Von Neumann Algebras

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Chapter 1

Von Neumann Algebras

Lecture Preview: In the first leture, we will cover the Bicommutant Theorem (Theorem 1.2.6) in detail. To prepare for this, you should familiarize yourself with the strong and weak operator topologies (Definition 1.1.2), and the commutant (Definition 1.2.1). The second lecture will focus on the structure of group von Neumann algebras (Section 1.3.3).

1.1 Strong and Weak Operator Topologies

Let $\mathcal{H}$ be a Hilbert space. There is a natural (metrizable) topology on $B(\mathcal{H})$ given by the operator norm. Studying this topology amounts to studying C^*-algebras. To study von Neumann algebras, we will need to consider two new topologies on $B(\mathcal{H})$. There will be several others later on that are also important, but these first two will suffice to define a von Neumann algebra.

The formal definitions of these topologies are given below, but from an analytic perspective it is much more important to understand what it means for a net to converge in these topologies. Let $(x_i)_{i\in I} \subset B(\mathcal{H})$ be a net and let $x \in B(\mathcal{H})$. Then $(x_i)_{i\in I}$ converges to x in the *strong operator topology* (SOT) if

$$\lim_{i\to\infty} \|(x - x_i)\xi\| = 0 \qquad \forall \xi \in \mathcal{H},$$

and $(x_i)_{i\in I}$ converges to x in the *weak operator topology* (WOT) if

$$\lim_{i\to\infty} \langle (x - x_i)\xi, \eta\rangle = 0 \qquad \forall \xi, \eta \in \mathcal{H}.$$

Viewing $\mathcal{H}$ as a metric space under its norm, SOT convergence can be thought of as "pointwise convergence." Compare this to convergence under the operator norm, which should be thought of as "uniform convergence."

Remark 1.1.1. Strong operator topology convergence and weak operator topology convergence are often referred to in the literature as *strong convergence* and *weak convergence*, respectively, but in these notes we will typically avoid this terminology.

Definition 1.1.2. The **strong operator topology** (**SOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, \ldots, \xi_n; \epsilon) := \{y \in B(\mathcal{H})\colon \|(x - y)\xi_j\| < \epsilon,\ j = 1, \ldots, n\},$$

for $x \in B(\mathcal{H})$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$.

The **weak operator topology** (**WOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n; \epsilon) := \{y \in B(\mathcal{H})\colon |\langle (x - y)\xi_j, \eta_j\rangle| < \epsilon,\ j = 1, \ldots, n\},$$

for $x \in B(\mathcal{H})$, $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$, and $\epsilon > 0$.

Operator norm convergence implies SOT convergence, which in turn implies WOT convergence (Exercise 1.1.1), but the converses are not true. Here are some simple counter-examples:

Example 1.1.3. Let m be the Lebesgue measure on $\mathbb{R}$. For a measurable subset $S \subset \mathbb{R}$, let the characteristic function 1_S act on $B(L^2(\mathbb{R}, m))$ by pointwise multiplication. Then $(1_{[-n,n]})_{n\in\mathbb{N}}$ SOT-converges to the identity, but not in operator norm. Indeed, for any $f \in L^2(\mathbb{R}, m)$ and any ϵ, there exists $N \in \mathbb{N}$ so that

$$\left(\int_{\mathbb{R}\setminus[-N,N]} |f|^2 \, dm\right)^{1/2} < \epsilon.$$

Thus, for any $n \geq N$ we have

$$\left\|(1 - 1_{[-n,n]})f\right\|_2 = \left(\int_{\mathbb{R}\setminus[-n,n]} |f|^2 \, dm\right)^{1/2} < \epsilon.$$

Thus this sequence of operators SOT-converges to 1. However, $1 - 1_{[-n,n]} = 1_{[-n,n]^c}$ is a projection and so $\|1_{[-n,n]} - 1\| = 1$ for all n. ■

Example 1.1.4. Consider the following unitary operator on $\ell^2(\mathbb{Z})$:

$$(U\xi)(n) := \xi(n+1) \qquad \xi \in \ell^2.$$

For $n \in \mathbb{N}$, let $x_n := U^n$. Then we claim that $(x_n)_{n\in\mathbb{N}}$ WOT-converges to the zero operator but does not SOT-converge. Indeed, fix $\xi, \eta \in \ell^2(\mathbb{Z})$. Let $\epsilon > 0$, then there exists $N \in \mathbb{N}$ sufficiently large so that

$$\left(\sum_{n\geq N} |\xi(n)|^2\right)^{1/2} < \epsilon$$

$$\left(\sum_{n<-N} |\eta(n)|^2\right)^{1/2} < \epsilon$$

Then for $m \geq 2N$ we have

$$\begin{aligned}
|\langle x_m\xi, \eta\rangle| &\leq \sum_{n\in\mathbb{Z}} |\xi(n+m)||\eta(n)| \\
&= \sum_{n<-N} |\xi(n+m)||\eta(n)| + \sum_{n\geq m-N} |\xi(n)||\eta(n-m)| \\
&\leq \|\xi\|\epsilon + \epsilon\|\eta\|.
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ WOT-converges to zero. However, since U is a unitary,

$$\|x_n\xi\| = \|U^n\xi\| = \|\xi\| \qquad \forall \xi \in \ell^2(\mathbb{Z}),$$

thus $(x_n)_{n\in\mathbb{N}}$ does not SOT-converge to zero. ■

You will explore how these topologies interact with the $*$-algebra structure of $B(\mathcal{H})$ in the exercises, but let us summarize things here. First, addition and scalar multiplication are both continuous with respect to both the SOT and WOT (see Exercise 1.1.5). Taking adjoints is continuous with respect to the WOT but not the SOT (see Exercises 1.1.6 and 1.1.7), though it is SOT continuous on normal operators (see Exercise 1.1.8). Finally, multiplication is not continuous with respect to either the WOT or the SOT, but on bounded subsets it is SOT continuous (see Exercises 1.1.10 and 1.1.11).

We leave the proof of the next proposition as an exercise (see Exercise 1.1.12).

Proposition 1.1.5. *Let $\{p_i : i \in I\} \subset B(\mathcal{H})$ be a set of pairwise orthogonal projections. If $\mathscr{F}$ is the collection of finite subsets of I ordered by inclusion, then then net $\left(\sum_{i\in F} p_i\right)_{F\in\mathscr{F}}$ converges in the SOT to a projection which we denote by $\sum_{i\in I} p_i$.*

Exercises

1.1.1. Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in operator norm to some $x \in B(\mathcal{H})$, then it converges in the strong operator topology to x. Show that if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges in the strong operator topology to some $x \in B(\mathcal{H})$, then it converges in the weak operator topology to x.

1.1.2. Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the strong operator topology. Show that

$$\|x\| \leq \limsup_{i\to\infty} \|x_i\|.$$

1.1.3. Show that $(x_i)_{\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the strong operator topology if and only if $((x - x_i)^*(x - x_i))_{i\in I}$ converges to zero in the weak operator topology.

1.1.4. Let (X, Ω, μ) be a σ-finite measure space and let $f \in L^\infty(X, \mu)$. Show that a net $(f_i)_{i\in I} \subset L^\infty(X, \mu)$ converges to f in the WOT as pointwise multiplication operators in $B(L^2(X, \mu))$ if and only if the net converges to f as elements of the dual space $L^1(X, \mu)^*$.

1.1.5. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ be nets indexed by the same directed set and let $x, y \in B(\mathcal{H})$.

(a) Suppose $(x_i)_{i\in I}$ and $(y_i)_{\in I}$ converge to x and y, respectively, in the SOT. Show that for any $\alpha \in \mathbb{C}$, the net $(\alpha x_i + y_i)_{i\in I}$ converges to $\alpha x + y$ in the SOT.

(b) Prove the corresponding statement for the WOT.

1.1.6. If $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the WOT, show that $(x_i^*)_{i\in I}$ converges to x^* in the WOT.

1.1.7. Consider the shift operator S on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $((S^*)^n)_{n\in\mathbb{N}}$ converges to zero in the SOT, but $(S^n)_{n\in\mathbb{N}}$ does not.

1.1.8. [1] In this exercise you will show that taking adjoints of normal operators is continuous with respect to the strong operator topology.

(a) Show that $y \in B(\mathcal{H})$ is normal if and only if $\|y\xi\| = \|y^*\xi\|$ for all $\xi \in \mathcal{H}$.

(b) Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a net of normal operators converging to a normal operator $x \in B(\mathcal{H})$ in the strong operator topology. Show that $(x_i^*)_{i\in I}$ converges to x^* in the strong operator topology.

1.1.9. Consider the operator $S_n \in B(\ell^2(\mathbb{N}))$ defined by

$$S_n e_j := \begin{cases} e_{j+1} & \text{if } 1 \leq j \leq n-1 \\ e_1 & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine a formula for S_n^* and show that S_n is normal.

(b) Show that $(S_n)_{n\in\mathbb{N}}$ converges in the strong operator topology to the shift operator $S \in B(\ell^2(\mathbb{N}))$, but $(S_n^*)_{n\in\mathbb{N}}$ does not converge to S^*.

(c) Reconcile the previous part with Exercise 1.1.8.

1.1.10. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ be nets indexed by the same directed set that converge in the strong operator topology. Show that if $\sup_{i\in I} \|x_i\| < \infty$, then $(x_i y_i)_{i\in I}$ converges in the strong operator topology.

1.1.11. Find an example of bounded nets $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ converging to $x, y \in B(\mathcal{H})$, respectively, in the WOT but such that $(x_i y_i)_{i\in I}$ does not converge to xy in the WOT. [**Hint:** consider Example 1.1.4.]

1.1.12. Prove Proposition 1.1.5: For each $i \in I$ let $\mathcal{K}_i := p_i\mathcal{H}$ and define

$$\mathcal{K} := \overline{\operatorname{span}} \bigcup_{i\in I} \mathcal{K}_i.$$

Letting $p \in B(\mathcal{H})$ be the projection onto $\mathcal{K}$, show that the net $(\sum_{i\in F} p_i)_{F\in\mathscr{F}}$ converges in the SOT to p.

[1] Thanks to Lydia de Wolf for pointing an error in a previous version of this exercise.

1.2 Bicommutant Theorem

Definition 1.2.1. Let $\mathcal{H}$ be a Hilbert space. For $x, y \in B(\mathcal{H})$, the **commutator** of x and y is denoted

$$[x, y] := xy - yx.$$

For a subset $X \subset B(\mathcal{H})$, the **commutant** of X, denoted X', is the set

$$X' := \{y \in B(\mathcal{H})\colon [x, y] = 0 \ \forall x \in X\}.$$

The **double commutant** of X is the set

$$X'' := (X')'$$

If $X \subset Y \subset B(\mathcal{H})$ is an intermediate subset, we call $X' \cap Y$ the **relative commutant** of X in Y.

