- **Corollary**: The additive identity $0 \in \mathbb{R}$ is unique

Pf: Suppose $0' \in \mathbb{R}$ is another neutral additive element; that is, $a + 0' = a$ for all $a \in \mathbb{R}$. But since $x + a = a$ has a unique solution by our lemma, we must have $0 = 0'$. $\boxtimes$

- **Prop^tion**: For any $a \in \mathbb{R}$, $a \cdot 0 = 0$

Pf: Let $a \in \mathbb{R}$, then

$$a \cdot 0 + a \cdot 0 \underset{\text{III}}{=} a \cdot (0 + 0) \underset{\text{IV}}{=} a \cdot (0)$$

But also, $0 + a \cdot 0 \underset{\text{IV}}{=} a \cdot 0$. So by lemma $a \cdot 0 = 0$ since both are solutions to $x + a \cdot 0 = a \cdot 0$. $\square$

- Using similar, meticulous arguments we can prove lots of other familiar arithmetic facts (see F1 – F10 in book).

Prop: For $a, b \in \mathbb{R}$

(i) If $a \neq 0$, $xa = b$ has a unique solution.

(ii) $-(-a) = a$

(iii) $(a^{-1})^{-1} = a$

(iv) $-(a+b) = -a + -b$

(v) $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$

(vi) $(-1) \cdot a = -a$

Pf Exercise. $\square$

Order Property VI

- **Def**: $\mathbb{R}_+$ is called the set of positive numbers. The set of $a \in \mathbb{R}$ s.t. $-a \in \mathbb{R}_+$ is called the set of negative numbers and is denoted $\mathbb{R}_-$

- These special subsets gives a way to compare the "size" of numbers, (order them)

For $a,b\in\mathbb{R}$ we write

$a>b$ if $a-b\in\mathbb{R}_+$ ("a is greater than b") /strictly greater

$a<b$ if $a-b\in\mathbb{R}_-$ ("a is less than b") /strictly less

(if $a-b\in\{0\}$, then $a=b$)

Also, we write:

$a\geq b$ if either $a>b$ or $a=b$
("a is greater than or equal to b")

$a\leq b$ if either $a<b$ or $a=b$
("b is ~~greater~~ less than or equal to b")

SOS

- Using Properties I–VI, we learn more about $\mathbb{R}$:

Skip

Prop For $a,b,c\in\mathbb{R}$ if $a>b$ and $b>c$, then $a>c$.

Pf: Since $a>b$ and $b>c$, we know $a-b,\ b-c\in\mathbb{R}_+$.

~~Hence~~

$a-c = a-b+b-c \in\mathbb{R}_+$ by VI.(1).

Therefore $a>c$. $\square$

Emphasize $c=d$ case

- **Prop** For $a,b,c,d\in\mathbb{R}$ if $a>b$ and $c\geq d$ then $a+c>b+d$.

Pf: By assumption $a-b\in\mathbb{R}_+$ and $c-d\in\mathbb{R}_+\cup\{0\}$

Thus

$$\mathbb{R}_+ \overset{\text{VI.(1)}}{\ni} a-b+c-d \overset{\text{II}}{=} a+c-b-d \overset{\text{III}}{=} a+c-(b+d)$$

So $a+c>b+d$. $\square$

- Remark: $a\in\mathbb{R}_+ \iff a>0$, $a\in\mathbb{R}_+\cup\{0\} \iff a\geq 0$, $a\in\mathbb{R}_- \iff a<0$, $a\in\mathbb{R}_-\cup\{0\} \iff a\in\mathbb{R}_-$..

