(iii) $|a|^2 = a^2$

(iv) $|a+b| \le |a| + |b|$ (triangle inequality)

(v) $|a-b| \ge \big||a| - |b|\big|$ (reverse triangle inequality)

Pf (i) – (iii) are easy or immediate. Verify at home.

(iv): First note

$$a \le |a| \text{ and } b \le |b| \text{ so } a+b \le |a| + |b|$$

and $-a \le |a|$ and $-b \le |b|$ so $-(a+b) \le |a| + |b|$.

Thus

$$|a+b| = \begin{cases} a+b & \text{if } a+b>0 \\ 0 & \text{if } a+b=0 \\ -(a+b) & \text{if } a+b<0 \end{cases} \le |a| + |b|.$$

(v) Using (iv) we have

$$|a| = |a-b+b| \le |a-b| + |b|$$

or

$$|a| - |b| \le |a-b|$$

Similarly

$$|b| = |b-a+a| \le |b-a| + |a|$$

or

$$|b| - |a| \le |b-a| = |-(b-a)| = |-b+a| = |a-b|$$

So

$$\big||a| - |b|\big| = \begin{cases} |a| - |b| & \text{if } |a|-|b|>0 \\ 0 & \text{if } |a|-|b|=0 \\ -(|a|-|b|) = |b|-|a| & \text{if } |a|-|b|<0 \end{cases} \le |a-b|.$$

$\square$

- The Δ & reverse-Δ inequalities are very important in analysis and will be used all the time.
- Remark about proving $|x| \le c \Leftarrow x \le c$ and $-x \le c$
- Least Upper Bound Property

Def: For a subset $S \subset \mathbb{R}$, $a \in \mathbb{R}$ is an upper bound for S if $a \ge s$ for all $s \in S$. If an upper bound for S exists, we say S is bounded from above.

Ex:
- $S = [-1,3]$ is bounded from above
  upper bounds?
- $S = \mathbb{N}$ is not bounded from above
- $S = \phi$?

Def: For a subset $S \subseteq \mathbb{R}$, say $y \in \mathbb{R}$ is a supremum or least upper bound (l.u.b.) of S if y is an upper bound for S, and for any other upper bound for S, say $a \in \mathbb{R}$, we have $y \le a$. We write $y = \sup(S)$.

Ex:
- $S = [-1,3]$, $\sup(S) = ?$
- $S = [-1,3)$, $\sup(S) = ?$
- ~~$S = \mathbb{N}$, write $\sup(\mathbb{N}) = +\infty$~~ (state as convention later)
- ~~$S = \phi$, $\sup(\phi) = -\infty$~~
- $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(S) = ?$

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- Draw graph of f, $S = \{f(t) : a \le t \le b\}$
  $\sup(S) = ?$

VII Any nonempty set $S \subseteq \mathbb{R}$ that is bounded from above has a least upper bound.

Convention: If S is not bdd from above we write $\sup(S) = +\infty$.
If $S = \phi$, we write $\sup(\phi) = -\infty$.

Thm: Let $S \subseteq \mathbb{R}$ be non-empty and bdd from above. For any $\varepsilon > 0$, $\exists s \in S$ s.t.
$$\sup(S) - \varepsilon < s \le \sup(S).$$

Pf: Let $\varepsilon > 0$. Suppose $\nexists s \in S$ s.t. $\sup(S) - \varepsilon < s$. Then $s \le \sup(S) - \varepsilon$ $\forall s \in S$. But that means $\sup(S)$ is an upper bound for S. Since $\sup(S)$ is the least upper bound, we have

$$\sup(S) \le \sup(S) - \varepsilon$$
$$0 \le -\varepsilon$$
$$\varepsilon \le 0$$

which contradicts $\varepsilon > 0$. Thus $\exists\, s \in S$ s.t. $\sup(S) - \varepsilon < s$. Since $\sup(S)$ is an upper bound, we also have $s \le \sup(S)$. $\square$

Def: For a subset $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is a lower bound for S if $a \le s$ for all $s \in S$. If a lower bound for S exists, we say S is bounded from below. If S is bdd from above and below, say S is bounded.

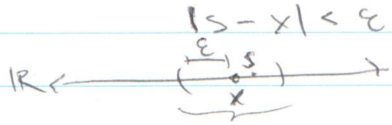
Def: For a subset $S \subset \mathbb{R}$, say $y \in \mathbb{R}$ is an infimum ~~of~~ a greatest lower bound (g.l.b.) of S if y is a lower bound for S, and for any other lower bound for S, say $a \in \mathbb{R}$, we have $a \le y$. We write $y = \inf(S)$

Thm If $S \subseteq \mathbb{R}$ is non-empty and bounded from below, then S has a greatest lower bound.
Pf Exercise.

