Boundary Algebras and Kitaev's Quantum Double Model

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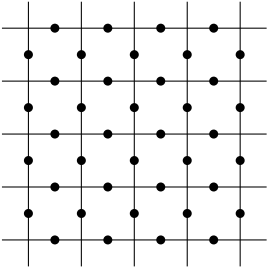
Spin systems!

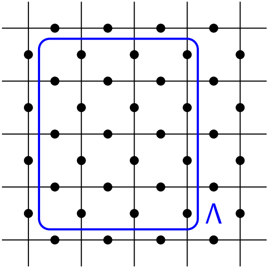


Consider infinite planar lattice, associate $\mathbb{C}^d$ to each vertex or edge

For each finite region Λ, have tensor product Hilbert space $\bigotimes_\Lambda \mathbb{C}^d$, algebra of operators $\mathfrak{A}(\Lambda) := \bigotimes_\Lambda M_d(\mathbb{C})$

Get UHF *quasilocal algebra* by taking inductive limit: $\mathfrak{A} := \bigotimes M_d(\mathbb{C})$

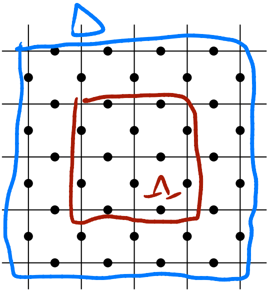
Nets of Projections



Definition

A *net of projections* is an assignment of a projection $p_\Lambda \in \mathfrak{A}(\Lambda)$ to each rectangle Λ satisfying that if $\Lambda \subseteq \Delta$, then $p_\Delta \leq p_\Lambda$.

What is Topological Order?



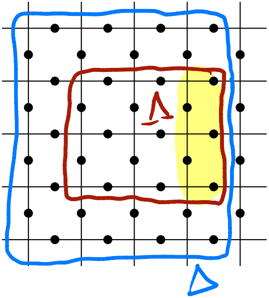
$\Lambda \ll \Delta$

If $\Lambda \ll \Delta$, then

$$p_\Delta \mathfrak{A}(\Lambda) p_\Delta = \mathbb{C} p_\Delta$$

(Bravyi-Hastings-Michalakis '10)

BUT CAN WE DO BETTER?



$\Lambda \Subset \Delta$

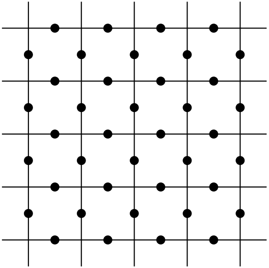
If $\Lambda \Subset \Delta$, then:

$p_\Delta \mathfrak{A}(\Lambda) p_\Delta = \mathfrak{B}(I) p_\Delta$, where
$I := \partial\Lambda \cap \partial\Delta$, and

for $x \in \mathfrak{B}(I)$ and $\Delta' \supset \Delta$, $x p_{\Delta'} = 0$ implies $x = 0$.

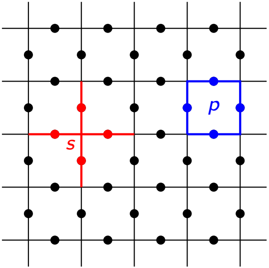
(Jones-Naaijkens-Penneys-W. '23)
Can take inductive limit of net of algebras $\mathfrak{B}(I)$ (boundary algebras) to get an AF C^*-algebra!

Example: Kitaev's Quantum Double Model

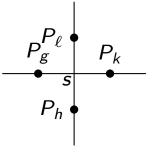


Let G be a finite group. We associate $\mathbb{C}^{|G|} = \text{span}\{|g\rangle | g \in G\}$ to each edge. We have left translation and right translation operators L_g and R_g acting on each $\mathbb{C}^{|G|}$ tensorand, as well as projections P_g onto $\text{span}\{|g\rangle\}$.