Observe that, regardless of the structure of X, X' is always a unital algebra. If X is closed under taking adjoints, then X' is a $*$-algebra. It also easily checked (algebraically) that:

$$\begin{aligned} X \subset X'' &= (X'')'' = \cdots \\ X' &= (X')'' = \cdots \end{aligned}$$

Note that inclusions are reversed under the commutant: $X \subset Y$ implies $Y' \subset X'$. Remarkably, the purely algebraic definition of the commutant has analytic implications. This culminates in The Bicommutant Theorem (Theorem 1.2.6).

Example 1.2.2. Let $\mathcal{H}$ be a Hilbert space. If $1 \in B(\mathcal{H})$ is the identity operator, then for any $\alpha \in \mathbb{C}$ and any $x \in B(\mathcal{H})$ one has $[x, \alpha 1] = 0$. Consequently, $\{\mathbb{C}1\}' = B(\mathcal{H})$.

Conversely, one also has $B(\mathcal{H})' = \mathbb{C}1$, which you will show in Exercise 1.2.1. As a special case of this, consider $\mathcal{H} = \mathbb{C}^n$ so that $B(\mathcal{H}) = M_n(\mathbb{C})$. To see that $M_n(\mathbb{C})' = \mathbb{C}1$, consider the matrices $E_{i,j} \in M_n(\mathbb{C})$ for $i, j = 1, \ldots, n$, where $E_{i,j}$ is the matrix with a one in the (i, j)-entry and zeros elsewhere. Note that $E_{i,j}E_{k,\ell} = \delta_{j=k}E_{i,\ell}$. Also, observe that that for any $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathbb{C})$,

$$E_{i,i}AE_{j,j} = A_{i,j}E_{i,j}$$

Thus if $A \in M_n(\mathbb{C})'$, then

$$A_{i,j}E_{i,j} = E_{i,i}AE_{j,j} = E_{i,i}E_{j,j}A = \delta_{i=j}E_{i,j}A.$$

This implies $A_{i,j} = 0$ unless $i = j$; that is, A is diagonal. We also have for any $i, j = 1, \ldots, n$

$$A_{i,i}E_{i,i} = E_{i,i}AE_{i,i} = E_{i,j}E_{j,i}AE_{i,i} = E_{i,j}AE_{j,i} = E_{i,j}E_{j,j}AE_{j,j}E_{j,i} = A_{j,j}E_{i,j}E_{j,j}E_{j,i} = A_{j,j}E_{i,i}.$$

So all the diagonal entries of A agree and so $A = A_{1,1}1 \in \mathbb{C}1$. ■

Example 1.2.3. For (X, Ω, μ) a σ-finite measure space, view $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ where $f \in L^\infty(X, \mu)$ acts by pointwise multiplication. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$, which you will show in Exercise 1.2.3. As a special case of this, consider $\mathbb{N}$ equipped with the counting measure. For $n \in \mathbb{N}$, let $e_n \in \ell^2(\mathbb{N})$ be the function defined by $e_n(k) = \delta_{n=k}$. Note that $e_n \in \ell^\infty(\mathbb{N})$ as well, and that for $f \in \ell^2(\mathbb{N})$ one has

$$[e_n f](k) = e_n(k)f(k) = \delta_{n=k}f(n) = [f(n)e_n](k),$$

that is: $e_n f = f(n)e_n$. Now, if $T \in \ell^\infty(\mathbb{N})'$ and $f \in \ell^2(\mathbb{N})$ we have

$$[T(f)](n) = e_n(n)[T(f)](n) = [e_nT(f)](n) = [T(e_nf)](n) = f(n)[T(e_n)](n).$$

So if we define $g\colon \mathbb{N} \to \mathbb{C}$ by $g(n) := [T(e_n)](n)$, then $T(f) = gf$. Also note that

$$|g(n)| = |[T(e_n)](n)| \leq \|T(e_n)\|_2 \leq \|T\|\|e_n\|_2 \leq \|T\|.$$

Thus $g \in \ell^\infty(\mathbb{N})$. ■

Definition 1.2.4. Let $\mathcal{K} \subset \mathcal{H}$ be a subspace. For $x \in B(\mathcal{H})$, we say $\mathcal{K}$ is **invariant** for x if $x\mathcal{K} \subset \mathcal{K}$. We say $\mathcal{K}$ is **reducing** for x if it is invariant for x and x^*. For a subset $X \subset B(\mathcal{H})$, we say $\mathcal{K}$ is **invariant** (resp. **reducing**) for X if it is invariant (resp. reducing) for all $x \in X$.

Note that if X is closed under taking adjoints, then a subspace is invariant for X if and only if it is reducing for X.

Lemma 1.2.5. *Let $M \subset B(\mathcal{H})$ be a $*$-subalgebra. Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace with $p \in B(\mathcal{H})$ the projection onto $\mathcal{K}$. Then $\mathcal{K}$ is reducing for M if and only if $p \in M'$.*

Proof. Assume $\mathcal{K}$ is reducing M. Let $x \in M$ and $\xi \in \mathcal{K}$. Then $x\xi \in \mathcal{K}$ so that

$$xp\xi = x\xi = px\xi.$$

If $\eta \in \mathcal{K}^\perp$, we have

$$\langle x\eta, \xi \rangle = \langle \eta, x^*\xi \rangle = 0,$$

since $x^*\xi \in \mathcal{K}$. Thus $x\eta \in \mathcal{K}^\perp$ and so $xp\eta = 0 = px\eta$. It follows that $xp = px$ so that $p \in M'$.

Conversely, suppose $p \in M'$. Let $x \in M$ and $\xi \in \mathcal{K}$. Then for $\eta \in \mathcal{K}^\perp$ we have

$$0 = \langle x\xi, p\eta \rangle = \langle px\xi, \eta \rangle = \langle xp\xi, \eta \rangle = \langle x\xi, \eta \rangle .$$

Thus $x\xi \in (\mathcal{K}^\perp)^\perp = \mathcal{K}$. Hence $M\mathcal{K} \subset \mathcal{K}$ so that $\mathcal{K}$ is reducing for M. $\square$

We have the following theorem due to von Neumann from 1929.

Theorem 1.2.6 (The Bicommutant Theorem)**.** *For a unital $*$-subalgebra $M \subset B(\mathcal{H})$, one has*

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''$$

Proof. We will show the following series of inclusions:

$$\overline{M}^{SOT} \subset \overline{M}^{WOT} \subset M'' \subset \overline{M}^{SOT}.$$

The first inclusion follows the fact that SOT-convergence implies WOT-convergence.

Now, suppose $x \in \overline{M}^{WOT}$, say with a net $(x_i)_{i\in I} \subset M$ converging to x in the WOT. Let $y \in M'$, then for any $\xi, \eta \in \mathcal{H}$ we have

$$\langle xy\xi, \eta \rangle = \lim_{i\to\infty} \langle x_i y\xi, \eta \rangle = \lim_{i\to\infty} \langle y x_i \xi, \eta \rangle = \langle yx\xi, \eta \rangle .$$

Since $\xi, \eta \in \mathcal{H}$ were arbitrary, we have $xy = yx$ and thus $x \in M''$.

Finally, suppose $x \in M''$. Note that to show $x \in \overline{M}^{SOT}$, it suffices to show for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$ that there exists $y \in M$ with

$$\|(x-y)\xi_j\| < \epsilon \qquad j = 1, \ldots, n.$$

Fix $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, and $\epsilon > 0$. For $y \in M$, define $\pi(y) \in B(\mathcal{H}^{\oplus n})$ by

$$\pi(y)(\eta_1, \ldots, \eta_n) := (y\eta_1, \ldots, y\eta_n).$$

If you view $\mathcal{H}^{\oplus n}$ as column vectors over $\mathcal{H}$ of height n, then $\pi(y)$ corresponds to the matrix

$$\begin{pmatrix} y & & 0 \\ & \ddots & \\ 0 & & y \end{pmatrix}.$$

With this perspective, one can show that $\pi(M)'$ consists of matrices of the form

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

where $a_{i,j} \in M'$ for $i,j = 1,\ldots,n$, and that $\pi(x)$ commutes with all such matrices since $x \in M''$ (see Exercise 1.2.6).

Now, let $\mathcal{S}$ denote the closure of the subspace $\{\pi(y)(\xi_1,\ldots,\xi_n) : y \in M\} \subset \mathcal{H}^{\oplus n}$. Then $\mathcal{S}$ is reducing for $\pi(M)$, and so if $p \in B(\mathcal{H}^{\oplus n})$ is the projection onto $\mathcal{S}$, then Lemma 1.2.5 implies $p \in \pi(M)'$ and so $p\pi(x) = \pi(x)p$. Note that $1 \in M$ implies $(\xi_1,\ldots,\xi_n) \in \mathcal{S}$. Thus we have

$$\pi(x)(\xi_1,\ldots,\xi_n) = \pi(x)p(\xi_1,\ldots,\xi_n) = p\pi(x)(\xi_1,\ldots,\xi_n) \in \mathcal{S}.$$

The definition of $\mathcal{S}$ then implies there exists $y \in M$ with

$$\|\pi(x)(\xi_1,\ldots,\xi_n) - \pi(y)(\xi_1,\ldots,\xi_n)\| < \epsilon.$$

Unpacking our notation, we see that

$$\|\pi(x)(\xi_1,\ldots,\xi_n) - \pi(y)(\xi_1,\ldots,\xi_n)\| = \|((x-y)\xi_1,\ldots,(x-y)\xi_n)\| = \left(\sum_{j=1}^n \|(x-y)\xi_j\|^2\right)^{1/2}.$$

Combining this with the previous inequality yields $\|(x-y)\xi_j\| < \epsilon$ for each $j = 1,\ldots,n$. □

The double commutant is given by a purely algebraic definition, whereas the SOT and WOT closures are purely analytic. Their equality in the above theorem tells us the following objects lie in the confluence of algebra and analysis:

Definition 1.2.7. We say a unital $*$-subalgebra $1 \in M \subset B(\mathcal{H})$ is a **von Neumann algebra** if $M = M''$ (equivalently, $M = \overline{M}^{SOT}$ or $M = \overline{M}^{WOT}$).

Recall from Example 1.2.2 that $B(\mathcal{H})' = \mathbb{C}1$ and that $\mathbb{C}1' = B(\mathcal{H})$. Hence $B(\mathcal{H})'' = B(\mathcal{H})$ and $\mathbb{C}1'' = \mathbb{C}1$ are examples of von Neumann algebras. Another example is $L^\infty(X,\mu)$ for a σ-finite measure space (X,Ω,μ), since

$$L^\infty(X,\mu)'' = L^\infty(X,\mu)' = L^\infty(X,\mu)$$

by Example 1.2.3. We will explore these and other examples in greater detail in the next section, but first we must define a few related concepts.