Convention: If S is not bdd from below, we write $\inf(S) = -\infty$. If $S = \emptyset$, we write, $\inf(S) = +\infty$.

Def: A set $S \subset \mathbb{R}$ is dense if for every $x \in \mathbb{R}$ and $\varepsilon > 0$, $\exists\, s \in S$ s.t.

$$|s - x| < \varepsilon \overset{\text{H.W}}{\iff} x - \varepsilon < s < x + \varepsilon$$



Thm: The rational numbers $\mathbb{Q} \subseteq \mathbb{R}$ are dense.

<u>Lemma 1</u> For any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > x$

<u>Pf:</u> We proceed by contradiction. Assume $\forall n \in \mathbb{N}$, $n \le x$. This means x is an upper bound for $\mathbb{N}$, and so by Property VII, $\sup(\mathbb{N})$ exists.

Now, $\forall n \in \mathbb{N}$, $n+1 \in \mathbb{N}$ which means

$$n+1 \le \sup(\mathbb{N}) \qquad \forall n \in \mathbb{N}$$
$$n \le \sup(\mathbb{N}) - 1 \qquad \forall n \in \mathbb{N}$$

But this means $\sup(\mathbb{N}) - 1$ is also an upperbound for $\mathbb{N}$. ~~Thus~~ As $\sup(\mathbb{N})$ is the <u>least</u> upper bound

$$\sup(\mathbb{N}) \le \sup(\mathbb{N}) - 1$$

a contradiction. Thus $\exists n \in \mathbb{N}$ s.t. $x < n$. □

Note that we also showed $\mathbb{N}$ is not bdd from above.

<u>Lemma 2</u> For any $\varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

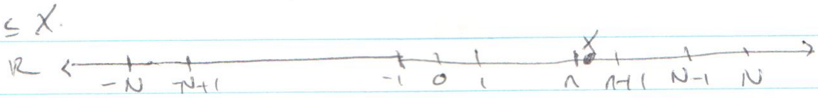
<u>Pf:</u> Let $\varepsilon > 0$. By Lemma 1, $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{\varepsilon} > 0$. By earlier props, $0 < \frac{1}{n} < \varepsilon$. □

<u>Lemma 3</u> For any $x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ s.t. $n \le x < n+1$.

<u>Pf:</u> Using Lemma 1, we can find $N \in \mathbb{N}$ s.t. $|x| < N$. From homework $-N < x < N$. Pick n to be the largest element of

$$\{-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N\}$$

s.t. $n \le x$.



□

<u>Lemma 4</u> For any $x \in \mathbb{R}$ and any $N \in \mathbb{N}$ $\exists n \in \mathbb{Z}$ such that $\frac{n}{N} \le x < \frac{n+1}{N}$

<u>Pf</u> Let $x \in \mathbb{R}$, $N \in \mathbb{N}$. By Lemma 3, $\exists n \in \mathbb{Z}$ s.t. $n \le N \cdot x < n+1$. Since $N > 0$, this implies

$$\frac{n}{N} \le x < \frac{n+1}{N}.$$

□

Proof of Thm: Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. By Lemma 2, $\exists N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \varepsilon$. Then by Lemma 4, $\exists n \in \mathbb{Z}$ s.t.

$$\frac{n}{N} \le x < \frac{n+1}{N}.$$

Subtracting $\frac{n}{N}$ from each side gives:

$$0 \le x - \frac{n}{N} < \frac{n+1}{N} - \frac{n}{N} = \frac{1}{N}$$

So

$$|x - \tfrac{n}{N}| = x - \tfrac{n}{N} < \tfrac{1}{N} < \varepsilon.$$

Since $\frac{n}{N} \in \mathbb{Q}$, we've shown $\mathbb{Q}$ is dense $\square$

Thm For every $x \in \mathbb{R}_+$, $\exists!$ element of $\mathbb{R}$ denoted $\sqrt{x}$, such that $(\sqrt{x})\cdot(\sqrt{x}) = x$.

