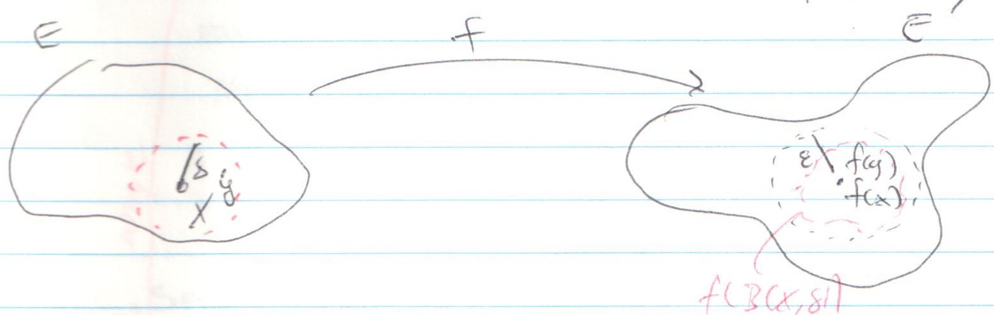
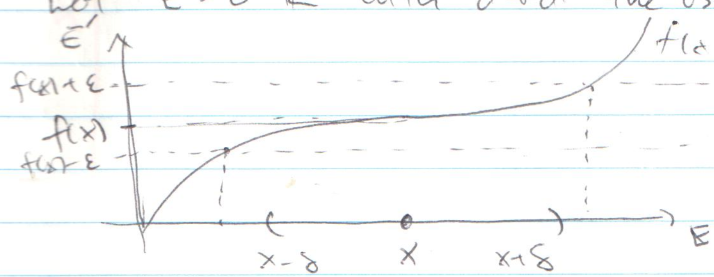
Continuous Functions IV

We now study "functions" on metric spaces that respect the metric space structure.
We use the term function here to refer to any map
$$f: E \to E'$$
where (E,d) and (E',d') are metric spaces.
That is, $\forall x \in E$, $f(x) \in E'$.

Def Let (E,d) and (E',d') be metric spaces, and let $f: E \to E'$ be a function.
For $x \in E$, we say f is continuous at x if
$\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall y \in E$ satisfying $d(x,y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.



Ex: Let $E = E' = \mathbb{R}$ with d & d' the usual metric



Remark: We can easily restate the def of cts at x using open balls:

$f: E \to E'$ is cts at $x \in E$ iff for every open ball $B(f(x), \varepsilon) \subseteq E'$ there exists an open ball $B(x, \delta)$ s.t.

$$f(B(x,\delta)) \subseteq B(f(x), \varepsilon)$$

Def: Let (E,d) and (E',d') be metric spaces and let $f: E \to E'$ be a function. For $S \subseteq E$ we say f is continuous on S if it cts at $x \in S$ for every $x \in S$. If f is cts on E, we simply say f is continuous

Ex: ① $E = E' = \mathbb{R}$ w/ usual metric. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous. Indeed, fix $x \in \mathbb{R}$ and let $\varepsilon > 0$.

scratch work

We will need to show $|f(x) - f(y)| < \varepsilon$ provided $|x-y|$ is small, so let's do some prelim. estimates:

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|$$
$$\leq |x-y| \cdot (|x| + |y|)$$
$$\leq |x-y| (|x| + |y-x+x|)$$
$$\leq |x-y| (2|x| + |y-x|)$$

So if $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq \delta (2|x| + \delta)$$

We can always pick $\delta \leq 1$, so that

$$|f(x) - f(y)| < \delta \cdot (2|x| + 1)$$

Thus if

$$\delta = \min\left\{\frac{\varepsilon}{2|x|+1}, 1\right\}$$

we obtain $|f(x) - f(y)| < \varepsilon$.

Set $\delta = \min\{\frac{\varepsilon}{2|x|+1}, 1\}$. Then if $y \in \mathbb{R}$ satisfies $|x-y| < \delta$, we have:

$$|f(x) - f(y)| \leq |x^2 - y^2| = |x-y|\,|x+y|$$
$$\leq |x-y| (2|x| + |x-y|) \leq \delta \cdot (2|x| + \delta)$$

$\le \delta(2|x|+1) \le \frac{\varepsilon}{2|x|+1}\cdot(2|x|+1) = \varepsilon.$

Thus f is cts at x. Since $x\in\mathbb{R}$ was arbitrary, f is cts on $\mathbb{R}$.