From the observation following Definition 1.2.1, we see that for M a von Neumann algebra, M' is also a von Neumann algebra. Consequently, so is $M \cap M'$ we which we give a name to here:

Definition 1.2.8. For M a von Neumann algebra, the **center of** M, denoted $\mathcal{Z}(M)$, is the von Neumann subalgebra $M \cap M'$. If $\mathcal{Z}(M) = \mathbb{C}1$, we say M is a **factor**. If $\mathcal{Z}(M) = M$, we say M is **abelian**.

For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a factor by Example 1.2.2, while for a σ-finite measure space (X,Ω,μ), $L^\infty(X,\mu)$ is abelian. There are examples where $\mathbb{C}1 \subsetneq \mathcal{Z}(M) \subsetneq M$, so factors and abelian von Neumann algebras only represent the two extremes on how much commutativity a von Neumann algebra permits.

We conclude by presenting a notion of what it means for two von Neumann algebras to be isomorphic.

Definition 1.2.9. We say two von Neumann algebras $M_1 \subset B(\mathcal{H}_1)$ and $M_2 \subset B(\mathcal{H}_2)$ are **spatially isomorphic** if there exists a unitary operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $UM_1U^* = M_2$. In this case we call $M_1 \ni x \mapsto UxU^* \in M_2$ a **spatial isomorphism**.

Exercises

1.2.1. Let $\mathcal{H}$ be a Hilbert space. Given $\xi,\eta \in \mathcal{H}$, recall that the rank one operator $\xi \otimes \bar{\eta} \in B(\mathcal{H})$ is defined by

$$(\xi \otimes \bar{\eta})(\zeta) := \langle \zeta, \eta \rangle \, \xi.$$

(a) Show that $x \in B(\mathcal{H})$ commutes with $\xi \otimes \bar{\eta}$ if and only if there exists $\lambda \in \mathbb{C}$ with $\xi \in \ker(x-\lambda)$ and $\eta \in \ker(x^* - \bar{\lambda})$.

(b) Show that $FR(\mathcal{H})' = \mathbb{C}$ and that $B(\mathcal{H})' = \mathbb{C}$.

1.2.2. For (X, Ω, μ) a σ-finite measure space and $f \in L^\infty(X, \mu)$, show that

$$L^2(X, \mu) \ni g \mapsto fg$$

defines a bounded linear operator on $L^2(X, \mu)$ with norm equal to $\|f\|_\infty$.
[**Hint:** for $\epsilon > 0$ consider $\{x \in X : |f(x)| \geq \|f\|_\infty - \epsilon\}$.]

1.2.3. For (X, Ω, μ) a σ-finite measure space, view $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ where $f \in L^\infty(X, \mu)$ acts by pointwise multiplication. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$. [**Hint:** first consider the case when μ is finite.]

1.2.4. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K} \subset \mathcal{H}$ a closed subspace, and $p \in B(\mathcal{H})$ the projection onto $\mathcal{K}$.

(a) Show that $\mathcal{K}$ is invariant for $x \in B(\mathcal{H})$ if and only if $pxp = xp$.

(b) Show that $\mathcal{K}$ is reducing for $x \in B(\mathcal{H})$ if and only if $xp = px$.

1.2.5. Let $\mathcal{H}$ be a Hilbert space and fix $n \in \mathbb{N}$. For all $T \in B(\mathcal{H}^{\oplus n})$, show that there exist $T_{i,j} \in B(\mathcal{H})$ for $i, j = 1, \ldots, n$ such that

$$T(\xi_1, \ldots, \xi_n) = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \cdots & T_{n,n} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

(In the above we are not distinguishing between row and column vectors.) Thus $B(\mathcal{H}^{\oplus n})$ can be identified with $n \times n$ matrices with entries in $B(\mathcal{H})$.

1.2.6. For $x \in B(\mathcal{H})$ and $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathbb{C})$, define $x \otimes A \in B(\mathcal{H}^{\oplus n})$ by

$$x \otimes A := \begin{pmatrix} A_{1,1}x & \cdots & A_{1,n}x \\ \vdots & \ddots & \vdots \\ A_{n,1}x & \cdots & A_{n,n}x \end{pmatrix}.$$

Let $X \subset B(\mathcal{H})$, and for each $i, j = 1, \ldots, n$ let $E_{i,j} \in M_n(\mathbb{C})$ be the matrix with a one in the (i, j)-entry and zeros elsewhere.

(a) Show that

$$\{x \otimes I_n : x \in X\}' = \left\{ \sum_{i,j=1}^n y_{i,j} \otimes E_{i,j} : y_{i,j} \in X' \ i, j = 1, \ldots, n \right\}.$$

(b) Show that

$$\left\{ \sum_{i,j=1}^n y_{i,j} \otimes E_{i,j} : y_{i,j} \in X' \ i, j = 1, \ldots, n \right\}' = \{x \in \otimes I_n : x \in X''\}.$$

1.2.7. Let $\pi \colon M_1 \to M_2$ be a spatial isomorphism.

(a) Show that π is an isometric $*$-isomorphism.

(b) Show that π is SOT and WOT continuous.

(c) Show that M_1' is spatially isomorphic to M_2'.

1.2.8. Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces, and for each $j = 1, \ldots, n$ define $\pi_j \colon B(\mathcal{H}_j) \to B(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$ by

$$\pi_j(x)(\xi_1, \ldots, \xi_n) = (0, \ldots, 0, x\xi_j, 0, \ldots, 0) \qquad (\xi_1, \ldots, \xi_n) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n.$$

(You can also think of $\pi_j(x)$ as an $n \times n$ matrix with x in the (j, j)-entry and zeros elsewhere).

(a) Show that π_j is an isometric $*$-homomorphism for each $j = 1, \ldots, n$.

(b) Let $M_j \subset B(\mathcal{H}_j)$ be a von Neumann algebra for each $j = 1, \ldots, n$. Show that

$$M_1 \oplus \cdots \oplus M_n = \left\{ \sum_{j=1}^{n} \pi_j(x_j) \colon x_j \in M_j,\ j = 1, \ldots, n \right\}$$

is a von Neumann algebra. (It is called the **direct sum** of $M_1, \ldots, M_n$.)

(c) Show that $\mathcal{Z}(M_1 \oplus \cdots \oplus M_n) = \mathcal{Z}(M_1) \oplus \cdots \oplus \mathcal{Z}(M_n)$.

(d) Show that $M_1 \oplus \cdots \oplus M_n$ is **not** a factor for $n \geq 2$.

1.2.9. Show that a von Neumann algebra M is abelian if and only if $M \subset M'$.

1.2.10. An abelian von Neumann algebra $A \subset B(\mathcal{H})$ is called **maximal abelian** if $A \subset B \subset B(\mathcal{H})$ for another abelian von Neumann algebra B implies $A = B$. Show that an abelian von Neumann algebra A is maximal abelian if and only if $A' = A$.

1.3 First Examples

1.3.1 $B(\mathcal{H})$ and Matrix Algebras

For any Hilbert space $\mathcal{H}$, we saw above that $B(\mathcal{H})$ is always a von Neumann algebra and a factor. In particular, if $\mathcal{H}$ is finite dimensional with $d := \dim(\mathcal{H})$, then $B(\mathcal{H})$ is simply the matrix algebra $M_d(\mathbb{C})$. Though an elementary example, $M_d(\mathbb{C})$ will eventually inform a great deal of our intuition about von Neumann algebras. We highlight a few important features below.

As factors, matrix algebras are as noncommutative as a von Neumann algebra can be. They also contain a lot of projections. For each pair $i, j = 1, \ldots, d$ let $E_{i,j} \in M_d(\mathbb{C})$ be the matrix with a one in the (i, j)-entry and zeros elsewhere. Then $E_{i,i}$ is projection for each $i = 1, \ldots, n$ and so is any sum of these matrices (see also Exercise 1.3.1).

Recall that the *unnormalized trace* on $M_d(\mathbb{C})$ is a linear functional $\mathrm{Tr} \colon M_d(\mathbb{C}) \to \mathbb{C}$ defined as

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} A_{i,i}.$$

The trace is invariant under cyclic permutation: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in M_d(\mathbb{C})$. In fact, up to a scalar, it is the unique linear functional on $M_d(\mathbb{C})$ with this property (see Exercise 1.3.2). Note that if $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{C}^d$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{d} \langle Ae_i, e_i \rangle .$$

In fact, the standard basis in the above formula can be replaced with *any* orthonormal basis $\{f_1, \ldots, f_d\}$ for $\mathbb{C}^d$. This is because if U is the unitary matrix whose columns are $f_1, \ldots, f_d$, then $Ue_i = f_i$ for each $i = 1, \ldots, d$. Consequently

$$\sum_{i=1}^{d} \langle Af_i, f_i \rangle = \sum_{i=1}^{d} \langle AUe_i, Ue_i \rangle = \sum_{i=1}^{d} \langle U^* AUe_i, e_i \rangle = \mathrm{Tr}(U^* AU) = \mathrm{Tr}(AUU^*) = \mathrm{Tr}(A).$$

One can even define a trace for $B(\mathcal{H})$ when $\mathcal{H}$ is infinite dimensional, but it will only be well-defined on the trace-class operators, which we revisit in Section 3.1.

1.3.2 Measure Spaces

For (X,Ω,μ) a σ-finite measure space, we saw above that $L^\infty(X,\mu) \subset B(L^2(X,\mu))$ is an abelian von Neumann algebra. In fact, it is *maximal abelian* in the sense that if $L^\infty(X,\mu) \subset A \subset B(\mathcal{H})$ for an abelian von Neumann algebra A, then $A = L^\infty(X,\mu)$ (see Exercises 1.2.3 and 1.2.10). As with matrix algebras, $L^\infty(X,\mu)$ will also eventually inform a great deal of our intuition. Indeed, it turns out that *all* abelian von Neumann algebras are of this form and we will see a partial proof of this in Section 2.2.

Despite the fact that $L^\infty(X,\mu)$ and $M_d(\mathbb{C})$ are radically different in terms of commutativity, there are still important similarities. $L^\infty(X,\mu)$ also has an abundence of projections. Indeed, for any measurable $E \subset X$, $1_E \in L^\infty(X,\mu)$ is a projection. In fact any projection in $L^\infty(X,\mu)$ is of this form (see Exercise 1.3.3). Consequently, the linear span of projections is exactly the set of μ-measurable simple functions, which we know from measure theory are $\|\cdot\|_\infty$ norm dense in $L^\infty(X,\mu)$. Using Exercise 1.2.2, we can then deduce that the linear span of projections is actually operator norm dense in $L^\infty(X,\mu)$. Additionally, when μ is a finite measure, $L^\infty(X,\mu) \subset L^1(X,\mu)$ and so

$$L^\infty(X,\mu) \ni f \mapsto \int_X f\ d\mu$$

is a natural linear functional on this von Neumann algebra, similar to the trace on $M_d(\mathbb{C})$.