(warning: ~~very~~ technical proof)

Proof. Consider the set

$$S = \{y \in \mathbb{R} : y^2 \le x\}$$

We first claim that S is bdd from above by $\max\{x, 1\}$. Indeed, if not then $\exists y \in S$ st $y > \max\{x, 1\}$. Recall $a > b > 0$, and $c > d > 0$ imply $ac > bd > 0$. Thus

$$y^2 = y \cdot y > y \cdot 1 > x \cdot 1 = x.$$

which contradicts $y \in S$. So $\max\{x, 1\}$ is an upper bound for S. Also note $S \neq \emptyset$ since $0 \in S$. Thus $\sup(S)$ exists and we claim $(\sup(S))^2 = x$. Write $a := \sup(S)$. Observe $\min\{x, 1\} \in S$ since

$$\min\{x, 1\}^2 \le \min\{x, 1\} \cdot 1 \le x \cdot 1 = x.$$

So $a \ge \min\{x, 1\} > 0$. Let $\varepsilon > 0$ be s.t. $0 < \varepsilon < a$. Then

$$0 < a - \varepsilon < a < a + \varepsilon$$

So

$$(a-\varepsilon)^2 < a^2 < (a+\varepsilon)^2 \qquad (*)$$

Now $a + \varepsilon \notin S$ since $a = \sup(S)$ is an upper bound.

so by def of S, $(a+\varepsilon)^2 > x$. Also, $\exists y \in S$ s.t. $a-\varepsilon < y < a$. (by earlier thm). But then

$$(a-\varepsilon)^2 < y^2 \leq x.$$

So we have $(a-\varepsilon)^2 < x < (a+\varepsilon)^2$

or $-(a+\varepsilon)^2 < -x < -(a-\varepsilon)^2$ $(**)$

Adding $(*)$ and $(**)$, we obtain

$$(a-\varepsilon)^2 - (a+\varepsilon)^2 < a^2 - x < (a+\varepsilon)^2 - (a-\varepsilon)^2$$

or $-[(a+\varepsilon)^2 - (a-\varepsilon)^2] < a^2 - x < (a+\varepsilon)^2 - (a-\varepsilon)^2$

From HW this means

$$|a^2 - x| < (a+\varepsilon)^2 - (a-\varepsilon)^2$$
$$= a^2 + 2\varepsilon a + \varepsilon^2 - a^2 + 2\varepsilon a - \varepsilon^2$$
$$= 4\varepsilon a.$$

Let $\tilde{\varepsilon} = 4\varepsilon a$. We've show

$$|a^2 - x| < \tilde{\varepsilon}$$

Since we can make ε arbitrarily small, $\tilde{\varepsilon}$ can be made arbitrarily small. So from HW, we know $a^2 = x$.

Lastly, we show x has a unique square root. Assume $a, \hat{a} \in \mathbb{R}$ both satisfy

$$a^2 = x = (\hat{a})^2$$

we know either $a > \hat{a}$, $a = \hat{a}$, or $a < \hat{a}$.
If $a > \hat{a}$, then $x = a^2 > \hat{a}^2 = x$, a contradiction.
Similarly for $a < \hat{a}$. Thus we must have $a = \hat{a}$ $\square$

Prop: $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Pf: We proceed by contradiction. Assume $\sqrt{2} \in \mathbb{Q}$, so that $\sqrt{2} = \frac{a}{b}$ for some $a \in \mathbb{Z}$, $b \in \mathbb{N}$. We may assume a, b have no common factors.
Then $\sqrt{2}\, b = a \to 2b^2 = a^2$. So a is even: i.e. $a = 2 \cdot c$ for some $c \in \mathbb{Z}$. But then

$$2b^2 = (2 \cdot c)^2 = 4 \cdot c^2$$
$$b^2 = 2 \cdot c^2$$

which means b is even too. But this contradicts our assumption that a, b had no common factors. $\square$

- This has two implications:
  - $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$
  - $\mathbb{Q}$ does not have the least upper bound property: Indeed

    $$S := \{x \in \mathbb{Q} : x \leq \sqrt{2}\}$$

    is bounded above (by 2).
    However, we claim S does not have a supremum in $\mathbb{Q}$. If $y \in \mathbb{Q}$ was a supremum, then note that $y \neq \sqrt{2}$ by the prop.

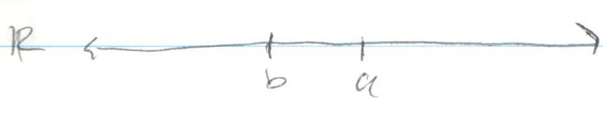
    Case 1: $y < \sqrt{2}$. By HW, $\mathbb{Q}$ dense $\Rightarrow \exists x \in \mathbb{Q}$ st $y < x < \sqrt{2}$. But then $x \in S$ and so y is not an upper bound for S. $\Rightarrow\Leftarrow$.