(2) Let ~~$E=E'=\mathbb{R}$ w/ usual metric $f:\mathbb{R}\to\mathbb{R}$~~

$E=E'=[0,+\infty)$, $f:\mathbb{R}\to\mathbb{R}$ $f(x)=\sqrt{x}$ is cts.

Fix $x\in[0,+\infty)$ and $\varepsilon>0$.

$$|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \le \frac{|x-y|}{\sqrt{x}}$$

So long as $x\neq 0$ this is ok. In which case we want $\delta=\sqrt{x}\cdot\varepsilon$

If $x=0$, we have

$$|f(x)-f(y)| = |0-\sqrt{y}| = \sqrt{y} = \sqrt{|y|} = \sqrt{|0-y|} < \sqrt{\delta}$$

So here we should pick $\delta=\varepsilon^2$.

~~Set~~ If $x\neq 0$, set $\delta=\sqrt{x}\cdot\varepsilon>0$. ~~Then~~ If $|x-y|<\delta$, then

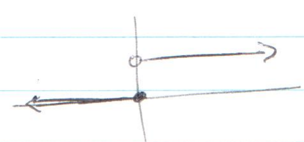
$|f(x)-f(y)| \le - — < \varepsilon$

If $x=0$, set $\delta=\varepsilon^2$. If $|0-y|<\delta$ then

$|f(x)-f(y)| \le - < \varepsilon$.

(3) $E=E'=\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}0 & x\le 0\\ 1 & 0<x\end{cases}$$



f is not cts at $x=0$:

$\exists\varepsilon>0$ s.t. $\forall\delta$ $\exists y\in E$ satisfying $d(x,y)<\delta$ but $d'(f(x),f(y))\ge\varepsilon$.

Set $\varepsilon=1$, and let $\delta>0$. Then if $y=\frac{\delta}{2}>0$ we have

$$|x-y| = |0-\tfrac{\delta}{2}| = \tfrac{\delta}{2} < \delta$$

but

$$|f(x)-f(y)| = |0-1| = 1 \ge \varepsilon.$$

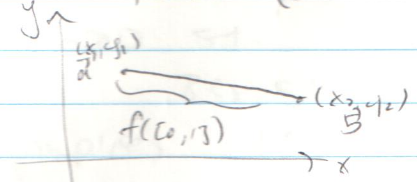
→ Check f is cts at all other points

10/20/2017 (4) $E=[0,1]$ w/usual metric, $E'=\mathbb{R}^2$ w/ 2-dim'l Euc metric.

Fix $\vec{a} = (x_1, y_1)$ & $\vec{b} = (x_2, y_2) \in \mathbb{R}^2$

$f : [0,1] \to \mathbb{R}^2$.

$$f(t) = \big((1-t)\cdot x_1 + t x_2 ,\ (1-t) y_1 + t y_2 \big)$$



$f(0) = (x_1, y_1) = \vec{a}$

$f(1) = (x_2, y_2) = \vec{b}$

Then f is cts on $[0,1]$. Fix $t \in [0,1]$ and let $\varepsilon > 0$. Let $\delta > 0$ to be determined later. If $s \in [0,1]$ satisfies $|t - s| < \delta$, then

$$d'(f(t), f(s)) = \sqrt{\big((1-t)x_1 + t x_2 - (1-s)x_1 - s x_2\big)^2 + \big((1-t)y_1 + t y_2 - (1-s)y_1 - s y_2\big)^2}$$

$$= \sqrt{\big((-t+s)\cdot x_1 + (t-s)\cdot x_2\big)^2 + \big((-t+s)\cdot y_1 + (t-s)\cdot y_2\big)^2}$$

$$= \sqrt{(s-t)^2 (x_1 - x_2)^2 + (s-t)^2 (y_1 - y_2)^2}$$

$$= |s - t| \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= |s - t| \cdot d'(\vec{a}, \vec{b}) < \delta \cdot d'(\vec{a}, \vec{b})$$

Thus if $\delta = \frac{\varepsilon}{d'(\vec{a}, \vec{b})}$, $d'(f(t), f(s)) < \varepsilon$.