1.3.3 Group von Neumann Algebras

Let Γ be a countable discrete group, which we can use to define a Hilbert space $\ell^2(\Gamma)$. Consider the left regular representation $\lambda\colon \Gamma \to B(\mathcal{H})$:

$$[\lambda(g)\xi](h) = \xi(g^{-1}h) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma$$

Equivalently, if for $g \in \Gamma$ we let $\delta_g \in \ell^2(\Gamma)$ be the function $\delta_g(h) = \delta_{g=h}$, then $\lambda(g)\delta_h = \delta_{gh}$ for all $h \in \Gamma$. The operators $\lambda(g)$ are in fact unitary operators with $\lambda(g)^* = \lambda(g^{-1})$, and in particular if $e \in \Gamma$ is the identity then $\lambda(e) = 1$. Denote $\mathbb{C}[\lambda(\Gamma)] := \text{span}\lambda(\Gamma)$, which we note is a unital $*$-subalgebra of $B(\ell^2(\Gamma))$.

Definition 1.3.1. The **group von Neumann algebra** for Γ is $L(\Gamma) := \mathbb{C}[\lambda(\Gamma)]''$.

These von Neumann algebras can be abelian, factors, or something in between. If Γ is an abelian group, then $\mathbb{C}[\lambda(\Gamma)]$ and consequently $L(\Gamma)$ are abelian. To understand when $L(\Gamma)$ is a factor, we require a definition:

Definition 1.3.2. We say that Γ is an **infinite conjugacy class** (**i.c.c.**) group if the conjugacy class $\{h^{-1}gh\colon h \in h \in \Gamma\}$ is infinite for all $g \in \Gamma \setminus \{e\}$.

Example 1.3.3.

(1) For $n \in \mathbb{N}$, the free group with n generators, $\mathbb{F}_n = \langle a_1,\ldots,a_n\rangle$, is an i.c.c. group.

(2) Let S_∞ denote the group of bijections $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $\pi(n) = n$ for all but finitely many $n \in \mathbb{N}$. This group can be viewed as the union of all permutation groups S_n, $n \in \mathbb{N}$, where $S_n \hookrightarrow S_{n+1}$ by fixing $n+1$. Then S_∞ is an i.c.c. group.

(3) Any finite or abelian group is not an i.c.c. group. ■

You will show in Exercise 1.3.7 that $L(\Gamma)$ is a factor if and only if Γ is an i.c.c. group.

As with our previous examples, $L(\Gamma)$ admits a natural linear functional $\tau\colon L(\Gamma) \to \mathbb{C}$ defined by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle\,.$$

Since $\tau(\lambda(g)) = \delta_{g=e}$, τ encodes the group relations; that is, $g_1g_2\cdots g_n = e$ for $g_1,\ldots,g_n \in \Gamma$ if and only if $\tau(\lambda(g_1)\cdots\lambda(g_n)) = 1$. Also, like the trace on $M_d(\mathbb{C})$, τ is invariant under cyclic permutations: $\tau(xy) = \tau(yx)$ for all $x,y \in L(\Gamma)$ (see Exercise 1.3.8). Because if this we call τ the **trace** on $L(\Gamma)$.

In constructing the group von Neumann algebra, one could instead use the right regular representation:

$$[\rho(g)\xi](h) = \xi(hg) \qquad \xi \in \ell^2(\Gamma),\ h \in \Gamma,$$

in which case one denotes by $R(\Gamma) := \mathbb{C}[\rho(\Gamma)]''$. There is a very natural relationship between $L(\Gamma)$ and $R(\Gamma)$ (see Theorem 1.3.7), but in order to witness it we require some additional terminology. Recall that for $\xi, \eta \in \ell^2(\Gamma)$ their convolution is defined by

$$(\xi * \eta)(g) = \sum_{h \in \Gamma} \xi(h)\eta(h^{-1}g).$$

From the Cauchy–Scwarz inequality, we have $|(\xi * \eta)(g)| \leq \|\xi\|_2 \|\eta\|_2$ for all $g \in \Gamma$. So $\xi * \eta \in \ell^\infty(G)$ with $\|\xi * \eta\|_\infty \leq \|\xi\|_2 \|\eta\|_2$.

Definition 1.3.4. We say $\xi \in \ell^2(\Gamma)$ is a **left** (resp. right) **convolver** if $\xi * \eta \in \ell^2(\Gamma)$ (resp. $\eta * \xi \in \ell^2(\Gamma)$) for all $\eta \in \ell^2(\Gamma)$. Denote the linear operator $\eta \mapsto \xi * \eta$ (resp. $\eta \mapsto \eta * \xi$) by $\lambda(\xi)$ (resp. $\rho(\xi)$). Denote $LC(\Gamma) := \{\lambda(\xi) \colon \xi \text{ is a left convolver}\}$ and $RC(\Gamma) := \{\rho(\xi) \colon \xi \text{ is a right convolver}\}$.

Observe that $\lambda(\delta_g) = \lambda(g)$ and $\rho(\delta_g) = \rho(g)$. We claim that $\lambda(\xi)$ is bounded for any left convolver ξ. By the Closed Graph Theorem, it suffices to show that if $(\eta_n)_{n \in \mathbb{N}} \subset \ell^2(\Gamma)$ satisfies $\eta_n \to 0$ and $\lambda(\xi)(\eta_n) \to \zeta$, then $\zeta = 0$. Since the $\|\cdot\|_2$ norm dominates the $\|\cdot\|_\infty$ norm, we see

$$\|\zeta\|_\infty = \lim_{n \to \infty} \|\lambda(\xi)(\eta_n)\|_\infty = \lim_{n \to \infty} \|\xi * \eta_n\|_\infty \leq \limsup_{n \to \infty} \|\xi\|_2 \|\eta_n\|_2 = 0.$$

Thus $\zeta = 0$ and so $\lambda(\xi)$ is bounded. A similar argument shows that $\rho(\xi)$ is bounded for any right convolver. Hence $LC(\Gamma), RC(\Gamma) \subset B(\ell^2(\Gamma))$.

Lemma 1.3.5. *$\xi \in \ell^2(\Gamma)$ is left (resp. right) convolver if and only if there exists $c > 0$ so that $\|\xi * \kappa\|_2 \leq c\|\kappa\|_2$ (resp. $\|\kappa * \xi\|_2 \leq c\|\kappa\|_2$) for all finitely supported $\kappa \in \ell^2(\Gamma)$.*

Proof. We will consider only left convolvers, since the proof for right convolvers is similar. The "only if" direction follows from the discussion preceding the lemma, where $c = \|\lambda(\xi)\|$.

Conversely, define for finitely supported $\kappa \in \ell^2(\Gamma)$ define $x\kappa := \xi * \kappa$. The hypothesis implies that x can be extended to a bounded operator on $\ell^2(\Gamma)$, which we also denote by x. Fix $\eta \in \ell^2(\Gamma)$. Given $\epsilon > 0$ there is a finite subset $F \subset \Gamma$ satisfying

$$\sum_{g \in \Gamma \setminus F} |\eta(g)|^2 < \epsilon^2.$$

In other words, if $\kappa := \eta 1_F$, then $\|\eta - \kappa\|_2 < \epsilon$. Since κ is finitely supported, we have $x\kappa = \xi * \kappa$ and so we estimate

$$\begin{aligned}
\|\xi * \eta - x\eta\|_\infty &\leq \|\xi * \eta - x\kappa\|_\infty + \|x(\kappa - \eta)\|_\infty \\
&\leq \|\xi * (\eta - \kappa)\|_\infty + \|x(\kappa - \eta)\|_2 \\
&\leq \|\xi\|_2 \|\eta - \kappa\|_2 + \|x\| \|\kappa - \xi\|_2 < (\|\xi\|_2 + \|x\|)\epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we have $\xi * \eta = x\eta \in \ell^2(\Gamma)$. Hence ξ is a left convolver. $\square$

Proposition 1.3.6. *$LC(\Gamma)$ and $RC(\Gamma)$ are von Neumann algebras.*

Proof. We will only consider $LC(\Gamma)$, the proof for $RC(\Gamma)$ being similar. We also leave checking that $LC(\Gamma)$ is a unital $*$-algebra as an exercise (see Exercise 1.3.9). By the Bicommutant Theorem, it suffices to show $LC(\Gamma)$ is SOT closed. Let $(\xi_i)_{i \in I} \subset \ell^2(\Gamma)$ be a net of left convolvers such that $(\lambda(\xi_i))_{i \in I}$ converges to some $x \in B(\ell^2(\Gamma))$ in the SOT. Observe that $\lambda(\xi_i)\delta_e = \xi_i$, so if we set $\xi := x\delta_e$ then $\xi_i = \lambda(\xi_i)\delta_e \to x\delta_e = \xi$. Using this and the SOT convergence of $(\lambda(\xi_i))_{i \in I}$ to x, we have for any $\eta \in \ell^2(\Gamma)$ that

$$\|\xi * \eta - x\eta\|_\infty \leq \|\xi * \eta - \xi_i * \eta\|_\infty + \|\xi_i * \eta - x\eta\|_\infty \leq \|\xi - \xi_i\|_2 \|\eta\|_2 + \|(\lambda(\xi_i) - x)\eta\|_2 \to 0.$$

Thus $\xi * \eta = x\eta \in \ell^2(\Gamma)$, which implies ξ is a left convolver and that $\lambda(\xi) = x$. Hence $x \in LC(\Gamma)$ and $LC(\Gamma)$ is SOT closed. $\square$

Theorem 1.3.7. *$R(\Gamma) = L(\Gamma)'$ and $L(\Gamma) = R(\Gamma)'$.*

Proof. We begin by showing

$$L(\Gamma) \subset LC(\Gamma) \subset RC(\Gamma)' \subset R(\Gamma)' \subset LC(\Gamma).$$

For any $g \in \Gamma$, $\lambda(g) = \lambda(\delta_g) \in LC(\Gamma)$. Hence $L(\Gamma) \subset LC(\Gamma)'' = LC(\Gamma)$ by Proposition 1.3.6, which gives the first inclusion. Note that a symmetric argument implies $R(\Gamma) \subset RC(\Gamma)$, and so taking commutants yields the third inclusion. The second inclusion follows from Exercise 1.3.10. Let $x \in R(\Gamma)'$ and set $\xi := x\delta_e$. Then for any $g \in \Gamma$ we have

$$x\delta_g = x(\rho(g)\delta_e) = \rho(g)(x\delta_e) = \rho(g)\xi = \xi * \delta_g,$$

where the last equality follows from a direct computation. Consequently, for any finitely supported $\kappa \in \ell^2(\Gamma)$ we have $\|\xi * \kappa\|_2 = \|x\kappa\|_2 \leq \|x\|\|\kappa\|_2$. Lemma 1.3.5 therefore implies that ξ is a left convolver. The above computation shows $x\delta_g = \xi * \delta_g = \lambda(\xi)\delta_g$, and since such vectors densely span $\ell^2(\Gamma)$ we have $x = \lambda(\xi)$. This gives the last inclusion.