    Case 2: $y > \sqrt{2}$. As before $\exists x \in \mathbb{Q}$ st. $\sqrt{2} < x < y$. But then x is an upper bound for S that is strictly smaller than y. $\Rightarrow\Leftarrow$. QED.

- The following idea/lemma will be useful throughout the course

Thm ("an epsilon of room")
Let $a, b \in \mathbb{R}$. If $a \leq b + \varepsilon$ $\forall \varepsilon > 0$, then $a \leq b$.

Pf: Suppose towards a contradiction that $a > b$. So



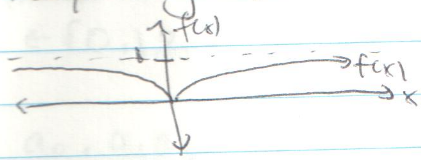
Let $\varepsilon = \frac{a-b}{2} > 0$. Then $b+\varepsilon = \frac{2b}{2} + \frac{a-b}{2} = \frac{a+b}{2} < \frac{2a}{2} = a$

or $b+\varepsilon < a$. But this contradicts $a \le b+\varepsilon$.

Thus we must have $a \le b$. $\square$

- The preceeding theorem will be useful when we want to show $a \le b$, but ~~[illegible]~~ is too hard to do directly. Thus we use the theorem to "give ourselves an ε of room" and prove the easier inequality of $a \le b+\varepsilon$.

<u>Ex</u>: Let $f(x) = \frac{x^2}{1+x^2}$



Define $S = \{f(x) : x \in \mathbb{R}\}$

<u>Claim</u>: $\sup(S) = 1$

<u>Pf</u>: We first check that 1 is an upper bound:

$$f(x) = \frac{x^2}{1+x^2} < \frac{x^2}{1+x^2} + \frac{1}{1+x^2} = \frac{1+x^2}{1+x^2} = 1.$$

So $\sup(S)$ exists and $\sup(S) \le 1$. To show $1 \le \sup(S)$, we'll use the theorem and check

$$1 \le \sup(S) + \varepsilon \quad \forall \varepsilon > 0.$$

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Let $\varepsilon > 0$. Set $x = \sqrt{\frac{1-\varepsilon}{\varepsilon}}$. Then

$$f(x) = \frac{\frac{1-\varepsilon}{\varepsilon}}{1+\frac{1-\varepsilon}{\varepsilon}} = \frac{1-\varepsilon}{\varepsilon + 1 - \varepsilon} = 1-\varepsilon$$

So $\sup(S) \ge f(x) = 1-\varepsilon$ or $1 \le \sup(S)+\varepsilon$.

Thus "ε-of room" thm $\Rightarrow 1 \le \sup(S) \Rightarrow \sup(S) = 1$. $\square$

- This example also demonstrates another common technique in analysis: to show $a=b$ we can show $a \le b$ and $a \ge b$ individually.

Decimal Expansions

For $n \in \mathbb{N}$, let $a_0 \in \mathbb{N} \cup \{0\}$ and $a_1, a_2, \ldots, a_n \in \{0,1,2,\ldots,9\}$

We define

$$a_0.a_1a_2\cdots a_n = a_0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n}$$

Ex: $3.027 = 3 + \frac{0}{10} + \frac{2}{100} + \frac{7}{1000}$

Def: $a_0.a_1a_2\cdots a_n$ is called a finite decimal expansion

Now let $a_0 \in \mathbb{N}\cup\{0\}$ and take an infinite list (sequence) $a_1, a_2, \ldots \in \{0,1,2,\ldots,9\}$

We define

$$a_0.a_1a_2\cdots = \sup\{a_0.a_1a_2\cdots a_n : n\in\mathbb{N}\}$$

Observe that the set is bdd above by a_0+1, so the supremum exists.

Ex: $a_0 = 0$, $a_n = 3$ $\forall n\in\mathbb{N}$.

$$0.33\cdots = \sup\{0.\underbrace{33\cdots3}_{n \text{ times}} : n\in\mathbb{N}\}$$

$$\text{“}=\text{”} \frac{1}{3}$$

Def: $a_0.a_1a_2\cdots$ is called an infinite decimal expansion.

→ Expand dec. exp for $\mathbb{R}$–

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Thm: Every $x\in\mathbb{R}_+\cup\{0\}$ has a decimal expansion.

