Spectral triples on a non-standard presentation of Effros-Shen AF algebras

Samantha Brooker
Arizona State University
(joint work with Konrad Aguilar and Jack Spielberg)

GOALS Research Showcase 2023

August 17, 2023

Spectral triples

Definition 1

A *spectral triple* is a triple (A, H, D) where A is a unital C^*-algebra, H is a Hilbert space which is a left A-module (ie, we think of $A \subseteq B(H)$ by way of a *-representation), and D is an unbounded self-adjoint operator on H such that

(a) the set $A_0 := \{a \in A : [D, a]$ is densely defined and extends to a bounded operator on $H\}$ is norm-dense in A, and

(b) $(1 + D^2)^{-1}$ is a compact operator.

Spectral triples on unital AF algebras

Christensen and Ivan [2] construct a spectral triple on a unital AF algebra A, where A has a faithful state τ and an inductive sequence of finite-dimensional subalgebras

$$\mathbb{C}1_A =: A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

such that $A = \overline{\cup_{n=0}^{\infty} A_n}$.

Let H be the GNS Hilbert space of A given by τ, with cyclic vector $\xi \in H$. Since τ is faithful, we can regard $A \subseteq B(H)$.

Spectral triples on unital AF algebras

Now we have:

- $\eta : a \in A \mapsto a\xi \in H$ (injective since τ is faithful)
- orthogonal projection P_n of H onto $\eta(A_n) \subseteq H$, which is a closed subspace since A_n is finite-dimensional
- $Q_n = P_n - P_{n-1}$.

Let $(\alpha_n)_{n=0}^{\infty}$ with $\alpha_0 = 0$ and $\alpha_n \to \infty$, then define an unbounded operator on H by

$$D = \sum_{n=1}^{\infty} \alpha_n Q_n.$$

$D = \sum_{n=1}^{\infty} \alpha_n Q_n$

If $a \in A_n$ and $m \geq n$, then a commutes with P_m, hence a commutes with Q_m for any $m > n$. Thus

$$[D, a] = \sum_{i=1}^{n} \alpha_i [Q_i, a].$$

Hence for any a in the norm-dense subset $\cup_{n=0}^{\infty} A_n \subseteq A$, the commutator $[D, a]$ is defined and bounded on $\eta(\cup_{i=0}^{\infty} A_i)$, which dense in H, so $[D, a]$ extends to a bounded map on H.

Effros-Shen algebras

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. By Elliott's theorem, there is a unique simple unital AF algebra $\mathcal{AF}_\theta$ such that $K_0(\mathcal{AF}_\theta) = \mathbb{Z} + \theta\mathbb{Z}$, $K_0(\mathcal{AF}_\theta)_+ = (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, \infty)$, and $[1]_0 = 1$. We will now describe $\mathcal{AF}_\theta$.

$\mathcal{AF}_\theta$

Suppose $\theta \in \mathbb{R}$ has continued fraction expansion $[a_0, a_1, \ldots]$. That is, $a_0 \in \mathbb{Z}$ and $a_1, \ldots, \in \mathbb{N} \setminus \{0\}$, and

$$\theta = \lim_{n \to \infty} [a_0, a_1, \ldots, a_n]$$

where

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

The continued fraction expansion of θ is used to define unital embeddings of finite-dimensional matrix algebras $\mathcal{A}_n$, forming an inductive sequence $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow \cdots$, and the inductive limit

$$\mathcal{AF}_\theta = \varinjlim \mathcal{A}_n$$

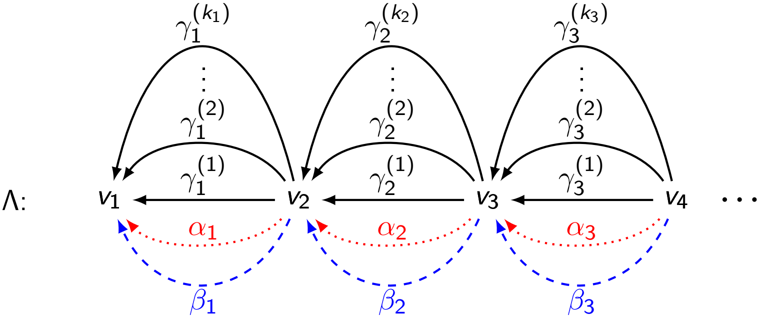
Another perspective

Mitscher and Spielberg [3] present an alternate view of $\mathcal{AF}_\theta$ as the C^*-algebra of a groupoid built from a *category of paths* Λ, which we can think of as the path space of a directed graph with some identifications.

