# Interiors, Closures, and Boundaries

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Let $(E, d)$ be a metric space, which we will reference throughout. In these exercises, we formalize for a subset $S \subset E$ the notion of its "interior", "closure", and "boundary," and explore the relations between them.

## 1 Definitions

We state for reference the following definitions:

**Definition 1.1.** Given a subset $S \subset E$, we say $x \in S$ is an **interior point of** $S$ if there exists $r > 0$ such that $B(x, r) \subset S$. The **interior of** $S$, denoted $S^\circ$, is the subset of $S$ consisting of the interior points of $S$.

**Definition 1.2.** Given a subset $S \subset E$, the **closure of** $S$, denoted $\overline{S}$, is the intersection of all closed sets containing $S$.

**Remark 1.3.** Note that there is always at least one closed set containing $S$, namely $E$, and so $\overline{S}$ always exists and $S \subset \overline{S}$. Moreover, as the intersection of closed sets, $\overline{S}$ is closed.

**Definition 1.4.** Given a subset $S \subset E$, we say $S$ is **dense in** $E$ if for all $x \in E$ and all $r > 0$, there exists $s \in S$ with $d(x, s) < r$. That is, $B(x, r) \cap S \neq \emptyset$.

**Definition 1.5.** Given a subset $S \subset E$, the **boundary of** $S$ is the set $\partial S := \overline{S} \setminus S^\circ$.

## 2 Exercises

1. For the following subsets, determine (without proof) their interiors, closures, and boundaries
   (a) $S = [0, 1]$ in $\mathbb{R}$ with the usual metric.
   (b) $S = (0, 1)$ in $\mathbb{R}$ with the usual metric.
   (c) $S = \mathbb{Z}$ in $\mathbb{R}$ with the usual metric.
   (d) $S = \mathbb{Q}$ in $\mathbb{R}$ with the usual metric.
   (e) $S = \mathbb{R}$ in $\mathbb{R}$ with the usual metric.
   (f) $S = [0, 1) \times [0, 1)$ in $\mathbb{R}^2$ with the 2-dimensional Euclidean metric.
2. Let $S \subset E$.
   (a) Show that $S^\circ$ is the union of all open subsets $U \subset S$.
   (b) Show that $S^\circ$ is open.
   (c) Show that $\partial S$ is closed.
3. Let $T \subset S \subset E$.
   (a) Show that $\overline{T} \subset \overline{S}$.
   (b) Show that $T^\circ \subset S^\circ$.

4. Let $S \subset E$.
   (a) Show that $S$ is open if and only if $S = S^\circ$.
   (b) Show that $S$ is closed if and only if $S = \overline{S}$.
5. Let $S \subset E$.
   (a) Show that $\overline{S} = ((S^c)^\circ)^c$.
   (b) Show that $S^\circ = \left(\overline{(S^c)}\right)^c$.
   (c) Show that $\partial S = \overline{S} \cap \overline{S^c}$.
   (d) Show $\partial S = \partial (S^c)$.
6. Let $S \subset E$
   (a) Show $\overline{S} = \{x \in E \colon B(x,r) \cap S \neq \emptyset \ \forall r > 0\}$.
   (b) Show $\partial S = \{x \in E \colon B(x,r) \cap S \neq \emptyset$ and $B(x,r) \cap S^c \neq \emptyset \ \forall r > 0\}$.
7. For $S \subset E$ show that the following are equivalent:
   (i) $S$ is dense in $E$.
   (ii) $(S^c)^\circ = \emptyset$.
   (iii) $\overline{S} = E$.
8. For $S \subset E$, show that $E$ is the disjoint union of $S^\circ$, $\partial S$, and $(S^c)^\circ$.
9. Let $S \subset E$.
   (a) Show that $S$ is closed if and only if $\partial S \subset S$.
   (b) Show that $S$ is open if and only if $\partial S \cap S = \emptyset$.
10. Let $A, B \subset E$.
    (a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
    (b) Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$.
    (c) For $E = \mathbb{R}$ with the usual metric, give examples of subsets $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ and $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.
11. Let $S \subset E$ be a connected set. Suppose $T \subset E$ satisfies $S \subset T \subset \overline{S}$. Show that $T$ is also connected.