$$|x_j^{(k)} - x_j^{(\ell)}| \le \sqrt{(x_j^{(k)} - x_j^{(\ell)})^2}$$
$$\le \sqrt{(x_1^{(k)} - x_1^{(\ell)})^2 + \cdots + (x_n^{(k)} - x_n^{(\ell)})^2}$$
$$= d(\vec{x}_k, \vec{x}_\ell)$$

Thus for $\varepsilon > 0$, if we let $N \in \mathbb{N}$ be s.t. $\forall k, \ell \ge N$

$$d(\vec{x}_k, \vec{x}_\ell) < \varepsilon$$

then $\forall k, \ell \ge N$

$$|x_j^{(k)} - x_j^{(\ell)}| < \varepsilon.$$

That is, $(x_j^{(k)})_{k \in \mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy sequence.
Since $\mathbb{R}$ is complete, it converges to some $x_j \in \mathbb{R}$.
We claim $(\vec{x}_k)_{k \in \mathbb{N}}$ converges to

$$\vec{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Indeed, let $\varepsilon > 0$. Let $N_1, N_2, \ldots, N_n \in \mathbb{N}$ be s.t.

$$\forall k \ge N_j \quad |x_j^{(k)} - x_j| < \frac{\varepsilon}{\sqrt{n}}$$

Then, if $N = \max\{N_1, \ldots, N_n\}$, $\forall k \ge N$ we have

$$d(\vec{x}_k, \vec{x}) = \sqrt{(x_1^{(k)} - x_1)^2 + \cdots + (x_n^{(k)} - x_n)^2} < \sqrt{\left(\tfrac{\varepsilon}{\sqrt{n}}\right)^2 + \cdots + \left(\tfrac{\varepsilon}{\sqrt{n}}\right)^2}$$
$$= \sqrt{\tfrac{\varepsilon^2}{n} + \cdots + \tfrac{\varepsilon^2}{n}} = \sqrt{\varepsilon^2} = \varepsilon.$$

So $\lim_{k \to \infty} \vec{x}_k = \vec{x}$ and $\mathbb{R}^n$ is complete. $\square$

__Cor:__ $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are complete. Pf: the equiv

10/6/2017

Compactness III.5

__Def__ In a metric space (E, d) for $S \subset E$, a collection $\{U_i\}_{i \in I}$ of open subsets $U_i \subseteq E$, $i \in I$, is called an open cover of S if

$$S \subset \bigcup_{i \in I} U_i$$

An open cover of E is simply called an open cover.
If $J \subseteq I$ and

$$S \subset \bigcup_{i \in J} U_i$$

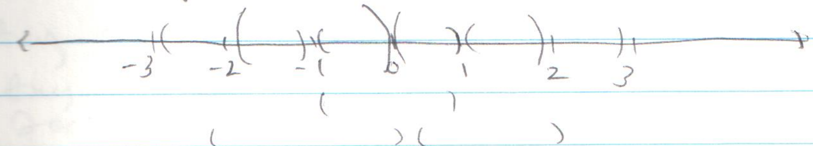
then $\{U_i\}_{i \in J}$ is called a subcover of $\{U_i\}_{i \in I}$ of S

Finally, we say $\{U_i\}_{i\in I}$ is a finite cover if $|I|<\infty$.

Ex: ① In $\mathbb{R}$ with the usual metric

$$\{B(n,1)\}_{n\in\mathbb{Z}}$$

is an open cover for $\mathbb{R}$:

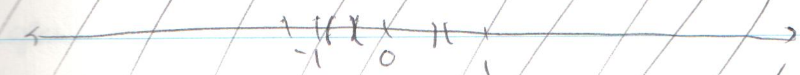


Does it have any subcovers?? of $\mathbb{R}$

~~However~~

~~$\{B(n,\frac{1}{2})\}_{n\in\mathbb{Z}}$~~

~~is not an open cover for $\mathbb{R}$:~~



~~since $m+\frac{1}{2}\notin\bigcup_{n\in\mathbb{Z}}B(n,\frac{1}{2})$ for any m~~

• In ~~$\mathbb{R}$ with usual metric $\{B(n,n)\}_{n\in\mathbb{N}}$~~ is an open cover

② In $\mathbb{R}^2$ w/ 2-dim'l Euc. metric

$$\{B((n,m),1)\}_{(n,m)\in\mathbb{Z}^2}$$

is an open cover.