<u>Prop</u>: Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be a function. Then f is cts if and only if for every open subset $U \subseteq E'$, the subset

$$f^{-1}(U) \subseteq E$$

is also open

<u>Pf</u>: ($\Longrightarrow$) Let $U \subseteq E'$ be open and fix $x \in f^{-1}(U)$. Then $f(x) \in U$ and so $\exists\, \varepsilon > 0$ s.t. $B(f(x), \varepsilon) \subseteq U$. By cty of f, $\exists\, \delta > 0$ s.t.

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U$$

That is, $B(x, \delta) \subseteq f^{-1}(U)$. Since $x \in f^{-1}(U)$ was arbitrary, we see that $f^{-1}(U)$ is open.

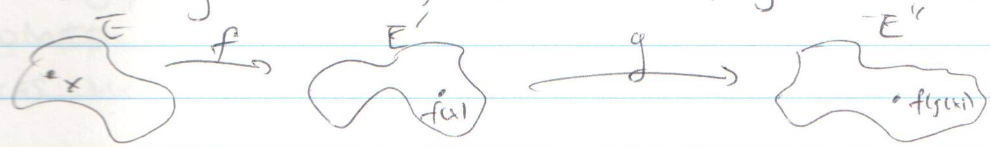
($\Longleftarrow$) Fix $x \in E$ and let $\varepsilon > 0$. Since

$B(f(x),\varepsilon) \subseteq E$ is open, $f^{-1}(B(f(x),\varepsilon))$ is open and in particular contains x. Thus $\exists\, \delta > 0$ s.t.

$$B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$$

which implies $f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$.
Since $\varepsilon > 0$ was arbitrary, f is cts at $x \in E$.
Since $x \in E$ ——— a ———, f is cts. □

<u>Prop</u> Let (E,d), (E',d'), and (E'',d'') be metric spaces, and let $f: E \to E'$ and $g: E' \to E''$ be functions. If f is cts at $x \in E$ and g is cts at $f(x) \in E'$, then ~~$f \circ g$ is cts at x~~ $f \circ g : E \to E''$ is cts at x. In particular, if f and g are cts, then so is $f \circ g$.



<u>Pf</u>: Let $\varepsilon > 0$. Since g is cts at $f(x)$, $\exists\, \delta_1 > 0$ s.t. if $z \in E'$ satisfies

$$d'(f(x),z) < \delta_1$$

then

$$d''(g(f(x)), g(z)) < \varepsilon.$$

Next, since f is cts at x, $\exists\, \delta > 0$ s.t. if $y \in E$ satisfies

$$d(x,y) < \delta$$

then

$$d'(f(x),f(y)) < \delta_1.$$

Thus, if $y \in E$ satisfies $d(x,y) < \delta$ then

$$d''(g(f(x)), g(f(y))) < \varepsilon.$$

That is, $g \circ f$ is cts at x. □

<u>Remark</u>: Our characterization of cty using

open sets gives us an easy way to check $g \circ f$ is cts: if $U \subseteq E''$ is open

$\Rightarrow g^{-1}(U) \subseteq E'$ is open

$\Rightarrow f^{-1}(g^{-1}(U)) \subseteq E$ is open

$\quad\quad \overset{\shortparallel}{(g \circ f)^{-1}(U)}$.

10/23/2017

Continuity and Limits, IV.2

Let (E,d) and (E',d') be metric spaces ~~and~~ ~~we want to define the notion of the limit of f "at x_0"~~. For $x_0 \in E$ if we consider $f: E \setminus \{x_0\} \to E'$ the "limit of f at x_0" should be a suggestion for $f(x_0)$ based on outputs of nearby points. One issue, what if x_0 has no nearby points (i.e. is isolated).

Recall (i) x_0 is isolated if $\{x_0\}$ is open $\overset{HW}{\iff} \exists r > 0$ s.t. $B(x,r)$ contains finitely many points

(ii) x_0 is a cluster point of E if $\forall r > 0$ $B(x,r)$ contains infinitely many points.

Cor: In a metric space (E,d), each point $x \in E$ is either isolated or a cluster point of E (but not both).

~~Defn~~. Since isolated points have no nearby points to influence f, we only define limits for cluster points of E.