Pf: By "Lemma 3" $\exists n_0\in\mathbb{Z}$ s.t.

$$n_0 \le x < n_0+1$$

Set $a_0 = n_0$. Note that we have

$$0 \le x - a_0 < 1$$

$$\rightsquigarrow 0 \le 10(x-a_0) < 10$$

We define the sequence $\{a_1, a_2, \ldots\}$ inductively. Suppose we have found, for $m\in\mathbb{N}$, $a_1, a_2, \ldots, a_m \in \{0,1,2,\ldots,9\}$ such that

$$0 \le 10^{m+1}(x - a_0.a_1a_2\cdots a_m) < 10 \qquad *$$

Using "Lemma 4", $\exists n_{m+1} \in \mathbb{N}$ s.t.

$$\frac{n_{m+1}}{10^{m+1}} \le x - a_0.a_1a_2\cdots a_m < \frac{n_{m+1}+1}{10^{m+1}}$$

So that

$$n_{m+1} \le 10^{m+1}(x - a_0.a_1a_2\cdots a_m) < n_{m+1}+1 \quad (**)$$

Using $(*)$ and $(**)$:

$$\left.\begin{aligned} 10^{m+1}(x-a_0.a_1a_2\cdots a_m) &< 10 \\ n_{m+1} &\le 10^{m+1}(x-a_0.a_1a_2\cdots a_m) \end{aligned}\right\} \Longrightarrow n_{m+1} < 10 \Rightarrow n_{m+1} \le 9$$

and

$$\left.\begin{aligned} 0 &\le 10^{m+1}(x-a_0.a_1a_2\cdots a_m) \\ 10^{m+1}(x-a_0.a_1a_2\cdots a_m) &< n_{m+1} \end{aligned}\right\} \Longrightarrow 0 < n_{m+1}+1 \Rightarrow n_{m+1} \ge 0$$

So $n_{m+1} \in \{0,1,2,\ldots,9\}$ and we set $a_{m+1} = n_{m+1}$.

From $(**)$ we also get:

$$0 \le 10^{m+1}(x - a_0.a_1a_2\cdots a_m a_{m+1}) < 1$$

$$\leadsto \quad 0 \le 10^{m+2}(x - a_0.a_1a_2\cdots a_m a_{m+1}) < 10$$

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So by induction we obtain an infinite sequence $(a_m)_{m\in\mathbb{N}} \subseteq \{0,1,2,\ldots,9\}$ satisfying $(*)$ for all $m \in \mathbb{N}$.

Define

$$y = a_0.a_1a_2\cdots = \sup\{a_0.a_1a_2\cdots a_m : m \in \mathbb{N}\}$$

We claim $x = y$. Indeed, the left-hand side (LHS) of $(*)$:

$$0 \le 10^{m+1}(x - a_0.a_1a_2\cdots a_m)$$

or $\quad 0 \le x - a_0.a_1a_2\cdots a_m$

or $\quad a_0.a_1a_2\cdots a_m \le x.$

So x is an upper bound for $\{a_0.a_1\cdots a_m : m \in \mathbb{N}\}$

$\Rightarrow y \le x.$

To see $x \le y$, we give ourselves an ε of room. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ be large enough so that

$$\frac{1}{10^{m+1}} < \varepsilon$$

which we can do using Lemma 4. Then the RHS of (*)

$$10^{m+1}(x - a_0.a_1a_2\cdots a_m) < 1$$

or $\quad x - a_0.a_1a_2\cdots a_m < \frac{1}{10^{m+1}}$

or $\quad x < a_0.a_1a_2\cdots a_m + \frac{1}{10^{m+1}}$

$$< y + \varepsilon$$

So $x < y+\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $x \le y \Rightarrow x = y$. $\square$

Skip

~~Don't Skip~~

- For $x \in \mathbb{R}_-$, we obtain a decimal expansion by finding one for $-x \in \mathbb{R}_+$. Say
$$-x = a_0.a_1a_2\cdots$$
Then we let
$$-a_0.a_1a_2\cdots$$
be the decimal expansion for x.

- Given $x = a_0.a_1a_2\cdots$ and $y = b_0.b_1b_2\cdots$ it is usually easy to tell them apart:
$$x = y \text{ if and only } a_n = b_n \ \forall n \in \mathbb{N}\cup\{0\}$$
Except in cases like
$$x = 0.99\cdots \quad \text{and} \quad y = 1.00\cdots$$
But because of the "$\le$" in (*), the decimal expansion we produced in the theorem will never have an infinite sequence of 9's.

Metric Spaces

Def: A metric space is a pair (E,d) consisting of a set E and a map
$$d: E \times E \to \mathbb{R}$$
such that: