Theorem 0.1. *Let $f : (E, d) \to (E', d')$ be a continuous map. If E is compact, then f is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. As f is continuous, for each $x \in E$ there exists $\delta_x > 0$ so that $d'(f(x), f(y)) < \frac{\varepsilon}{2}$ whenever $y \in B(x, \delta_x)$. Clearly

$$E = \bigcup_{x \in E} B(x, \tfrac{\delta_x}{2}).$$

As E is compact, there exist finitely many points $\{x_1, \ldots, x_J\} \subset E$ such that

$$E = \bigcup_{j=1}^{J} B(x_j, \tfrac{\delta_j}{2}).$$

Let $\delta = \min\{\frac{\delta_j}{2} : 1 \leq j \leq J\}$, which is well defined since there are only finitely many j's. If $x, y \in E$ satisfy $d(x, y) < \delta$, then x belongs to some $B(x_j, \frac{\delta_j}{2})$ and by the triangle inequality y belongs to the ball $B(x_j, \delta_j)$:

$$d(x_j, y) \leq d(x_j, x) + d(x, y) < \frac{\delta_j}{2} + \delta \leq \delta_j.$$

Then, using triangle inequality once more, we conclude that

$$d'(f(x), f(y)) \leq d'(f(x), f(x_j)) + d'(f(x_j), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Remark. The point of covering E with balls of radius $\frac{1}{2}\delta_x$ (instead of the balls $B(x, \delta_x)$ I initially tried and failed to use in lecture) is that x and y belong to *the same ball* $B(x_j, \delta_j)$, thus $f(x)$ and $f(y)$ can be compared using the triangle inequality by comparing both to $f(x_j)$; and by the definition of $B(x_j, \delta_j)$, each latter difference is less than $\frac{\varepsilon}{2}$.