open sets gives us an easy way to check $g \circ f$ is cts: if $U \subseteq E''$ is open
$\Rightarrow g^{-1}(U) \subseteq E'$ is open
$\Rightarrow f^{-1}(g^{-1}(U)) \subseteq E$ is open
$\overset{\shortparallel}{(g \circ f)^{-1}(U)}$.

10/23/2017

Continuity and Limits, IV.2

Let (E,d) and (E',d') be metric spaces ~~and let f be a function~~. For $x_0 \in E$ ~~we want to ~~define~~ capture the notion of the limit of f "at x_0".~~ if we consider $f: E \setminus \{x_0\} \to E'$ the "limit of f at x_0" should be a suggestion for $f(x_0)$ based on outputs of nearby points. One issue, what if x_0 has no nearby points (i.e. is isolated).

Recall (i) x_0 is isolated if $\{x_0\}$ is open
$\overset{HW}{\iff}$ ~~B~~ $\exists r>0$ s.t. $B(x,r)$ ~~has~~ contains finitely many points
(ii) x_0 is a cluster point of E if $\forall r>0$ $B(x,r)$ contains infinitely many points.

Cor: In a metric space (E,d), each point $x \in E$ is either isolated or a cluster point of E (but not both).

~~Def~~. Since isolated points have no nearby points to influence f, we only define limits for cluster points of E.

Def Let (E,d) and (E',d') be metric spaces and let $x_0 \in E$ be a cluster point of E. For $f: E \setminus \{x_0\} \to E'$ a function, a point $y \in E'$ is called a limit point of f at x_0 if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in E$ ~~whenever~~ satisfying $d(x,x_0) < \delta$ then $d'(f(x),y) < \varepsilon$.

~~Example~~

Remark: The above def is equivalent to saying that the new function $\tilde{f}: E \to E'$ defined by
$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases}$$
is cts at x_0.

Prop Let (E,d) and (E',d') be metric spaces, let $x_0 \in E$ be a cluster point of E, and let $f: E \setminus \{x_0\} \to E'$ be a function. If ~~the~~ a limit point of f at x_0 exists, then it is unique. Hence we may say "the limit point of f at x_0".

Pf: Suppose $y, z \in E'$ are both limit points of f at x_0. Let $\varepsilon > 0$. Then $\exists \delta_1, \delta_2 > 0$ s.t. for $x \in E \setminus \{x_0\}$

if $d(x,x_0) < \delta_1$, then $d'(f(x),y) < \varepsilon/2$

if $d(x,x_0) < \delta_2$, then $d'(f(x),z) < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Since x_0 is a cluster point of E, $\exists x \in B(x,\delta) \setminus \{x_0\}$. Hence for this x we have
$$d'(y,z) \le d'(y,f(x)) + d'(f(x),z) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$
Since $\varepsilon > 0$ was arbitrary, $d'(y,z) = 0$, which means $y = z$. Thus the limit point of f at x_0 is unique. $\square$

similar to ! lim x_n seq

- We will use the following notation for the limit point of f at x_0: (if it exists)

$$\lim_{x \to x_0} f(x) \quad \left(= \lim_{y \to x_0} f(y) = \lim_{t \to x_0} f(t) \right)$$

- Remark: When we use this notation, there are always a few things we are implicitly assuming: (i) $\exists$ metric spaces (E,d) and (E',d')
  (ii) $x_0 \in E$ is a cluster point of E
  (iii) $f: E \setminus \{x_0\} \to E'$ is a function

- Remark: Even if f is already defined at x_0, we may be interest in the limit of f at x_0. In this we define

$$\tilde{f}: E \setminus \{x_0\} \to E'$$

by $\tilde{f}(x) = f(x)$ for $x \in E \setminus \{x_0\}$ and consider $\lim_{x \to x_0} \tilde{f}(x)$ (if it exists)

But, we will just write

$$\lim_{x \to x_0} f(x)$$

for this and simply remind ourselves that this quantity is not determined by $f(x_0)$.

- Prop: Let (E,d) and (E',d') be metric spaces, let $x_0 \in E$, and let $f: E \to E'$ be a function. Then f is cts at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$

Pf: Immediate $\square$

- Remark: If $S \subseteq E$, but x_0 is a cluster point of S, and $f: S \to E'$ is a function, we will still ~~use~~ write

$$\lim_{x \to x_0} f(x)$$

to denote the limit (if it exists). However, we may refer to this as the limit of f at x_0 on/in/along S.

- Let us now connect ~~match~~ continuity ~~definition notion~~ ~~of~~ of a function with our previous notion of a limit of a sequence

Prop: Let (E,d) and (E',d') be metric spaces. Then a function $f: E \to E'$ is cts at $x_0 \in E$ iff $\forall (x_n)_{n \in \mathbb{N}} \subset E$ converging to x_0 we also have $(f(x_n))_{n \in \mathbb{N}} \subset E'$ converges to $f(x_0)$.

Pf: $(\Longrightarrow)$ Suppose f is cts at x_0. Let $(x_n)_{n \in \mathbb{N}} \subset E$ be a sequence converging to x_0. Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$d'(f(x_n), f(x_0)) < \varepsilon.$$

Since f cts, $\exists \delta > 0$ s.t. if $d(x, x_0) < \delta$ then $d'(f(x), f(x_0)) < \varepsilon$. Since $\lim_{n \to \infty} x_n = x_0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $d(x_n, x_0) < \delta$.

Thus for all $n \geq N$

$$d(f(x_n), f(x_0)) < \varepsilon.$$

That is, $\lim_{n \to \infty} f(x_n) = f(x_0)$.

$(\Longleftarrow)$ We proceed by contrapositive: suppose f is not cts at x_0. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in E$ s.t. $d(x, x_0) < \delta$ but $d'(f(x), f(x_0)) \geq \varepsilon$. Let $x_n \in E$ correspond to $\delta = \frac{1}{n}$. Then

$$d(x_n, x_0) < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

but $\quad d'(f(x_n), f(x_0)) \geq \varepsilon \quad \forall n \in \mathbb{N}.$

Clearly $\lim_{n \to \infty} x_n = x_0$, while $(f(x_n))_{n \in \mathbb{N}}$ does not converge. $\square$