Skip

which we can do using Lemma 4. Then the RHS of (*)

$$10^{m+1}(x - a_0.a_1a_2\cdots a_m) < 1$$

or $$x - a_0.a_1a_2\cdots a_m < \frac{1}{10^{m+1}}$$

or $$x < a_0.a_1a_2\cdots a_m + \frac{1}{10^{m+1}}$$

$$< y + \varepsilon$$

So $x < y + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $x \le y \Rightarrow x = y$. □

~~Don't Skip~~

- For $x \in \mathbb{R}_-$, we obtain a decimal expansion by finding one for $-x \in \mathbb{R}_+$, say

$$-x = a_0.a_1a_2\cdots$$

Then we let

$$-a_0.a_1a_2\cdots$$

be the decimal expansion for x.

- Given $x = a_0.a_1a_2\cdots$ and $y = b_0.b_1b_2\cdots$ it is usually easy to tell them apart:

$$x = y \text{ if and only } a_n = b_n \ \forall n \in \mathbb{N} \cup \{0\}$$

Except in cases like

$$x = 0.99\cdots \quad \text{and} \quad y = 1.00\cdots$$

But because of the "$\le$" in (*), the decimal expansion we produced in the theorem will never have an infinite sequence of 9's.

Metric Spaces

Def: A metric space is a pair (E, d) consisting of a set E and a map

$$d : E \times E \to \mathbb{R}$$

such that:

(1) $d(x,y) \geq 0$ for all $x,y \in E$
(2) $d(x,y) = 0$ iff $x = y$
(3) $d(x,y) = d(y,x)$
(4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in E$
(triangle inequality) $\rightsquigarrow$ 

the map d is called a metric

Note that:
- E is just a set, so there may be no operations to do on points in E (e.g. "$x+y$" may not make sense for $x,y \in E$)
- $d(x,y)$ is the "distance" between the points $x,y \in E$

Examples:

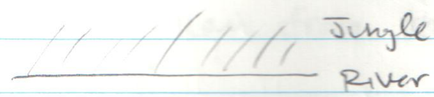
(a) $E = \mathbb{R}$, $d(x,y) := |x-y|$ for $x,y \in \mathbb{R}$
Check def
(1) ✓ by def'n of $|\cdot|$
(2) ✓ by def'n of $|\cdot|$
(3) ✓ since $|x-y| = |-(x-y)| = |y-x|$
(4) ✓ by previous version of Δ-inequality

(b) $E = \mathbb{Q} \subseteq \mathbb{R}$ with d as above. In general, if $E_1 \subseteq E$ for (E,d) a metric space, then $(E_1, d|_{E_1 \times E_1})$ is a metric space, where $d|_{E_1 \times E_1}$ is the restriction of d to $(x,y) \in E_1 \times E_1$

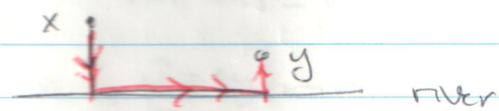
(c) E = set of URL's on the internet
$d(x,y)$ = ~~# clicks~~ smallest number of clicks or keystrokes to

get your web browser from site x to site y.

(d) "Jungle River Metric"

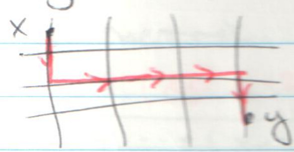
E: 

To go from x to y, cannot travel left or right, only up or down. Can travel left or right only on the river:



(e) "Taxi Cab Metric"

E: City Streets



: $d(x,y)$

(f) Trivial Metric Space

E = any set

$$d(x,y) = \begin{cases} 0 & x=y \\ 1 & \text{otherwise} \end{cases}$$

9/8/2017
recall def. of metric space

(g) For any $n \in \mathbb{N}$, n-dimensional Euclidean space is the set

$$\mathbb{R}^n := \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$$

For $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ define the n-dimensional Euclidean metric by

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$$

Let's verify that $\mathbb{R}^n$ with above d is a metric space:

(1), (2) ~~are clear~~ expand

(3) follows because $(x_i-y_i)^2 = (-(x_i-y_i))^2 = (y_i-x_i)^2$

(4) this will follow from the "Cauchy-Schwarz Inequality"

Prop: (Cauchy-Schwarz Inequality)
For any $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$

$$|a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n| \leq \sqrt{a_1^2+a_2^2+\cdots+a_n^2} \cdot \sqrt{b_1^2+b_2^2+\cdots+b_n^2}$$

Cor: For $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$, $\vec{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ we have

$$d(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{x})$$

In particular, the n-dim'l Euclidean metric is actually a metric.