The inclusions established above show, $LC(\Gamma) = RC(\Gamma)' = R(\Gamma)'$. A symmetric argument yields $RC(\Gamma) = LC(\Gamma)' = L(\Gamma)'$. Using the Bicommutant Theorem, these equalities imply

$$R(\Gamma) = (R(\Gamma)')' = LC(\Gamma)' = L(\Gamma)'.$$

Taking commutants then gives $R(\Gamma)' = L(\Gamma)$. □

Remark 1.3.8. If G is a locally compact group (e.g. $\mathbb{R}$), it is still possible to define $L(G)$ using the left regular representation of G on $L^2(G, \mu)$, where μ the left-invariant Haar measure on G. However, in the mini-courses we will restrict ourselves to the discrete case.

Group von Neumann algebras remain far from fully understood. On the one hand, by a deep result of Alain Connes, all amenable i.c.c. groups yield the same group von Neumann algebra. This von Neumann algebra (which we will define by other means in a later chapter) is called the *hyperfinite* II_1 *factor*, but we will not have time in the mini-course to delve into Connes' proof.

On the other hand, the famous *Free Group Factor Isomorphism Problem*, which is still open, asks whether or not $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$, where $\mathbb{F}_k$ is the free group with k generators. A very active area of research in von Neumann algebras is focused on how much of Γ is "rememebered" by $L(\Gamma)$. The best results to date have relied on a collection of techniques known as Popa's deformation/ridigity theory.

Exercises

1.3.1. Consider the following 2×2 matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

Show that they are all projections and that their span is all of $M_2(\mathbb{C})$. Can you find 9 projections in $M_3(\mathbb{C})$ that span?

1.3.2. Suppose $\varphi\colon M_d(\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying $\varphi(AB) = \varphi(BA)$ for all $A, B \in M_d(\mathbb{C})$. Show that $\varphi = \varphi(1)\frac{1}{n}\mathrm{Tr}$. [**Hint:** show that $\varphi(E_{i,j}) = 0$ for $i \neq j$ and that $\varphi(E_{i,i})$ does not depend on $i = 1, \ldots, n$.]

1.3.3. For $f \in L^\infty(X,\mu) \subset B(L^2(X,\mu))$, show that f is a projection if and only if $f = 1_E$ for some measurable $E \subset X$. [**Hint:** show that $\mu\{x \in X \colon f(x) \notin \{0,1\}\} = 0$.]

1.3.4. Let Γ be a discrete group with left regular representation $\lambda\colon \Gamma \to B(\ell^2\Gamma)$. For $g \in \Gamma$, show that $\lambda(g)$ is a unitary operator with $\lambda(g)^* = \lambda(g^{-1})$.

1.3.5. Verify the claims in Example 1.3.3.

1.3.6. Let Γ be an infinite countable discrete group. Let $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ be a sequence that never repeats. Show that the sequence of unitaries $(\lambda(g_n))_{n\in\mathbb{N}}$ converges to zero in the WOT.

1.3.7. Let Γ be a countable discrete group.

(a) For $x \in L(\Gamma)$ and $g \in \Gamma$, show that
$$\Gamma \ni h \mapsto \langle x\delta_{g^{-1}h}, \delta_h \rangle$$
is a constant map.

(b) Denote the value of the constant map in the previous part by $c_g(x)$. Show that
$$x\delta_e = \sum_{g\in\Gamma} c_g(x)\delta_g,$$
and hence $\sum_g |c_g(x)|^2 < \infty$.

(c) For $x \in \mathcal{Z}(L(\Gamma))$, show that $c_g(x) = c_{h^{-1}gh}(x)$ for all $g, h \in \Gamma$, and that $c_g(x) = 0$ whenever $\{h^{-1}gh \colon h \in \Gamma\}$ is infinite.

(d) Prove that $L(\Gamma)$ is a factor if and only if Γ is an i.c.c. group.

1.3.8. Let Γ be a countable discrete group and let τ be the trace on $L(\Gamma)$.

(a) Show that $\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g))$ for all $g, h \in \Gamma$.

(b) Show that τ is WOT continuous.

(c) Prove that $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$.

1.3.9. Let Γ be a countable discrete group. In this exercise, you will show that $LC(\Gamma)$ and $RC(\Gamma)$ are $*$-algebras.

(a) Show that $1 = \lambda(e) \in LC(\Gamma) \cap RC(\Gamma)$ where $e \in \Gamma$ is the identity.

(b) If $\xi, \eta \in \ell^2(\Gamma)$ are left (resp.) convolvers, show that $\xi * \eta$ is a left (resp.) convolver.

(c) For $\lambda(\xi), \lambda(\eta) \in LC(\Gamma)$, show that $\lambda(\xi)\lambda(\eta) = \lambda(\xi * \eta) \in LC(\Gamma)$.

(d) For $\rho(\xi), \rho(\eta) \in RC(\Gamma)$, show that $\rho(\xi)\rho(\eta) = \rho(\xi * \eta) \in RC(\Gamma)$.

1.3.10. For a left convolver ξ and a right convolver η, show that $\lambda(\xi)\rho(\eta) = \rho(\eta)\lambda(\xi)$.

Chapter 2

Borel Functional Calculus and Abelian von Neumann Algebras

Let $\mathcal{H}$ be a Hilbert space and let $x \in B(\mathcal{H})$ be a *normal* operator: $[x, x^*] = 0$. This implies that $\mathbb{C}[x, x^*]$—the set of polynomials in x, x^*, and 1—is an abelian $*$-algebra and consequently its norm-closure, which we denote by $C^*(x)$, is an abelian C^*-algebra. The Gelfand transform then yields an isometric $*$-isomorphism

$$\Gamma \colon C^*(x) \to C(\sigma(x)).$$

This gives us a way to apply continuous functions on $\sigma(x)$ to the operator x: for $f \in C(\sigma(x))$ define $f(x) := \Gamma^{-1}(f)$. Since it is an isometric $*$-isomorphism, this definition respects the $*$-algebra structure and norm of $C(\sigma(x))$:

$$(f+g)(x) = f(x) + g(x) \qquad (f \cdot g)(x) = f(x)g(x) \qquad \|f(x)\| = \|f\|_\infty.$$

We call this the *continuous functional calculus*. In this chapter, we will extend this functional calculus to bounded Borel functions on $\sigma(x)$. While $f(x) \in C^*(x)$ when f is continuous, it may not be the case if f is only assumed to be bounded and Borel. However, we do always have $f(x) \in W^*(x)$, where

$$W^*(x) := \mathbb{C}[x, x^*]''$$

is the von Neumann algebra generated by x. Recall from the Bicommutant Theorem that $W^*(x)$ is equivalent to both the SOT and WOT closures of $\mathbb{C}[x, x^*]$, and since norm-convergence implies SOT and WOT convergence we have $C^*(x) \subset W^*(x)$.

At the end of the chapter, we will then use the Borel functional calculus to produce a (partial) classification of abelian von Neumann algebras.

Lecture Preview: In the first lecture, we will prove the Borel Functional Calculus (Theorem 2.1.3) in detail. You should familiarize yourself with the following proof ingredients ahead of time: the Riesz Representation Theorem (see [Theorem 2.16, GOALS Prerequisite Notes]), Proposition 2.1.1, and Lemma 2.1.2. In the second lecture, we will give the classification of abelian von Neumann algebras (Theorem 2.2.6). It is important to be comfortablw with Definitions 2.2.1 and 2.2.3 and Corollary 2.2.5. You might also find Examples 2.2.8 and 2.2.9 illuminating.

2.1 Borel Functional Calculus

We will use Borel measures to extend from continuous functions to bounded Borel functions. Since $\sigma(x)$ for $x \in B(\mathcal{H})$ is a compact subset of $\mathbb{C}$, $C(\sigma(x))$ falls under the scope of the Riesz Representation Theorem (see [Theorem 2.16, GOALS Prerequisite Notes]), which gives us easy access to Borel measures, as seen in the following proposition.

Proposition 2.1.1. *Let $x \in B(\mathcal{H})$ be a normal operator. For any $\xi, \eta \in \mathcal{H}$, there exists a unique regular Borel measure $\mu_{\xi,\eta} \in M(\sigma(x))$ satisfying $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$ and*

$$\langle f(x)\xi, \eta\rangle = \int_{\sigma(x)} f \ d\mu_{\xi,\eta} \qquad \forall f \in C(\sigma(x)). \tag{2.1}$$

Moreover, we have $\overline{\mu_{\xi,\eta}} = \mu_{\eta,\xi}$ for all $\xi, \eta \in \mathcal{H}$ and

$$\begin{aligned} \mu_{\alpha\xi_1+\xi_2,\eta} &= \alpha\mu_{\xi_1,\eta} + \mu_{\xi_2,\eta} && \forall \alpha \in \mathbb{C},\ \xi_1, \xi_2, \eta \in \mathcal{H} \\ \mu_{\xi,\beta\eta_1+\eta_2} &= \bar{\beta}\mu_{\xi,\eta_1} + \mu_{\xi,\eta_2} && \forall \beta \in \mathbb{C},\ \xi, \eta_1, \eta_2 \in \mathcal{H}. \end{aligned}$$

Proof. Observe that for $f \in C(\sigma(x))$

$$|\langle f(x)\xi, \eta\rangle| \leq \|f(x)\|\|\xi\|\|\eta\| = \|f\|_\infty\|\xi\|\|\eta\|.$$

Thus $f \mapsto \langle f(x)\xi, \eta\rangle$ is a bounded linear functional on $C(\sigma(x))$ with norm at most $\|\xi\|\|\eta\|$. The Riesz Representation Theorem implies there exists $\mu_{\xi,\eta} \in M(\sigma(x))$ satisfying $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$ and (2.1). Since $M(\sigma(x)) = C(\sigma(x))^*$, this measure is uniquely determined by (2.1). Using this uniqueness, one obtains the remaining properties via the conjugate symmetry, linearity, and conjugate linearity (respectively) of the inner product. □

For a locally compact Hausdorff space X we denote by $B(X)$ the collection of bounded Borel measurable functions $f\colon X \to \mathbb{C}$, which we equip with the supremum norm $\|f\|_\infty$. Any $f \in B(X)$ is integrable with respect to any $\mu \in M(X)$. In particular, for any Borel measurable subset $S \subset X$, we have $1_S \in B(X)$ and for any $\mu \in M(X)$ we have

$$\mu(S) = \int_X 1_S \ d\mu.$$

In the context of the above proposition, any reasonable definition of $f(x) \in B(\mathcal{H})$ for $f \in B(\sigma(x))$ should satisfy

$$\langle f(x)\xi, \eta\rangle = \int_{\sigma(x)} f \ d\mu_{\xi,\eta}.$$

The above discussion tells us we can already make sense of the right-hand side, and the following lemma tells us precisely how to produce $f(x) \in B(\mathcal{H})$ satisfying the above equation.