Details of the Model



We define A_s to be the operator

$$A_s := \sum_{gh=\ell k}$$ 

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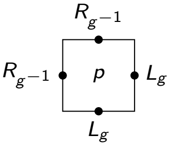
define $B_p^{(g)}$ to be the operator

$$B_p^{(g)} :=$$ 

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and define $B_p := \frac{1}{|G|} \sum_{g \in G} B_p^{(g)}$.

Local Hamiltonians and Ground State

Note that A_s and B_p are commuting projections!
For a rectangle Λ, we have a local Hamiltonian

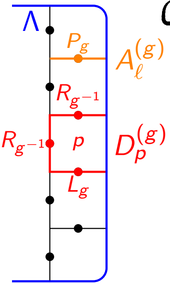
$$H_\Lambda := \sum_{s \subseteq \Lambda} (I - A_s) + \sum_{p \subseteq \Lambda} (I - B_p).$$

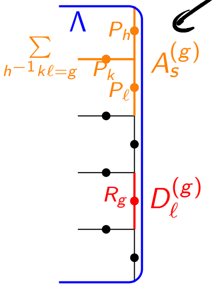
The projection onto the ground state space is

$$p_\Lambda := \prod_{s \subseteq \Lambda} A_s \prod_{p \subseteq \Lambda} B_p.$$

These projections satisfy the topological order axioms!
(Naaijkens '12)

What are the boundary algebras?





$n = |I|$

$$\mathfrak{B}(I) := \mathrm{C}^*\left\{A_\ell^{(g_1)}, D_p^{(g_2)}\right\} \cong \mathrm{End}_{\mathsf{Hilb}(G)}\left(\bigoplus_{g\in G}\mathbb{C}_g\right)^{\otimes n}$$

$$\mathfrak{B}(I) := \mathrm{C}^*\left\{A_s^{(g_1)}, D_\ell^{(g_2)}\right\} \cong \mathrm{End}_{\mathsf{Rep}(G)}\left(\bigoplus_{\sigma\in\mathsf{Irr}(\mathsf{Rep}(G))}\dim(\sigma)\,\sigma\right)^{\otimes n}$$

Can I get a less categorical description?

For rough case:

$$\mathfrak{B}(I) \cong \mathrm{End}_{\mathrm{Hilb}(G)}\left(\bigoplus_{g\in G}\mathbb{C}_g\right)^{\otimes n} \cong \bigoplus_{g\in G} M_{|G|^{n-1}}(\mathbb{C}).$$

For smooth case:

$$\mathfrak{B}(I) \cong \mathrm{End}_{\mathrm{Rep}(G)}\left(\bigoplus_{\sigma\in\mathrm{Irr}(\mathrm{Rep}(G))}(\dim\sigma)\sigma\right)^{\otimes n}$$

$$= \left\{T\colon \left(\mathbb{C}^{|G|}\right)^{\otimes n} \to \left(\mathbb{C}^{|G|}\right)^{\otimes n} \,\middle|\, \left[T, \bigotimes_{i=1}^{n} L_g\right] = 0\ \forall g\in G\right\}.$$

In both cases, inductive limit is $M_{|G|^\infty}$

~~These algebras are very different—inductive limits have different K-theory!~~ in case where G nonabelian

The canonical state

We get a canonical state $\psi\colon \mathfrak{A} \to \mathbb{C}$ given by

$$p_\Delta x p_\Delta = \psi(x) p_\Delta$$

for $x \in \mathfrak{A}(\Lambda)$ and $\Delta \gg \Lambda$. This is the unique translation-invariant ground state for the quantum double model (Naaijkens '12).
Can extend ψ to state on $\mathfrak{B} := \varinjlim \mathfrak{B}(I)$.
For both rough and smooth boundaries, ψ is a trace on $\mathfrak{B}$, so von Neumann completion is the hyperfinite II_1 factor!
Gives an independent proof of Ogata's 2022 result that cone algebras are type II for this model!