Pf: We compute (do $n=2$)

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$$

$$= \sqrt{(x_1-z_1+z_1-y_1)^2 + \cdots + (x_n-z_n+z_n-y_n)^2}$$

$$= \sqrt{(x_1-z_1)^2 + \cdots + (x_n-z_n)^2 + 2\{(x_1-z_1)\cdot(z_1-y_1) + \cdots + (x_n-z_n)(z_n-y_n)\} + (z_1-y_1)^2 + \cdots + (z_n-y_n)^2}$$

So applying the CS-inequality with

$$a_i = x_i - z_i$$
$$b_i = z_i - y_i$$

we have, continuing the above comp.

continued

$$\leq \sqrt{(x_1-z_1)^2 + \cdots + (x_n-z_n)^2 + 2\sqrt{(x_1-z_1)^2 + \cdots + (x_n-z_n)^2} \cdot \sqrt{(z_1-y_1)^2 + \cdots + (z_n-y_n)^2} + (z_1-y_1)^2 + \cdots + (z_n-y_n)^2}$$

$$= \sqrt{d(\vec{x}, \vec{z})^2 + 2\, d(\vec{x}, \vec{z}) \cdot d(\vec{z}, \vec{y}) + d(\vec{z}, \vec{y})^2}$$

$$= \sqrt{(d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y}))^2} = d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y}). \quad \square$$

Proof (Cauchy-Schwarz Inequality) (Do $n=2$)

Fix $\vec{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\vec{b}=(b_1,\dots,b_n)\in\mathbb{R}^n$

Note that for any $\alpha,\beta\in\mathbb{R}$ we have

$$0\le(\alpha a_1-\beta b_1)^2+\cdots+(\alpha a_n-\beta b_n)^2$$
$$=\alpha^2(a_1^2+\cdots+a_n^2)-2\alpha\beta(a_1b_1+\cdots+a_nb_n)+\beta^2(b_1^2+\cdots+b_n^2)$$

or

$$2\alpha\beta(a_1b_1+\cdots+a_nb_n)\le\alpha^2(a_1^2+\cdots+a_n^2)+\beta^2(b_1^2+\cdots+b_n^2)$$

Choose

$$\alpha=\sqrt{b_1^2+\cdots+b_n^2}\qquad\beta=\pm\sqrt{a_1^2+\cdots+a_n^2}$$

Then the above inequality becomes

$$\pm\cancel{2}\sqrt{b_1^2+\cdots+b_n^2}\sqrt{a_1^2+\cdots+a_n^2}\,(a_1b_1+\cdots+a_nb_n)\le\cancel{2}(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)$$

or

$$\pm(a_1b_1+\cdots+a_nb_n)\le\sqrt{a_1^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+\cdots+b_n^2}$$

This implies

$$|a_1b_1+\cdots+a_nb_n|\le\sqrt{a_1^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+\cdots+b_n^2}\quad\blacksquare$$

Prop (Reverse triangle inequality)

Let (E,d) be a metric space. For any $x,y,z\in E$ we have

$$|d(x,y)-d(y,z)|\le d(x,z)$$

real #'s, so subtraction makes sense.

Pf: Using the triangle inequality we have

$$d(x,y)\le d(x,z)+d(z,y)$$
$$=d(x,z)+d(y,z)$$

or

$$d(x,y)-d(y,z)\le d(x,z)\qquad *$$

Using the triangle inequality again we have:

$$d(y,z)\le d(y,x)+d(x,z)$$
$$=d(x,y)+d(x,z)$$

or

$$d(y,z)-d(x,y)\le d(x,z)$$

or $$-(d(x,y)-d(y,z))\le d(x,z)\qquad **$$

so combining (*) and (**) we have

$$|d(x,y) - d(y,z)| \le d(x,z) \qquad \square$$

Open and closed sets (III.2)

Def: For (E,d) a metric space, $x \in E$, and $r>0$ we define the open ball in E of center x and radius r as the set

$$B(x,r) := \{y \in E : d(x,y) < r\}$$

9/11/2017

We define the closed ball in E of center x and radius r as the set

$$B[x,r] := \{y \in E : d(x,y) \le r\}$$

We'll say "ball" to refer to an either open or closed ball, when the distinction does not matter.

Picture: (27)

- Note that since $d(x,x)=0$, we always have

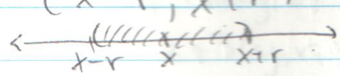
$$x \in B(x,r) \qquad x \in B[x,r]$$

for any $r>0$.

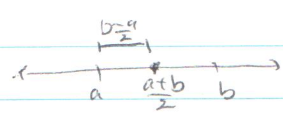
Ex: • $(E,d) = (\mathbb{R}, |\cdot|)$

$B(x,r) = \{y \in \mathbb{R} : |x-y| < r\}$ → $x-r < y < x+r$

$= (x-r, x+r)$ interval notation



Also, the interval (a,b):

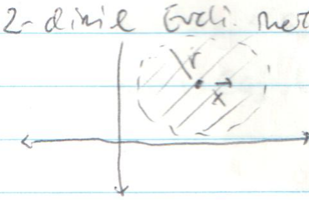


so $(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$. $B[x,r] = [x-r, x+r]$

$[a,b] = B\left[\frac{a+b}{2}, \frac{b-a}{2}\right]$

- $(E,d) = (\mathbb{R}^2$, 2-dim'l Eucl. metric$)$

$B(\vec{x}, r)$



$B[x,r]$: