Schreier's Formula for some Free Probability Invariants
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Motivation

The Nielsen-Schreier Theorem

If F is a free group, then any $H \leq F$ is a free group. In particular, if $F \cong \mathbb{F}_n$ for $n \in \mathbb{N}$ and $[F : H] < \infty$, then $H \cong \mathbb{F}_k$ where

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One expects given a finite index inclusion $M_0 \subset M_1$, for any generating set S_0 for M_0 there exists a generating set S_1 for M_1 so that

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Theorem [Shlyakhtenko 2022]

For subfactors of the form $M_0 \subset M_1 = M_0 \rtimes G$ with G a finite abelian group, one has for a given $\varepsilon > 0$ the existence of a generating sets S_0 for M_0 and S_1 for M_1 for which

$$\delta^\star(S_1) - 1 \leq |G|^{-1}(\delta^\star(S_0) - 1) + \varepsilon.$$

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- Recently, due to MIP*=RE, it is possible to for the left most inequality to be strict inequality
- It is known that when M is embedded in the ultrapower of the hyperfinite II_1 factor, then we have
$$\dim \overline{\mathrm{Der}_c(A, \tau)} \leq \delta_0(X) \leq \delta^*(X) \leq \delta^\star(X) \leq \Delta(A, \tau),$$
where $A = \mathbb{C}\langle X \rangle$.

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- For $m \in (A \otimes A^\circ)''$ and $d \in$ $\mathrm{Der}(A, \tau)$, we have $d \cdot m \in$ Der (A, τ) defined as

$$[d \cdot m](a) = d(a)m,$$

and so $\mathrm{Der}(A, \tau)$ and $\mathrm{Der}(B \subset A, \tau)$ are right $(A \otimes A^\circ)''$-module

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Theorem [GG. 2023]

Let $G \overset{\alpha}{\curvearrowright} (M, \tau)$ be a trace-preserving action of a finite group G on a tracial von Neumann algebra and let $A \subset M$ be a finitely generated unital $*$-subalgebra which is globally invariant under α. Then we have the following $(A \otimes A^\circ)''$- isomorphism

$$\text{Der}(\mathbb{C}[G] \subset A \rtimes_\alpha^{\text{alg}} G, \tau) \to \bigoplus_{g \in G} (\text{Der}(A, \tau))_{1 \otimes \alpha_{g^{-1}}}.$$

Results cont.

Theorem [Charlesworth + Nelson 2022]

Let (M, τ) be a tracial von Neumann algebra with a finitely generated unital $*$-subalgebra $A \subset M$. For a finite dimensional unital $*$-subalgebra $B \subset A$, one has

$$\dim \mathrm{Der}(A, \tau)_{(A \otimes A^\circ)''} = \dim \mathrm{Der}(B, \tau)_{(B \otimes B^\circ)''} + \dim \mathrm{Der}(B \subset A, \tau)_{(A \otimes A^\circ)''}.$$

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In the case of the crossed product,

$$\dim \operatorname{Der}(A \rtimes_\alpha^{\text{alg}} G, \tau) = \dim \operatorname{Der}(\mathbb{C}[G], \tau) + \dim \operatorname{Der}(\mathbb{C}[G] \subset A \rtimes^{\text{alg}} G, \tau)_{((A \rtimes_\alpha^{\text{alg}} G) \otimes (A \rtimes_\alpha^{\text{alg}} G)^\circ)''}$$

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Corollary [GG. 2023]

$$\begin{aligned} \dim \mathrm{Der}(A \rtimes_\alpha^{\text{alg}} G, \tau)_{((A \rtimes_\alpha^{\text{alg}} G) \otimes (A \rtimes_\alpha^{\text{alg}} G)^\circ)''} - 1 \\ &= \frac{1}{[(A \rtimes_\alpha^{\text{alg}} G)'' : A'']} (\dim \mathrm{Der}(A, \tau)_{(A \otimes A^\circ)''} - 1) \\ &= \frac{1}{|G|} (\dim \mathrm{Der}(A, \tau)_{(A \otimes A^\circ)''} - 1) \end{aligned}$$

Results cont.

Theorem B [GG. 2023]

Let $G \overset{\alpha}{\curvearrowright} (M, \tau)$ be a trace-preserving action of a finite group G on a tracial von Neumann algebra and let $A \subset M$ be a finitely generated unital $*$-subalgebra which is globally invariant under α. Then

$$\dim \overline{\mathrm{Der}_c(\mathbb{C}[G] \subset A \rtimes_\alpha^{\mathrm{alg}} G, \tau)}_{((A \rtimes_\alpha^{\mathrm{alg}} G) \otimes (A \rtimes_\alpha^{\mathrm{alg}} G)^\circ)''} = \frac{1}{|G|} \dim \overline{\mathrm{Der}_c(A, \tau)}_{(A \otimes A^\circ)''}.$$

Furthermore, we have

$$\dim \overline{\mathrm{Der}_c(A \rtimes_\alpha^{\mathrm{alg}} G, \tau)}_{((A \rtimes_\alpha^{\mathrm{alg}} G) \otimes (A \rtimes_\alpha^{\mathrm{alg}} G)^\circ)''} - 1 = \frac{1}{|G|} (\dim \overline{\mathrm{Der}_c(A, \tau)}_{(A \otimes A^\circ)''} - 1).$$

Theorem C [GG. 2023]

Let $G \overset{\alpha}{\curvearrowright} (M, \tau)$ be a trace-preserving action of a finite abelian group G on a tracial von Neumann algebra and let $A \subset M$ be a finitely generated unital $*$-subalgebra which is globally invariant under α. Then

$$\Delta(\mathbb{C}[G] \subset A \rtimes_{\alpha}^{\text{alg}} G, \tau) = \frac{1}{|G|}\Delta(A, \tau).$$

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Corollary [GG. 2023]

Let $G \overset{\alpha}{\curvearrowright} (M, \tau)$ be a trace-preserving action of a finite abelian group G on a tracial von Neumann algebra and let $A \subset M$ be a finitely generated unital $*$-subalgebra which is globally invariant under α. If we have $\dim \mathrm{Der}_c(A, \tau)_{(A \otimes A^\circ)''} = \Delta(A, \tau)$ and A'' can be embedded in the ultrapower of the hyperfinite II_1 factor, then for any generating set Y of $A \rtimes_{\alpha}^{alg} G$, one has

$$\dim \overline{\mathrm{Der}_c(A \rtimes_{\alpha}^{\text{alg}}, \tau)}_{(A \rtimes_{\alpha}^{\text{alg}} G) \otimes (A \rtimes_{\alpha}^{\text{alg}} G)^\circ)''} = \delta_0(Y) = \delta^*(Y) = \delta^\star(Y) = \Delta(A \rtimes_{\alpha}^{\text{alg}} G, \tau).$$