The category of paths is used to define a groupoid, which yields a C^*-algebra via a standard construction (see, for instance, [4]). Mitscher and Spielberg compute the Elliott invariant of the resulting C^*-algebra to identify it as an Effros-Shen algebra.

A category of paths for $\mathcal{AF}_\theta$

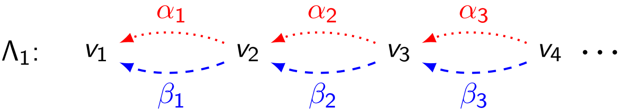
We describe the example Λ from [3]:



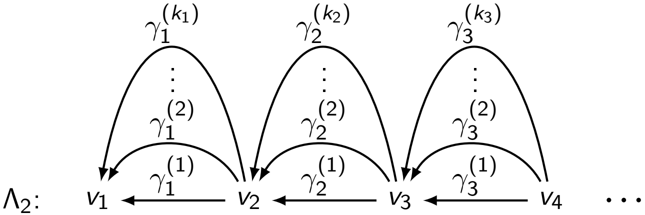
The category of paths Λ is an amalgamation of two categories of paths Λ_1 and Λ_2...

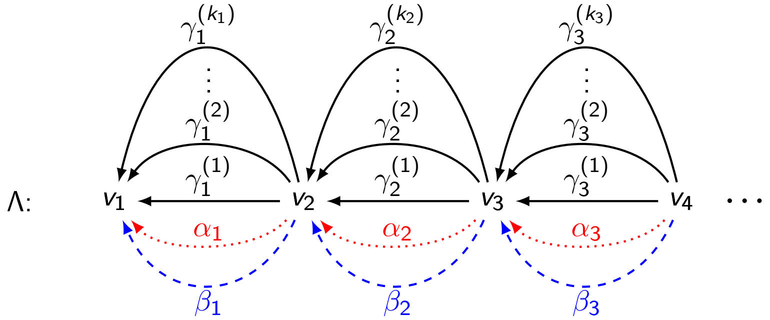
Λ_1 and Λ_2

Λ_1 consists of the red and blue edges, $\{\alpha_i, \beta_i : i = 1, 2, \dots\}$ which commute among themselves: $\alpha_i \beta_{i+1} = \beta_i \alpha_{i+1}$.



Λ_2 consists of the remaining edges $\{\gamma_i^{(1)}, \dots, \gamma_i^{(k_i)} : i = 1, 2, \dots\}$, where $(k_i)_{i=1}^{\infty}$ is a sequence of nonnegative integers, with infinitely many $k_i \neq 0$. Λ_2 has k_i edges from v_{i+1} to v_i:



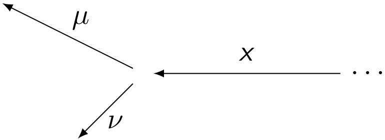


While the α's and β's can commute among themselves, the γ-edges act as "closed gates" through which the α's and β's cannot move. For instance,

$$\alpha_1\beta_2\alpha_3\cdot\gamma_4^{(1)}\cdot\beta_5\beta_6\alpha_7 = \alpha_1\alpha_2\beta_3\cdot\gamma_4^{(1)}\cdot\alpha_5\beta_6\beta_7$$
$$= v_1\alpha^2\beta\gamma_4^{(1)}\alpha\beta^2.$$

The groupoid $G(\Lambda)$

The category of paths Λ consists of the finite paths under the identifications just described. From there, we get the groupoid $G(\Lambda)$, whose elements can be thought of as *path-switchers* of *generalized infinite paths:*



- $\mu, \nu \in \Lambda$ are finite paths, $x \in \Lambda^{\infty}$ is a (generalized) infinite path
- the element $[\mu, \nu, x] \in G(\Lambda)$ has source $\nu x \in \Lambda^{\infty}$ and range $\mu x \in \Lambda^{\infty}$