Does it have any subcovers?

For what values $r\geq 0$

$$\{B((n,m),r)\}_{(n,m)\in\mathbb{Z}^2}$$

(i) ~~stop being~~ is not an open cover of $\mathbb{R}$? $\frac{\sqrt{2}}{2}$

(ii) start having subcovers? does it have any finite subcovers? $\sqrt{2}$

③ Let $S=[0,1]\subseteq\mathbb{R}$, equipped with usual metric. Then

$$\{B(a,\tfrac{1}{2})\}_{a\in\mathbb{Q}\cap[0,1]}$$

is an open cover: Any subcovers! Any finite subcovers?

4 For $S = (0,1] \subseteq \mathbb{R}$ w/ usual metric
$$\{ (\tfrac{1}{n}, 1) \}_{n \in \mathbb{N}}$$
is an open cover for $(0,1)$.
Any subcovers?
Any finite subcovers?
Does this cover $(0,1]$?

Def: For (E,d) a metric space, a subset $S \subset E$ is compact if every open cover for S, has a finite subcover.
That is, whenever
$$S \subset \bigcup_{i \in I} U_i$$
for open sets $U_i \subset E$, $\exists\ i_1, i_2, \ldots, i_n \in I$ s.t.
$$S \subset U_{i_1} \cup \cdots \cup U_{i_n}$$
~~(Note: S compact iff every open cover has a finite subcover)~~

Non-Ex: From Ex. 4, we saw that $(0,1) \subseteq \mathbb{R}$ has an open cover w/out a finite subcover.
Consequently $(0,1)$ is not compact.

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Ex: we will eventually see $[0,1]$ is compact. - turning non-ex to ex of finite subcover.

Prop: Let (E,d) be a metric space with a compact subset $S \subset E$. Then any closed subset $V \subset S$ is also compact.

Pf: Let $\{U_i\}_{i \in I}$ be an open cover for V:
$$V \subset \bigcup_{i \in I} U_i$$
Since V is closed, V^c is open and clearly
$$S \subset E = V^c \cup V$$

Consequently
$$S \subset V^c \cup \bigcup_{i\in I} U_i$$
which means $\{V^c\} \cup \{U_i\}_{i\in I}$ is an open cover for S. write $U_{0} := V^c$ for and set $I_0 = I \cup \{0\}$. So
$$\{V^c\} \cup \{U_i\}_{i\in I} = \{U_i\}_{i\in I_0}$$
Now, S is compact, so there is a finite subcover: $i_1, \dots, i_n \in I_0$ s.t.
$$S \subset U_{i_1} \cup \dots \cup U_{i_n}.$$
Note that $V \subset S$, so $\{U_{i_1}, \dots, U_{i_n}\}$ also cover V. if $i_1, \dots, i_n \in I$, then we are done. If $i_j = 0$ for some j, note that $V \cap V^c = \emptyset$ implies
$$V \subset U_{i_1} \cup \dots \cup U_{i_{j-1}} \cup U_{i_{j+1}} \cup \dots \cup U_{i_n}!$$
i.e. we have produced a finite subcover. □

Prop: For (E,d) a metric space and $S \subset E$ a compact subset, S is bounded.

