d') be metric spaces, and let $f: E \to E'$ be a cts function. If E is compact, then f is bounded.

<u>Pf</u>: By the theorem, $f(E)$ is compact and consequently is bounded. $\square$

Cor. Let (E,d) be a nonempty compact metric space. Then a cts function $f: E \to \mathbb{R}$ attains both its minimum & maximum values. That is, $\exists\, x_1, x_2 \in E$ s.t.

$$f(x_1) = \inf_{x \in E} f(x)$$

$$f(x_2) = \sup_{x \in E} f(x).$$

Pf: By the theorem, $f(E) \subseteq \mathbb{R}$ is compact and therefore is closed and bounded. Hence $\inf(f(E))$ and $\sup(f(E))$ both exist and are contained in $f(E)$. $\square$

10/27/2013

~~Recall defn cty~~

We now explore a stronger notion of cty. Recall that, in general, for a function cts at a point x_0, given $\varepsilon > 0$ the δ in the defin of cty may depend on ε and x.

E.g. When proving $f(x) = x^2$ was cts for $\varepsilon > 0$ we set:

$$\delta = \min\{1, \frac{\varepsilon}{2|x|+1}\}$$

Def: Let (E,d) and (E',d') be metric spaces, and let $f: E \to E'$ be a function. We say f is uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. whenever $x, y \in E$ satisfy

$$d(x,y) < \delta$$

then

$$d'(f(x), f(y)) < \varepsilon.$$

(In particular δ does not depend on a particular $x_0 \in E$)

For $S \subseteq E$, we say f is uniformly cts on S if $f|_S: S \to E'$ is uniformly cts.

Remark: f unif cts $\Rightarrow$ f cts.

Ex (1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is uniformly cts:
Given $\varepsilon > 0$, set $\delta = \varepsilon$. Then $|x-y| < \delta$ implies
$$|f(x) - f(y)| = |x-y| < \delta = \varepsilon.$$

(2) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$ is unif. cts.
Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{2}$. Then if
$$|x-y| < \delta$$
we have
$$|f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right|$$
$$= \left| \frac{y^2 - x^2}{(1+x^2)(1+y^2)} \right|$$
$$= |x-y| \left| \frac{x+y}{(1+x^2)(1+y^2)} \right|$$
$$\le |x-y| \left(\frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)} \right)$$
$$\le |x-y| \left(\frac{|x|}{1+x^2} + \frac{|y|}{1+y^2} \right)$$
Claim: $\frac{|x|}{1+x^2} \le 1$. Indeed if $|x| \le 1$ then $\frac{|x|}{1+x^2} \le \frac{|x|}{1} \le 1$.
Otherwise $|x| > 1 \Rightarrow \frac{|x|}{1+x^2} \le \frac{|x|}{x^2} = \frac{1}{|x|} < 1$.
$$\le |x-y| (1+1)$$
$$< 2\delta = \varepsilon.$$

(3) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is cts on $(0,1)$ but not unif cts.
To see cty, note that $g(x) = 1$ and $h(x) = x$ are cts on $(0,1)$ and that $h(x) \neq 0$.
So by previous prop, $f = \frac{g}{h}$ is cts on $(0,1)$.
To see not unif. cty, ~~fix~~ fix $\varepsilon > 0$ which we determine later. Now let $\delta > 0$

be arbitrary. We need to find $x, y \in (0,1)$ s.t.

$$|x-y| < \delta$$

but

$$|f(x) - f(y)| \geq \varepsilon.$$

Let's work with f first:

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| \overset{*}{=} \frac{|y-x|}{x \cdot y}$$

Since we have full control over x, y, we can choose $y = \frac{x}{2}$ so that

$$\overset{*}{=} \frac{|\frac{x}{2} - x|}{x \cdot \frac{x}{2}} = \frac{x/2}{x \cdot x/2} = \frac{1}{x}$$

Then this is $\geq \varepsilon \iff x \leq \frac{1}{\varepsilon}$.

We also have to ensure $|x-y| < \delta$:

$$|x-y| = |x - \tfrac{x}{2}| = \frac{x}{2} < \delta$$

$$x < 2\delta$$

Thus we pick $x \in (0,1)$ s.t. $x < \min\{2\delta, \frac{1}{\varepsilon}\}$ and $y = \frac{x}{2}$.

Note that we got this to work for any $\varepsilon > 0$. We only needed it to work for one value of ε, so $f(x) = \frac{1}{x}$ is really not unif. cts.

(4) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Our proof that it is cts does not show that f is not unif cts. It may be that we simply were not clever enough to find a δ that didn't depend on x_0.

Exercise: Show f is not unif. cts.

Thm Let (E,d) and (E',d') be metric spaces, and let $f: E \to E'$ be a cts function. If E is compact, then f is unif. cts

Pf: Let $\varepsilon > 0$. For each $x \in E$, the cty of f at x implies $\exists\, \delta_x > 0$ s.t.

$$d'(f(x), f(y)) < \frac{\varepsilon}{2}$$

whenever $d(x,y) < \delta_x$.

Observe that $\{B(x, \frac{\delta_x}{2})\}_{x \in E}$ is an open cover of E. Since E is compact, there is a finite subcover: $\exists\, x_1, \dots, x_n \in E$ s.t.

$$E \subset B(x_1, \tfrac{\delta_{x_1}}{2}) \cup \dots \cup B(x_n, \tfrac{\delta_{x_n}}{2})$$

Set

$$\delta = \min\{\tfrac{\delta_{x_1}}{2}, \dots, \tfrac{\delta_{x_n}}{2}\} > 0.$$

Now, suppose ~~st~~ $x, y \in E$ satisfy $d(x,y) < \delta$. Since we have our finite subcover, $x \in B(x_j, \frac{\delta_{x_j}}{2})$ for some $j \in \{1, \dots, n\}$.

~~[illegible]~~

Hence $d(x, x_j) < \frac{\delta_{x_j}}{2} < \delta_{x_j}$

Also

$$\begin{aligned} d(y, x_j) &\le d(y, x) + d(x, x_j) \\ &< \delta + \frac{\delta_{x_j}}{2} \\ &\le \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}. \end{aligned}$$

Consequently

$$\begin{aligned} d'(f(x), f(y)) &\le d'(f(x), f(x_j)) + d'(f(x_j), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

That is, f is unif. cts. □

Remark: The theorem says a cts. function having a compact domain is sufficient for it to be unif. cts. However, it is not necessary: $f(x) = x$ is unif cts on $\mathbb{R}$.

Ex: $f(x) = \frac{1}{x}$ was not unif. cts on $(0,1)$, but by the theorem it is unif cts on $[a,b]$ for any $0<a<b<1$.

~~Ex: Suppose $a,b \in \mathbb{R}$, $a<b$ and $f:(a,b] \longrightarrow \mathbb{R}$ is uniformly cts. Then the limit of f at a along $(a,b]$ exists.~~

- State: compact $\Leftrightarrow$ seq. compact $\Leftrightarrow$ complete + totally bounded

Prop: let (E,d) and (E',d') be metric spaces, and let $f: E \longrightarrow E'$ be a unif. cts function. ~~then~~ If $(x_n)_{n\in\mathbb{N}} \subseteq E$ is a Cauchy sequence, then so is $(f(x_n))_{n\in\mathbb{N}} \subseteq E'$.

Pf: Exercise. $\square$

Remark: The converse of this proposition is false: $f(x)=x^2$ is not unif. cts on $\mathbb{R}$. However $\mathbb{R}$ is complete, so $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is cauchy $\Leftrightarrow$ it converges. Since f is cts, $(f(x_n))_{n\in\mathbb{N}}$ is conv. $\Rightarrow$ it is Cauchy.

Ex: Suppose $a,b \in \mathbb{R}$ satisfy $a<b$ and let

$$f: (a,b] \longrightarrow \mathbb{R}$$

be a unif. cts function. Then the limit of f at a along $(a,b]$ exists.

Indeed, observe that

$$\left(a+\tfrac{b-a}{n}\right)_{n\in\mathbb{N}} \subseteq (a,b]$$

is a Cauchy sequence. Since f is unif. cts the prop $\Rightarrow$ $\left(f\left(a+\tfrac{b-a}{n}\right)\right)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is Cauchy. Since $\mathbb{R}$ is complete, it converges to some $y \in \mathbb{R}$.

Claim: $\lim_{x\to a} f(x) = y$.

Indeed, let $\varepsilon>0$. Before we determine $\delta>0$, Note that $\exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$\left| f\left(a + \frac{b-a}{n}\right) - y \right| < \varepsilon/2$$

Let $\delta > 0$ be s.t. if $x, y \in (a,b]$ satisfy

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| < \frac{\varepsilon}{2},$$

which exists by uniform cty of f on $(a,b]$.

Suppose $x \in (a,b]$ satisfies

$$|x - a| < \delta.$$

Let $n \in \mathbb{N}$ be s.t. $n \geq N$ and s.t.

$$a < a + \frac{b-a}{n} < x$$

Hence

$$\left| a + \frac{b-a}{n} - x \right| < \delta$$

and so

$$|f(x) - y| \leq \left| f(x) - f\left(a + \frac{b-a}{n}\right) \right| + \left| f\left(a + \frac{b-a}{n}\right) - y \right|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{x \to a} f(x) = y$. QED.

Continuous Functions on Connected Sets IV.5

Thm: Let (E,d) and (E',d') be metric spaces, and let $f: E \to E'$ be a cts function. If $S \subseteq E$ is connected, then so is $f(S)$.

Pf: We proceed by contrapositive. Suppose $f(S)$ is disconnected. Then $\exists$ disjoint, non-empty subsets $A, B \subseteq f(S)$ that are open rel. to $f(S)$ and $A \cup B = f(S)$. list

Define

$$A_1 := f^{-1}(A) \cap S \subseteq E$$
$$B_1 := f^{-1}(B) \cap S$$

We claim: (i) $A_1 \cap B_1 = \emptyset$

(ii) $A_1 \neq \emptyset \neq B_1$