Prop For $a\in\mathbb{R}$, $a^2 = a\cdot a \geq 0$ ($\in\mathbb{R}_+\cup\{0\}$)

Pf We handle the cases $a\in\mathbb{R}_+$, $a=0$, $a\in\mathbb{R}_-$ separately

- If $a\in\mathbb{R}_+$, then $a\cdot a\in\mathbb{R}_+$ by VI.(1)
- If $a=0$, then $0\cdot 0=0\geq 0$. by ~~earlier~~ earlier prop.
- If $a\in\mathbb{R}_-$, then $-a\in\mathbb{R}_+$ and so

$$
\begin{aligned}
a\cdot a &= [-(-a)]\cdot[-(-a)] \\
&= [(-1)\cdot(-a)]\cdot[(-1)\cdot(-a)] \\
&= (-1)\cdot(-1)\cdot(-a)\cdot(-a) \\
\text{IV} \quad &= (-(-1))\cdot(-a)\cdot(-a) \\
\text{VI}.(i) \quad &= (1)\cdot(-a)\cdot(-a) = (-a)\cdot(-a) \in \mathbb{R}_+
\end{aligned}
$$

- Again, there is much more we can ~~proof~~ prove (see 01-09)

Prop (i) If $a>b$ and $b>c$, then $a>c$.

(ii) If $a>b>0$, $c\geq d>0$, then $ac>bd$

(iii) $(pos)+(pos)=pos$

$neg + neg = neg$

$(pos)\cdot(pos) = pos$

$(pos)\cdot(neg) = neg$

$(neg)\cdot(neg) = pos$

(iv) If $a>0$, then $a^{-1}>0$

(v) If $a>b>0$, then $b^{-1}>a^{-1}>0$

Pf (i)-(iv) Exercise.

(v) Since $a>b>0$, $a\cdot b>0 \Rightarrow (ab)^{-1}>0$ by (iii)

But $(ab)^{-1}=a^{-1}b^{-1}$. Then by (i)

$$\underset{b^{-1}}{\overset{\parallel}{(ab)^{-1}a}} > \underset{a^{-1}}{\overset{\parallel}{(ab)^{-1}b}}$$

$\boxed{}$

Def For $a\in\mathbb{R}$, we define the absolute value of a by

$$|a| = \begin{cases} a & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -a & \text{if } a<0 \end{cases}$$

Observe that $|-a|=|a|$

Prop For $a,b\in\mathbb{R}$

(i) $|a|\geq 0$ with $|a|=0$ if and only if $a=0$

(ii) $|ab|=|a|\cdot|b|$

(iii) $|a|^2 = a^2$

(iv) $|a+b| \le |a| + |b|$ (triangle inequality)

(v) $|a-b| \ge ||a| - |b||$ (reverse triangle inequality)

Pf (i)–(iii) are easy or immediate. Verify at home.

(iv): First note

$$a \le |a| \text{ and } b \le |b| \text{ so } a+b \le |a|+|b|$$

and $$-a \le |a| \text{ and } -b \le |b| \text{ so } -(a+b) \le |a|+|b|.$$

Thus

$$|a+b| = \begin{cases} a+b & \text{if } a+b>0 \\ 0 & \text{if } a+b=0 \\ -(a+b) & \text{if } a+b<0 \end{cases} \le |a|+|b|.$$

(v) Using (iv) we have

$$|a| = |a-b+b| \le |a-b| + |b|$$

or

$$|a| - |b| \le |a-b|$$

Similarly

$$|b| = |b-a+a| \le |b-a| + |a|$$

or

$$|b| - |a| \le |b-a| = |-(b-a)| = |-b+a| = |a-b|$$

So

$$||a|-|b|| = \begin{cases} |a|-|b| & \text{if } |a|-|b|>0 \\ 0 & \text{if } |a|-|b|=0 \\ -(|a|-|b|) = |b|-|a| & \text{if } |a|-|b|<0 \end{cases} \le |a-b|.$$

$\square$

- The Δ & reverse-Δ inequalities are very important in analysis and will be used all the time.
- Remark about proving $|x| \le c \Leftarrow x \le c$ and $-x \le c$
- Least Upper Bound Property

Def: For a subset $S \subset \mathbb{R}$, $a \in \mathbb{R}$ is an upper bound for S if $a \ge s$ for all $s \in S$. If an upper bound for S exists, we say S is bounded from above.