$\mathcal{AF}_\theta$

Let $\sigma \in \mathbb{R} \setminus \mathbb{Q}$ have simple continued fraction expansion $\sigma = [c_0, c_1, \ldots]$. Mitscher and Spielberg define a sequence of integers $(k_i)_{i=1}^{\infty}$ as follows ([3] Theorem 5.12): Let k_1 be arbitrary, and for $p \geq 0$,

$$k_i = \begin{cases} 0 & : c_1 + c_3 + \cdots + c_{2p-1} + 2 < i < c_1 + c_3 + \cdots + c_{2p+1} \\ c_{2p} & : i = c_1 + c_3 + \cdots + c_{2p-1} + 2. \end{cases}$$

Visually we may represent this by

$$(k_i)_{i=1}^{\infty} = (k_1, c_0, \underbrace{0, \ldots, 0}_{c_1 - 1}, c_2, \underbrace{0, \ldots, 0}_{c_3 - 1}, c_4, \underbrace{0, \ldots, 0}_{c_5 - 1}, c_6, \ldots).$$

Let Λ_2 be determined by the sequence $(k_i)_{i=1}^{\infty}$, and let $G = G(\Lambda)$. Let $\theta = [0, 1, k_1, 1, k_2, 1, \ldots]$. Then $C^*(G)$ is isomorphic to the Effros-Shen algebra $\mathcal{AF}_\theta$ [3, Theorem 7.2].

Inductive limit structure

We consider a sequence of subgroupoids $(G_i)_{i=1}^{\infty}$ of G, where i determines the length of the switched part of a path-switcher: $G_i = \{[\mu, \nu, x] \in G : |\mu| = |\nu| \leq i\}$. Then [3] Theorem 5.1 states:

Theorem 2

$C^*(G)$ is the limit of the inductive sequence

$$C^*(G_1) \to C^*(G_2) \to \cdots$$

where the connecting maps are induced from the inclusion maps $C_c(G_i) \hookrightarrow C_c(G_{i+1})$.

We are interested in using this inductive limit structure to construct a spectral triple on $\mathcal{AF}_\theta$ in the style of [2].

Challenge

However, the subalgebras $C^*(G_i)$ are *not* finite-dimensional, which is why we cannot directly apply Christensen and Ivan's construction using this inductive sequence.

On the other hand, the category of paths description provides some concrete footholds that make this surprisingly tractable.

Why?

Aguilar and Latrémolière [1] endow $\mathcal{AF}_\theta$ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ with quantum metric structure by constructing *Lip-norms*.

Spectral triples are another way of (potentially) getting Lip-norms:

Given a spectral triple (A, H, D), the map $L : a \mapsto \|[D, a]\|$ is a densely defined seminorm on A, and we can consider the Monge-Kantorovich (pseudo)-metric on $\mathcal{S}(A)$:

$$d(\phi, \psi) := \sup\{|\phi(a) - \psi(a)| : a \in A, \|[D, a]\| \leq 1\}.$$

If this metrizes the weak*-topology on $\mathcal{S}(A)$ (and satisfies some other conditions), then L is a Lip-norm.

New quantum metrics?

If the seminorm L_θ we get from the spectral triple $(\mathcal{AF}_\theta, H, D)$ is in fact a Lip-norm, we will have a new (and, we think, different) way of viewing quantum metric structure on the Effros-Shen algebras compared to the work of Aguilar and Latrémolière.

Thank You!

[1] K. Aguilar and F. Latrémolière, "Quantum ultrametrics on AF algebras and the Gromov-Hausdorff propinquity," *Studia Math.*, vol. 231, no. 2, pp. 149–193, 2015, ISSN: 0039-3223,1730-6337. DOI: 10.4064/sm227-2-5. [Online]. Available: https://doi.org/10.4064/sm227-2-5.

[2] E. Christensen and C. Ivan, "Spectral triples for AF C^*-algebras and metrics on the Cantor set," *J. Operator Theory*, vol. 56, no. 1, pp. 17–46, 2006, ISSN: 0379-4024,1841-7744.

[3] I. Mitscher and J. Spielberg, "AF C^*-algebras from non-AF groupoids," *Trans. Amer. Math. Soc.*, vol. 375, no. 10, pp. 7323–7371, 2022, ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8723. [Online]. Available: https://doi.org/10.1090/tran/8723.

[4] I. Putnam, *Lecture notes on C^*-algebras*, Jan. 2019.