Pf: Fix any $x \in E$ and set
$$U_n := B(x, n) \quad \forall n \in \mathbb{N}.$$
Since for any $y \in E$ we have $d(x,y) < \infty$, $\exists\, n \in \mathbb{N}$ s.t. $d(x,y) < n$. Hence $y \in U_n$. Thus $\{U_n\}_{n\in\mathbb{N}}$ is an open cover for E and in particular is an open cover for S. Since S is compact, there is a finite subcover for S: $n_1, \dots, n_d \in \mathbb{N}$ s.t.
$$S \subset U_{n_1} \cup \dots \cup U_{n_d}.$$
However, noting $U_n \subseteq U_{n+1}$, if we let $N = \max\{n_1, \dots, n_d\}$, then
$$S \subset U_{n_1} \cup \dots \cup U_{n_d} \subseteq U_N = B(x, N).$$

Hence S is bounded. $\square$

It will turn out ~~that~~ in $\mathbb{R}^n$, V closed bdd $\Leftrightarrow$ compact.

<u>Prop</u>: (Nested Set Property)

Let $S_1, S_2, S_3, \ldots$ be a sequence of non-empty, closed subsets of compact metric space (E, d) satisfying

$$S_1 \supset S_2 \supset S_3 \supset \cdots$$

Then

$$\bigcap_{n \in \mathbb{N}} S_n \neq \phi.$$

<u>Pf</u>: Suppose, TAC, that $\bigcap_{n \in \mathbb{N}} S_n = \phi$. Then

$$E = \phi^c = \left(\bigcap_{n \in \mathbb{N}} S_n\right)^c = \bigcup_{n \in \mathbb{N}} S_n^c.$$

Note that each S_n^c is open since S_n is closed. Thus $\{S_n^c\}_{n \in \mathbb{N}}$ is an open cover for E. As E is compact, $\exists$ a finite subcover: $n_1, \ldots, n_d \in \mathbb{N}$ s.t.

$$E \subset S_{n_1}^c \cup \cdots \cup S_{n_d}^c$$

In fact we have $E = \downarrow$. So

$$\phi = E^c = (S_{n_1}^c \cup \cdots \cup S_{n_d}^c)^c = S_{n_1} \cap \cdots \cap S_{n_d}.$$

However, if $N = \max\{n_1, \ldots, n_d\}$, then

$$S_{n_1} \cap \cdots \cap S_{n_d} = S_N$$

which was assumed to be non-empty. $\square$

<u>~~Cor~~</u> <u>EX</u>: We can use this prop to give another proof that $\mathbb{R}$ is <u>not</u> compact. Indeed, consider

$$S_n := [n, +\infty) \quad \forall n \in \mathbb{N}$$

Then each S_n is closed and $S_1 \supseteq S_2 \supseteq \cdots$

However

$$\bigcap_{n \in \mathbb{N}} S_n = \phi$$

Indeed, if $x \in \bigcap S_n$, then $x \geq n \;\forall n \in \mathbb{N}$. That is, x is an upper bound for $\mathbb{N}$, which we know cannot exist.

already done

~~E=(0,1), $S_n = (0, \frac{1}{n}]$~~

The nicest implication of compactness has to do with sequences.

Def: In a metric space (E,d), for $S \subseteq E$ we say a point $x \in E$ is a cluster point of S if $\forall r > 0$ the intersection
$$B(x,r) \cap S$$
contains infinitely many points.

Thm Let (E,d) be a compact metric space. If $S \subset E$ is infinite, then S has at least one cluster point.

Pf: Let $S \subset E$ be infinite. Assume, TAC, that S has no cluster points. This means $\forall x \in E$, x is not a cluster point of S, and $\exists\, r_x > 0$ s.t.
$$B(x, r_x) \cap S$$
is finite (or empty). Now
$$\{B(x, r_x)\}_{x \in E}$$
is clearly an open cover for E. As E is compact, $\exists$ a finite subcover:
$$E \subset B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \cdots \cup B(x_n, r_{x_n}).$$
Since $S \subseteq E$, this covers S too, and so
$$S = \bigcup_{j=1}^{n} [B(x_j, r_{x_j}) \cap S]$$
But each
$$B(x_j, r_{x_j}) \cap S$$
contains a finite number of points and consequently so does their finite union S. This contradicts S being infinite, and so S must have a cluster point. $\square$

10/11/2017

Cor Let (E,d) be a compact metric space.

Then any sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence

Pf: Let $S = \{x_n : n \in \mathbb{N}\}$.

Case 1: S is finite, say $S = \{y_1, \dots, y_d\}$. Then for some $j = 1, \dots, d$ there are infinitely many $n \in \mathbb{N}$ s.t. $x_n = y_j$. Let $(n_k)_{k\in\mathbb{N}}$ be the strictly increasing sequence of indices satisfying

$$x_{n_k} = y_j \quad \forall k \in \mathbb{N}.$$

Then $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and $\lim_{k\to\infty} x_{n_k} = y_j$.

Case 2: S is infinite.
Then by the thm, S has at least one cluster point. Say $x \in E$ is a cluster point of S. We will inductively define a subsequence converging to x as follows:

Since x is a cluster pt,

$$B(x, 1) \cap S$$

has infinitely many elements. Define

$$n_1 = \min\{n \in \mathbb{N} : x_n \in B(x,1)\}.$$

So x_{n_1} is the first entry of our subsequence.
Assume we have found $n_1 < n_2 < \dots < n_k$ s.t.

$$x_{n_l} \in B(x, \tfrac{1}{l}) \quad \text{each } l = 1, \dots, k.$$

Since

$B(x, \frac{1}{k+1}) \cap S$ is infinite

$B(x, \frac{1}{k+1}) \setminus \{x_{n_1}, \dots, x_{n_k}\}$ ~~$\{ \dots x_n \in B(x, \frac{1}{k+1})\}$~~ is non-empty and so we set

$$n_{k+1} = \min\{n > n_k : x_n \in B(x, \tfrac{1}{k+1})\}$$

Thus by induction we obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$x_{n_k} \in B(x, \tfrac{1}{k}) \quad \forall k \in \mathbb{N}$$
$$\iff d(x_{n_k}, x) < \tfrac{1}{k}.$$

Clearly $\lim_{k\to\infty} x_{n_k} = x$, and so we are done $\square$

Cor: A compact metric space (E,d) is complete

Pf: Suppose $(x_n)_{n\in\mathbb{N}} \subseteq E$ is a Cauchy sequence. The previous corollary implies it has a convergent subsequence. But, as we have seen this implies $(x_n)_{n\in\mathbb{N}}$ converges. Hence (E,d) is complete. $\square$

Cor: ~~In~~ a metric space (E,d), if $S\subset E$ is compact then S is closed.

Pf: Suppose $(x_n)_{n\in\mathbb{N}} \subseteq S$ converges to some $x\in E$. Since ~~(E,d)~~ is compact, $(S, d|_{S\times S})$ is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$y := \lim_{k\to\infty} x_{n_k} \in S.$$

But then the subsequence converges to y in (E,d) too. Since every subsequence of a convergent sequence converges to the same limit, we obtain

$$x = y \in S. \qquad \square$$

Our next goal is to proof the Bolzano—Weierstrass theorem which says that in $\mathbb{R}^n$ w/ the n-dim'l Euclidean metric, any closed and bounded set is compact. We first require a lemma:

10/13/2017

Lemma: Fix $n\in\mathbb{N}$ and let $(\mathbb{R}^n, d)$ be the n-dim'l Euc. metric space. Let V be a bounded subset of $(\mathbb{R}^n,d)$. Then $\forall \varepsilon>0$, V is contained in the union of a finite number of closed balls of radius ε.

<u>Pf:</u> Given $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, for each coordinate $j = 1, \dots, n$ $\exists a_j \in \mathbb{Z}$ s.t.

$$\frac{a_j}{m} \le x_j < \frac{a_j + 1}{m} \iff 0 \le x_j - \frac{a_j}{m} < \frac{1}{m}$$

where $m \in \mathbb{N}$ will be determined later.
Then we have for $\vec{a} := \left(\frac{a_1}{m}, \dots, \frac{a_n}{m}\right)$:

$$d(\vec{x}, \vec{a}) = \sqrt{\left(x_1 - \frac{a_1}{m}\right)^2 + \dots + \left(x_n - \frac{a_n}{m}\right)^2}$$

$$< \sqrt{\left(\frac{1}{m}\right)^2 + \dots + \left(\frac{1}{m}\right)^2} = \frac{\sqrt{n}}{m}$$

Thus if we choose $m \in \mathbb{N}$ s.t. $\frac{\sqrt{n}}{m} < \varepsilon$ (i.e. $m > \frac{\sqrt{n}}{\varepsilon}$) then

$$\vec{x} \in B[\vec{a}, \varepsilon]$$ (closed ball w/ center $\vec{a}$ and radius ε)

(in fact it is in the open ball, but we won't need that)

Now, for V bounded, $\exists \vec{y} \in \mathbb{R}^n$ and $R > 0$ s.t.

$$V \subseteq B[\vec{y}, R]$$

Replacing R with (No) $M \ge R + d(\vec{y}, \vec{0})$ where $\vec{0} = \underbrace{(0, \dots, 0)}_{n \text{ times}}$, we can ~~assume $\vec{y} = \vec{0}$.~~ have ~~That is,~~

$$V \subseteq B[\vec{0}, M]$$

~~for some R.~~ We claim that if we produce for each $\vec{x} \in V$ an $\vec{a}$ as above, we will only every use a finite number of values for each $a_1, a_2, \dots, a_n$. Indeed, for $\vec{x} \in V$ and $\vec{a}$ as above we have:

$$\left|\frac{a_j}{m}\right| = \left|x_j + \left(\frac{a_j}{m} - x_j\right)\right| \le |x_j| + \left|\frac{a_j}{m} - x_j\right| \le d(\vec{x}, \vec{0}) + \frac{1}{m}$$
$$\le M + \frac{1}{m}$$

so $|a_j| \le mM + 1$ for each $j = 1, \dots, n$ and there are a finite # of integers a_j satisfying this. Thus if let $A \subseteq \mathbb{Z}$ be the finite set of integers that satisfy this for any $j=1,\dots,n$, then by our choice of m that

$$V \subseteq \underbrace{\bigcup_{a_1,\dots,a_n \in A}}_{\text{finite}} B\left[\left(\frac{a_1}{m},\dots,\frac{a_n}{m}\right),\varepsilon\right]$$

$\square$

<u>Thm</u> Fix $n \in \mathbb{N}$ and let $(\mathbb{R}^n, d)$ be the n-dim'l Euc. metric space. Then any closed and bounded $V \subseteq \mathbb{R}^n$ is compact.

<u>Pf</u> Suppose, towards a contradiction, that V is <u>not</u> compact. Then $\exists$ an open cover $\{U_i\}_{i\in I}$ of V with no finite subcover. Using the previous lemma for $\varepsilon = \frac{1}{2}$ yields closed balls $B_1,\dots,B_{d}$ of radius $\frac{1}{2}$ s.t.

$$S \subseteq B_1 \cup B_2 \cup \dots \cup B_d$$

Hence

$$S = (B_1 \cap S) \cup (B_2 \cap S) \cup \dots \cup (B_d \cap S)$$

Now, for some $j=1,\dots,d$ it must be that $B_j \cap S$ cannot be covered by finitely many U_i, $i \in I$. Denote:

$$S_1 := B_j \cap S.$$

Observe that S_1 is closed, bounded, and for any $x,y \in S_1 \subseteq B_j$ we have

$$d(x,y) \leq 1.$$

Applying the previous lemma to S_1 and $\varepsilon = \frac{1}{4}$, and repeating the above argument, we obtain $S_2 \subseteq S_1$ which is closed, bounded, not covered by finitely many U_i, $i \in I$, and for any $x,y \in S_2$

$$d(x,y) \leq \frac{1}{2}.$$

Iterating this argument for $\varepsilon = \frac{1}{2N}$, $N \geq 3$ we obtain a sequence of closed bounded sets

$$S \supset S_1 \supset S_2 \supset S_3 \supset \cdots$$

none of which are covered by finitely many of the U_i, $i \in I$, and s.t.

$\forall x, y \in S_m$

$$d(x,y) \leq \frac{1}{m}$$

In particular, no S_m is empty (otherwise it would easily be covered by a finite number of U_i, $i \in I$).

Let $p_m \in S_m$ for each $m \in \mathbb{N}$.

Observe that $(p_m)_{m \in \mathbb{N}}$ is a Cauchy sequence. Indeed, for $\varepsilon > 0$, if $M \in \mathbb{N}$ is s.t. $\frac{1}{M} < \varepsilon$, then for any $m_1, m_2 \geq M$ we have

$$p_{m_1}, p_{m_2} \in S_M$$

$\Rightarrow$

$$d(p_{m_1}, p_{m_2}) \leq \frac{1}{M} < \varepsilon.$$

Because $(\mathbb{R}^n, d)$ is <u>complete</u>, $(p_m)_{m \in \mathbb{N}}$ converges to some $p \in \mathbb{R}^n$. Since $(p_m)_{m \in \mathbb{N}} \subset S$ and S is closed, $p \in S$.

Consequently, $p \in U_{i_0}$ for some $i_0 \in I$.

Since U_{i_0} is an open set, $\exists\, \varepsilon > 0$ s.t.

$$B(p, \varepsilon) \subseteq U_{i_0}.$$

Let $M \in \mathbb{N}$ be s.t. $\forall m \geq M$

$$d(p_m, p) < \varepsilon/2.$$

Picking $n \geq M$ and s.t. $\frac{1}{n} < \frac{\varepsilon}{2}$ we have for any $x \in S_n$ that

$$d(x,p) \leq d(x, p_n) + d(p_n, p)$$
$$< \frac{1}{n} + \frac{\varepsilon}{2}$$
$$< \varepsilon.$$

~~Hence~~ Since $x \in S_n$ was arbitrary, we have

$$S_n \subseteq B(p, \varepsilon) \subseteq U_{i_0}.$$

But this contradicts S_m not being covered by a finite union of the $U_i, i \in I$. $\square$

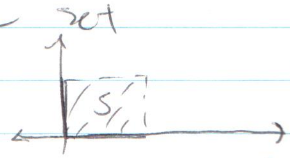
10/16/2017

Connectedness III.6

Before discussing "connectedness" we discuss the notion of relative open and closedness.

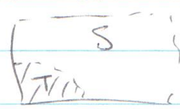
Ex: $(\mathbb{R}^2, d)$ be the 2-dim'l Euc. metric space. Consider the set

$$S = \{(x,y) : 0 \le x < 1,\ 0 \le y < 1\}$$



One can check that S is neither open nor closed in $(\mathbb{R}^2, d)$ ~~Hence~~. However, in $(S, d|_{S\times S})$ S is automatically open and closed. Consider $B((x_0,y_0), r, d)$ in $(S, d|_{S\times S})$

If we look at:

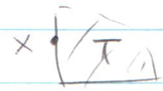


$$T = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1/4,\ x \ge 0,\ y \ge 0\} \subseteq S$$

single example

Then T is neither open or closed in $(\mathbb{R}^2, d)$ but in $(S, d|_{S\times S})$ it is open.

For



The open balls in $\underline{\underline{S}}$ centered at x cannot leak out to the left because when we restrict to S there is nothing to the left.

The above example leads to the following definition

Def: In a metric space (E, d), for $T \subset S \subset E$ we say T is open (resp. closed) relative to S