Lemma 2.1.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $q\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is linear in the first coordinate, conjugate linear in the second coordinate, and there exists $C > 0$ such that $|q(\xi, \eta)| \leq C\|\xi\|\|\eta\|$ for all $\xi, \eta \in \mathcal{H}$. Then there exists a unique $x \in B(\mathcal{H})$ satisfying*

$$\langle x\xi, \eta\rangle = q(\xi, \eta) \qquad \forall \xi, \eta \in \mathcal{H},$$

and $\|x\| \leq C$.

We leave the proof as an exercise (see Exercise 2.1.2), but remark that it is similar to the proof of [Theorem 1.36, GOALS Prerequisite Notes]. The map q is called a *bounded sesquilinear form*, and the above lemma is sometimes called the Riesz Representation Theorem (for Bounded Sesquilinear Forms).

Theorem 2.1.3 (Borel Functional Calculus)**.** *Let $x \in B(\mathcal{H})$ be a normal operator. There exists a contractive $*$-homomorphism*

$$B(\sigma(x)) \ni f \mapsto f(x) \in W^*(x).$$

In particular, for $f \in C(\sigma(x))$ the operator $f(x)$ is the same operator given by the continuous functional calculus.

Proof. Fix $f \in B(\sigma(x))$. For $\xi, \eta \in \mathcal{H}$ define

$$q(\xi, \eta) := \int_{\sigma(x)} f \ d\mu_{\xi,\eta},$$

where $\mu_{\xi,\eta}$ is as in Proposition 2.1.1. The same proposition implies q is linear in the first coordinate, conjugate linear in the second coordinate, and satisfies

$$|q(\xi,\eta)| \leq \int_{\sigma(x)} |f|\ d|\mu_{\xi,\eta}| \leq \|f\|_\infty \|\mu_{\xi,\eta}\| \leq \|f\|_\infty \|\xi\|\|\eta\|.$$

Thus Lemma 2.1.2 implies there exists $y \in B(\mathcal{H})$ with $\|y\| \leq \|f\|_\infty$ and

$$\langle y\xi,\eta\rangle = q(\xi,\eta) = \int_{\sigma(x)} f\ d\mu_{\xi,\eta} \qquad \forall \xi,\eta \in \mathcal{H}.$$

Define $f(x) := y$.

Thus $B(\sigma(x)) \ni f \mapsto f(x)$ is contractive. For all $\xi,\eta \in \mathcal{H}$ we have

$$\begin{aligned}\langle (f+g)(x)\xi,\eta\rangle = \int_{\sigma(x)} f+g\ d\mu_{\xi,\eta} &= \int_{\sigma(x)} f\ d\mu_{\xi,\eta} + \int_{\sigma(x)} g\ d\mu_{\xi,\eta}\\ &= \langle f(x)\xi,\eta\rangle + \langle g(x)\xi,\eta\rangle = \langle (f(x)+g(x))\xi,\eta\rangle\,,\end{aligned}$$

which implies $(f+g)(x) = f(x)+g(x)$. It is similarly shown that $(fg)(x) = f(x)g(x)$ and $\bar{f}(x) = f(x)^*$. So $f \mapsto f(x)$ is a contractive $*$-homomorphism. Note that—by construction—if $f \in C(\sigma(x))$ then $f(x)$ agrees with the operator given by the continuous functional calculus.

It remains to show that this $*$-homomorphism is valued in $W^*(x) = \mathbb{C}[x,x^*]''$. Observe that for $y \in \mathbb{C}[x,x^*]'$, $f \in C(\sigma(x))$, and $\xi,\eta \in \mathcal{H}$ we have

$$0 = \langle (yf(x) - f(x)y)\xi,\eta\rangle = \langle f(x)\xi, y^*\eta\rangle - \langle f(x)y\xi,\eta\rangle = \int_{\sigma(x)} f\ d\mu_{\xi,y^*\eta} - \int_{\sigma(x)} f\ d\mu_{y\xi,\eta}.$$

Since $f \in C(\sigma(x))$ was arbitrary and $\mu_{\xi,y^*\eta}, \mu_{y\xi,\eta} \in M(\sigma(x)) = C(\sigma(x))^*$, we must have $\mu_{\xi,y^*\eta} = \mu_{y\xi,\eta}$. Consequently, for $f \in B(\sigma(x))$ we have

$$\langle (yf(x) - f(x)y)\xi,\eta\rangle = \int_{\sigma(x)} f\ d\mu_{\xi,y^*\eta} - \int_{\sigma(x)} f\ d\mu_{y\xi,\eta} = 0$$

for all $y \in \mathbb{C}[x,x^*]'$ and all $\xi,\eta \in \mathcal{H}$. It follows that $yf(x) - f(x)y = 0$ for all $y \in \mathbb{C}[x,x^*]'$ so that $f(x) \in \mathbb{C}[x,x^*]'' = W^*(x)$. $\square$

For $x \in B(\mathcal{H})$ normal, let $S \subset \sigma(x)$ be Borel measurable. Then $1_S \in B(\sigma(x))$ and $1_S = \overline{1_S} = 1_S^2$ imply $1_S(x) = 1_S(x)^* = 1_S(x)^2$; that is, $1_S(x)$ is a projection. Consequently, if $f \in B(\sigma(x))$ is a simple function, then $f(x)$ is a linear combination of projections. From this we can deduce that projections are ubiquitious in von Neumann algebras:

Corollary 2.1.4. *A von Neumann algebra is the norm closure of the span of its projections.*

Proof. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra, and let $x \in M$. By considering the real and imaginary parts of x (Re $(x) = \frac{1}{2}(x+x^*)$ and Im $(x) = \frac{i}{2}(x^* - x)$) we may assume x is self-adjoint. In particular, x is normal and hence $f(x) \in W^*(x) \subset M$ for all $f \in B(\sigma(x))$ by the Borel functional calculus. Thus the discussion preceding the statement of the corollary implies that approximating the identity function on $\sigma(x)$ uniformly by simple functions gives, via the Borel functional calculus, a uniform approximation of x by linear combinations of projections in M. $\square$

Contrast this result with the fact that there exist C^*-algebras with no non-trivial projections. Indeed, if X is compact Hausdorff space, and X is connected, then $C(X)$ has exactly two projections: 0 and 1. Non-commutative examples exist as well.

It is not true in general that the $*$-homomorphism in the Borel functional calculus is injective. For example, if there exists a subset $S \in \sigma(x)$ such that $\mu_{\xi,\eta}(S) = 0$ for all $\xi,\eta \in \mathcal{H}$ then we will have $f(x) = g(x)$ so long as f and g agree on $\sigma(x) \setminus S$. This concept is explored further in Exercise 2.1.4.

Exercises

2.1.1. Let $x \in B(\mathcal{H})$ be normal. For $\xi \in \mathcal{H}$, let $\mu_{\xi,\xi}$ be as in Proposition 2.1.1. Show that $\mu_{\xi,\xi}$ is a positive measure.

2.1.2. Prove Lemma 2.1.2: First fix $\xi \in \mathcal{H}$ and show for all $\eta \in \mathcal{H}$ that $q(\xi, \eta) = \langle \xi_1, \eta \rangle$ for some $\xi_1 \in \mathcal{H}$. Then show that $x(\xi) := \xi_1$ defines a bounded operator $x \in B(\mathcal{H})$.

2.1.3. Let $x \in B(\mathcal{H})$ be a normal operator and let Ω be the Borel σ-algebra on $\sigma(x)$.

(a) Show that $1_\emptyset = 0$ and $1_{\sigma(x)} = 1$.

(b) Show that $1_{S \cap T} = 1_S 1_T$ for all $S, T \in \Omega$.

(c) Let $\{S_n \in \Omega : n \in \mathbb{N}\}$ be a collection of pairwise disjoint subsets and let $S := \bigcup_{n=1}^\infty S_n$. Show that

$$1_S = \sum_{n=1}^\infty 1_{S_n},$$

where series is the SOT-limit of the net of partial sums (see Proposition 1.1.5).

(The map $S \mapsto 1_S$ is called a **projection valued measure**.)

2.1.4. Let $x \in B(\mathcal{H})$ be a normal operator. We say a Borel measurable subset $S \subset \sigma(x)$ is x-**null** if $1_S(x) = 0$. For $f \in B(\sigma(x))$, define

$$x.im(f) := \{z \in \mathbb{C} : \text{for all } \epsilon > 0,\ \{w \in \sigma(x) : |f(w) - z| \leq \epsilon\} \text{ is not } x\text{-null}\}.$$

and

$$\|f\|_{\infty,x} := \sup_{z \in x.im(f)} |z|.$$

(a) Show that S is x-null if and only if $\mu_{\xi,\eta}(S) = 0$ for all $\xi, \eta \in \mathcal{H}$.

(b) Show that $f(x) = 0$ if and only if $\|f\|_{\infty,x} = 0$.

(c) Show that $\|f(x)\| = \|f\|_{\infty,x}$.

(d) Show that $\sigma(f(x)) \subset x.im(f)$.

2.2 Abelian von Neumann Algebras

In this section we will prove that abelian von Neumann algebras are of the form $L^\infty(X, \mu)$ for some measure space (X, Ω, μ). This result often inspires the following platitude: "Von Neumann algebras are noncommutative measure spaces." Nevertheless, this perspective is quite helpful in developing one's intuition for von Neumann algebras, and by the end of GOALS you will probably be like

For the sake of simplicity, we will restrict ourselves the case when the Hilbert space contains a *cyclic* vector.

Definition 2.2.1. Let $A \subset B(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is said to be **cyclic** for A if the subspace $A\xi$ is dense in $\mathcal{H}$.

To motivate this definition, suppose $x \in B(\mathcal{H})$ is normal and $\xi_0 \in \mathcal{H}$ is cyclic for $\mathbb{C}[x, x^*]$. Let $\mu := \mu_{\xi_0,\xi_0}$ be as in Proposition 2.1.1. Note that μ is a positive measure (see Exercise 2.1.1). For any $a, b \in \mathbb{C}[x, x^*]$ and any $S \subset \sigma(x)$ have

$$\mu_{a\xi_0,b\xi_0}(S) = \int_{\sigma(x)} 1_S \; d\mu_{a\xi_0,b\xi_0} = \langle 1_S(x)a\xi_0, b\xi_0\rangle = \langle (b^*1_S(x)a)\xi_0, \xi_0\rangle = \int_{\sigma(x)} \bar{q}1_S p \; d\mu$$

where p and q are polynomials such that $p(x, x^*) = a$ and $q(x, x^*) = b$. Thus if $\mu(S) = 0$, then the above computation implies $\mu_{a\xi_0,b\xi_0}(S) = 0$. That is, $\mu_{a\xi_0,b\xi_0} \ll \mu$. Furthermore, since ξ_0 is cyclic for $\mathbb{C}[x, x^*]$, given any $\xi, \eta \in \mathcal{H}$ and any $\epsilon > 0$ we can find $a, b \in \mathbb{C}[x, x^*]$ so that $\|a\xi_0 - \xi\|, \|b\xi_0 - \eta\| < \epsilon$. Proposition 2.1.1 implies

$$\|\mu_{\xi,\eta} - \mu_{a\xi_0,b\xi_0}\| \leq \|\mu_{\xi-a\xi_0,\eta}\| + \|\mu_{a\xi_0,\eta-b\xi_0}\| < \epsilon\|\eta\| + \|a\xi_0\|\epsilon < \epsilon(\|\eta\| + \|\xi\| + \epsilon),$$

and it follows that $\mu_{\xi,\eta} \ll \mu$. One consequence of this is that $1_S(x) = 0$ for $S \subset \sigma(x)$ Borel if and only if $\mu(S)$ (see Exercise 2.1.4.(a)). Another consequence (which we will prove below) is that $W^*(x)$ can be identified with $L^\infty(\sigma(x), \mu)$, where a bounded Borel function $f \in L^\infty(\sigma(x), \mu)$ is identified with $f(x)$.

Example 2.2.2. Let Γ be a discrete group and let $\lambda, \rho \colon \Gamma \to B(\ell^2(\Gamma))$ be the left and right regular representations. Define algebras $A := \text{span}\lambda(\Gamma)$ and $B := \text{span}\rho(\Gamma)$ Then $\delta_e \in \ell^2(\Gamma)$ is cyclic for both A and B since $\lambda(g)\delta_e = \delta_g = \rho(g)\delta_e$ for all $g \in \Gamma$. Moreover since A and B commute, if $a \in A$ and $a\delta_e = 0$ then $a = 0$. Indeed, for any $b \in B$ we have

$$ab\xi_0 = ba\xi_0 = 0.$$

Since $B\xi_0$ is dense in $\mathcal{H}$, it must be that $a = 0$. ■

The previous example highlights a related concept:

Definition 2.2.3. Let $A \subset B(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is said to be **separating** for A if $x\xi = 0$ for $x \in A$ implies $x = 0$.

The observation we made in Example 2.2.2 is an instance of a more general fact.

Proposition 2.2.4. *Let $A \subset B(\mathcal{H})$ be a subalgebra. If $\xi \in \mathcal{H}$ is cyclic for A, then it is separating for its commutant A'. If A is a unital $*$-subalgebra and ξ is separating for A', then ξ is cyclic for A. Consequently, for a von Neumann algebra $M \subset B(\mathcal{H})$, a vector is cyclic (resp. separating) for M if and only if it is separating (resp. cyclic) for M'.*

Proof. Let $\xi \in \mathcal{H}$ be cylic for A and suppose $y \in A'$ is such that $y\xi = 0$. Then for all $x \in A$ we have

$$yx\xi = xy\xi = 0.$$

Since ξ is cyclic for A, $\{x\xi \colon x \in A\}$ is dense in $\mathcal{H}$. Thus $y = 0$, and so ξ is separating for A'.

Now suppose A is a unital $*$-subalgebra and ξ is separating for A'. Let $p \in B(\mathcal{H})$ be the projection onto $\mathcal{K} := (A\xi)^\perp$. To see that ξ is cyclic for A it suffices to show $p = 0$. Indeed, $p = 0$ is equivalent to $\mathcal{K} = \{0\}$ and therefore

$$\overline{A\xi} = ((A\xi)^\perp)^\perp = \mathcal{K}^\perp = \{0\}^\perp = \mathcal{H}$$

(see [Exercise 1.18, GOALS Prerequisite Notes]). Now, for $x_1, x_2 \in A$ and $\eta \in \mathcal{K}$ we have

$$\langle x_1\eta, x_2\xi\rangle = \langle \eta, x_1^* x_2\xi\rangle = 0,$$

since $x_1^* x_2 \in A$. Thus $x_1\eta \in \mathcal{K}$, and hence $A\mathcal{K} \subset \mathcal{K}$. That is, $\mathcal{K}$ is reducing for A and so Lemma 1.2.5 implies $p \in A'$. Note that $\xi \in A\xi$ since A is unital, and hence $p\xi = 0$. Since ξ is separating for A', this implies $p = 0$. The final observations follow from M being a unital $*$-subalgebra and $M = (M')'$. □

Corollary 2.2.5. *If $A \subset B(\mathcal{H})$ is an abelian algebra, then every cyclic vector for A is also separating for A.*

Proof. If $\xi \in \mathcal{H}$ is cyclic for A, then by the proposition it is separating for A'. In particular, it is separating for $A \subset A'$. □

Recall that for a an abelian C^*-algebra A, the Gelfand transform gives an isometric $*$-isomorphism

$$\Gamma \colon A \to C_0(\sigma(A)),$$

where $\sigma(A)$ is a locally compact Hausdorff space formed by the spectrum of A: the set of all $*$-homomorphims from A to $\mathbb{C}$. In particular, if A is unital then $\sigma(A)$ is compact and the image of the Gelfand transform is $C(\sigma(A))$.

Theorem 2.2.6. *Let $A \subset B(\mathcal{H})$ be an abelian von Neumann algebra with a cyclic vector $\xi_0 \in \mathcal{H}$. For any SOT dense unital C^*-subalgebra $A_0 \subset A$, there exists a positive regular Borel measure $\mu \in M(\sigma(A_0))$ and a spatial isomorphism*

$$\Gamma^* \colon A \to L^\infty(\sigma(A_0), \mu)$$

satisfying

$$\langle x\xi_0, \xi_0\rangle = \int_{\sigma(A_0)} \Gamma^*(x)\ d\mu \qquad \forall x \in A.$$

Moreover, Γ^* extends the Gelfand transform $\Gamma \colon A_0 \to C(\sigma(A_0))$.

Proof. Let $\Gamma : A_0 \to C(\sigma(A_0))$ be the Gelfand transform. Define $\phi \colon A \to \mathbb{C}$ by $\phi(x) = \langle x\xi_0, \xi_0\rangle$ for $x \in A$. For $f \in C(\sigma(A_0))$ we have

$$|\phi(\Gamma^{-1}(f))| = |\left\langle \Gamma^{-1}(f)\xi_0, \xi_0\right\rangle| \leq \|\Gamma^{-1}(f)\|\|\xi_0\|^2 = \|f\|_\infty \|\xi_0\|^2.$$

Thus $\phi \circ \Gamma^{-1} \in C(\sigma(A_0))^*$, and so the Riesz Representation Theorem implies there exists a regular Borel measure $\mu \in M(\sigma(A_0))$ so that

$$\phi \circ \Gamma^{-1}(f) = \int_{\sigma(A_0)} f\ d\mu.$$

Observe that for a positive function $f \in C(\sigma(A_0))$, we have

$$\int_{\sigma(A_0)} f\ d\mu = \int_{\sigma(A_0)} \sqrt{f}^2\ d\mu = \ \phi \circ \Gamma^{-1}(\sqrt{f}^2)) = \left\langle \Gamma^{-1}(\sqrt{f}^2)\xi_0, \xi_0 \right\rangle = \left\| \Gamma^{-1}(\sqrt{f})\xi_0 \right\|^2 \geq 0.$$

Hence μ is a positive measure.

Define $U_0 \colon A_0\xi_0 \to C(\sigma(A_0)) \subset L^2(\sigma(A_0), \mu)$ by

$$U_0(x\xi_0) = \Gamma(x) \qquad x \in A_0.$$

Since ξ_0 is separating for A by Corollary 2.2.5, this is well-defined. Moreover, for $x, y \in A_0$

$$\langle U_0(x\xi_0), U_0(y\xi_0)\rangle_{L^2(\sigma(A_0),\mu)} = \int_{\sigma(A_0)} \Gamma(x)\overline{\Gamma(y)}\ d\mu = \int_{\sigma(A_0)} \Gamma(y^*x)\ d\mu = \phi(y^*x) = \langle y^*x\xi_0, \xi_0\rangle = \langle x\xi_0, y\xi_0\rangle .$$

Thus U_0 is an isometry on $A_0\xi_0$. Note that ξ_0 is cyclic for A_0 because it is cyclic for A and A_0 is SOT dense in A. Hence $A_0\xi_0$ is dense in $\mathcal{H}$ and so we can extend U_0 to an isometry $U \colon \mathcal{H} \to L^2(\sigma(A_0), \mu)$. Since $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0), \mu)$, U is surjective and hence a unitary.

Define a spatial isomorphism $\Gamma^* \colon A \to B(L^2(\sigma(A_0), \mu))$ via $\Gamma^*(x) = UxU^*$. For $x \in A_0$ and $g \in C(\sigma(A_0))$ we have

$$\Gamma^*(x)g = UxU^*g = Ux\left(\Gamma^{-1}(g)\xi_0\right) = U\Gamma^{-1}(\Gamma(x)g)\xi_0 = \Gamma(x)g.$$

By the density of $C(\sigma(A_0)) \subset L^2(\sigma(A_0), \mu)$, it follows that $\Gamma^*(x) = \Gamma(x)$ (where we are viewing $\Gamma(x \in B(L^2(\sigma(A_0), \mu))$ as a pointwise multiplication operator). Thus Γ^* extends the Gelfand transform.

Finally, towards proving $\Gamma^*(A) = L^\infty(\sigma(A_0), \mu)$ we first observe $L^\infty(\sigma(A_0), \mu) = \overline{\Gamma^*(A_0)}^{WOT}$. Indeed,

$$\Gamma^*(A_0) = \Gamma(A_0) = C(\sigma(A_0)) \subset L^\infty(\sigma(A_0), \mu),$$

so that $\overline{\Gamma^*(A_0)}^{WOT} = \overline{C(\sigma(A_0)}^{WOT}$. Recall that by Exercise 1.1.4, the WOT on $L^\infty(\sigma(A_0), \mu)$ corresponds to the weak* topology induced by $L^1(\sigma(A_0), \mu)^* = L^\infty(\sigma(A_0), \mu)$, and $C(\sigma(A_0))$ is dense in this topology by Exercise 2.2.2. Thus

$$\overline{\Gamma^*(A_0)}^{WOT} = \overline{C(\sigma(A_0)}^{WOT} = L^\infty(\sigma(A_0), \mu).$$

Hence, to finish the proof it suffices to prove the following inclusions:

$$\overline{\Gamma^*(A_0)}^{WOT} \subset \Gamma^*(A) \subset \overline{\Gamma^*(A_0)}^{WOT}.$$

To see the first inclusion, suppose $(\Gamma^*(x_i))_{i\in I} \subset \Gamma^*(A_0)$ WOT-converges to some $T \in B(L^2(\sigma(A_0), \mu))$. Then for all $\xi, \eta \in \mathcal{H}$ we have

$$\langle U^*TU\xi, \eta\rangle = \langle TU\xi, U\eta\rangle = \lim_{i\to\infty} \langle Ux_iU^*U\xi, U\eta\rangle = \lim_{i\to\infty} \langle x_i\xi, \eta\rangle .$$

Thus $(x_i)_{i\in I}$ WOT-converges to $U^*TU \in B(\mathcal{H})$. Since $A = \overline{A_0}^{WOT}$, $x := U^*TU \in A$ and $\Gamma^*(x) = UxU^* = T$. So the first inclusion holds. To see the second inclusion, observe that if $(x_i)_{i\in I} \in A$ is a net WOT-converging to $x \in A$, then for any $f, g \in L^2(\sigma(A_0), \mu)$ we have

$$\langle(\Gamma^*(x) - \Gamma^*(x_i))f, g\rangle_{L^2(\sigma(A_0),\mu)} = \langle U(x - x_i)U^*f, g\rangle_{L^2(\sigma(A_0),\mu)} = \langle(x - x_i)U^*f, U^*g\rangle \to 0.$$

Since $\overline{A_0}^{WOT} = A$ (by the Bicommutant Theorem), this implies $\Gamma^*(A) = \Gamma^*(\overline{A_0}^{WOT}) \subset \overline{\Gamma^*(A_0)}^{WOT}$. □

Remark 2.2.7. Observe that if we take $A_0 = A$ in the proof of the previous theorem, then it follows that

$$L^\infty(\sigma(A), \mu) = \Gamma^*(A) = \Gamma(A) = C(\sigma(A)).$$

That is, the μ-measurable essentially bounded functions coincide with the continuous functions on $\sigma(A)$. This should be taken as an indication that the spectrum of a commutative C^*-algebra A is strange when A is also a von Neumann algebra. Indeed, these are *Stonean spaces* and are examples of extremally disconnected spaces.

Let us explore Theorem 2.2.6 when $A = W^*(x)$ for $x \in B(\mathcal{H})$ a normal operator and relate it to the Borel functional calculus. A natural choice for A_0 is $C^*(x)$ (the *unital* C^*-algebra generated by x), which is SOT dense in $W^*(x)$ because $\mathbb{C}[x,x^*] \subset C^*(x)$ is SOT dense. Recall that in this case, $\sigma(C^*(x)) = \sigma(x)$. Suppose $\xi_0 \in \mathcal{H}$ is a cyclic vector for $W^*(x)$, and let $\mu \in M(\sigma(x))$ be as in Theorem 2.2.6. Note that since

$$\int_{\sigma(x)} f \; d\mu = \langle f(x)\xi_0, \xi_0 \rangle \qquad \forall f \in C(\sigma(x)),$$

we have $\mu = \mu_{\xi_0,\xi_0}$ where μ_{ξ_0,ξ_0} is defined as in Proposition 2.1.1. Now, since μ is a Borel measure, $L^\infty(\sigma(x),\mu) = B(\sigma(x))/\sim$ where the equivalence relation is μ-almost everywhere equivalence. We claim that for $f \in B(\sigma(x))$ with $[f] \in L^\infty(X,\mu)$, the operator $f(x)$ defined by the Borel functional calculus equals $(\Gamma^*)^{-1}([f])$ where Γ^* is as in Theorem 2.2.6. Indeed, for all $g \in C(\sigma(x))$ we have

$$\left\langle (f(x) - (\Gamma^*)^{-1}([f]))\xi_0, g(x)\xi_0 \right\rangle = \int_{\sigma(x)} f\bar{g} \; d\mu_{\xi_0,\xi_0} - \int_{\sigma(x)} [f]\bar{g} \; d\mu = \int_{\sigma(x)} (f - [f])\bar{g} \; d\mu = 0.$$

The above computation implies $(f(x) - (\Gamma^*)^{-1}([f]))\xi_0 = 0$ because ξ_0 is cyclic for $C^*(x) = \Gamma^{-1}(C(\sigma(x))$ (by virtue of $C^*(x)$ being SOT dense in $W^*(x)$). But ξ_0 is separating for $W^*(x)$ by Corollary 2.2.5, so we have $f(x) = (\Gamma^*)^{-s}([f])$ as claimed. All of which is to say, when $A = W^*(x)$ and $A_0 = C^*(x)$ the $*$-isomorphism in Theorem 2.2.6 respects the Borel functional calculus.

Theorem 2.2.6 also allows us to better understand group von Neumann algebras for commutative groups. We consider a few examples below.

Example 2.2.8. For $n \in \mathbb{N}$ with $n \geq 2$, let

$$\Gamma := \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}.$$

Since Γ is an abelian group, $L(\Gamma)$ is an abelian von Neumann algebra and from Example 2.2.2 we know that δ_0 is a cyclic vector for $L(\Gamma)$. Let $x \in L(\Gamma)$ be the unitary operator corresponding to the group generator $1 \in \mathbb{Z}/n\mathbb{Z}$, so that $L(\Gamma) = W^*(x)$. Since $xx^* = 1 = x^*x$ (i.e. x is normal), from the above discussion we know

$$L(\Gamma) \cong L^\infty(\sigma(x), \mu)$$

for a regular Borel measure $\mu \in M(\sigma(x))$. Observe that the matrix representation of x with respect to the basis $\delta_0, \delta_1, \ldots, \delta_{n-1}$ is the permutation matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

So [Example 3.15.(1), GOALS Prerequisite Notes] implies $\sigma(x)$ is the set of eigenvalues of the above matrix: $\{\exp(\frac{2\pi i k}{n}) \colon k = 0, 1, \ldots, n-1\}$ (**Exercise:** confirm this). Denote $\zeta_k = \exp(\frac{2\pi i k}{n})$ for $k = 0, 1, \ldots, n-1$, then

$$e_k := \frac{1}{\sqrt{n}}\left(\delta_0 + \zeta_k^{-1}\delta_1 + \cdots + \zeta_k^{-(n-1)}\delta_{n-1}\right)$$

is a unit eigenvector of x with eigenvalue ζ_k. Since $z1_{\{\zeta_k\}}(z) = \zeta_k 1_{\{\zeta_k\}}(z)$ for $z \in \mathbb{C}$, the Borel functional calculus implies $x1_{\{\zeta_k\}}(x) = \zeta_k 1_{\{\zeta_k\}}(x)$. That is, $1_{\{\zeta_k\}}(x)$ is the projection onto the ζ_k eigenspace. As this space is spanned by the unit vector e_k, we have $1_{\{\zeta_k\}}(x) = e_k \otimes \overline{e_k}$. Thus we have

$$\mu(\{\zeta_k\}) = \int_{\sigma(x)} 1_{\{\zeta_k\}} \; d\mu = \left\langle 1_{\{\zeta_k\}}(x)\delta_0, \delta_0 \right\rangle = \langle e_k \otimes \overline{e_k}\delta_0, \delta_0 \rangle = \langle \langle \delta_0, e_k \rangle e_k, \delta_0 \rangle = |\langle \delta_0, e_k \rangle|^2 = \frac{1}{n}.$$

Hence μ is the uniform probability distribution on $\{\zeta_k \colon k = 0, 1, \ldots, n-1\}$. ■

Example 2.2.9. Consider the abelian von Neumann algebra $L(\mathbb{Z})$. As in the previous example, $\delta_0 \in \ell^2(\mathbb{Z})$ is a cyclic vector for $L(\mathbb{Z})$. Let $x \in L(\mathbb{Z})$ be the unitary operator corresponding to $1 \in \mathbb{Z}$, so that $L(\mathbb{Z}) = W^*(x)$. Let

$$\mathbb{T} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\},$$

then $\mathbb{Z}$ and $\mathbb{T}$ are Pontryagin duals to each other via

$$\mathbb{Z} \times \mathbb{T} \ni (n, \zeta) \mapsto \zeta^n.$$

This duality allows us to define a unitary $U : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}, m)$ (where m is the normalized Lebesgue measure on $\mathbb{T}$) via

$$[U(\xi)](\zeta) = \sum_{n\in\mathbb{Z}} \xi(n)\zeta^n \qquad \xi \in \ell^2(\mathbb{Z}),\ \zeta \in \mathbb{T}.$$

If $f : L^\infty(\mathbb{T}, m)$ is the identity function $f(\zeta) = \zeta$, we have

$$[Ux\xi](\zeta) = \sum_{n\in\mathbb{Z}} [x\xi](n)\zeta^n = \sum_{n\in\mathbb{Z}} \xi(n-1)\zeta^n = \zeta \sum_{n\in\mathbb{Z}} \xi(n-1)\zeta^{n-1} f(z)[U\xi](\zeta).$$

Hence $UxU^* = f$. Using an argument similar to the on Theorem 2.2.6, one then obtains $L(Z) \cong L^\infty(\mathbb{T}, m)$. We leave the details for you to check in Exercise 2.2.3. ■

Exercises

2.2.1. Let $\mathcal{H}$ be a Hilbert space and let $p \in B(\mathcal{H})$ be a non-trivial projection: $p \neq 0$ and $p \neq 1$. Show that the algebra $A := pB(\mathcal{H})p$ has no cyclic vectors.

2.2.2. Let X be a compact Hausdorff space and let $\mu \in M(X)$ be a positive regular Borel measure. Show that $C(X)$ is weak* dense in $L^\infty(X, \mu)$ by showing that if $f \in L^1(X, \mu)$ satisfies

$$\int_X fg \, d\mu = 0 \qquad \forall g \in C(X)$$

then $f = 0$.

2.2.3. Fill in the remaining details of Example 2.2.9: first show that $UL(\mathbb{Z})U^* = \overline{\mathbb{C}[f, \bar{f}]}^{WOT} = \overline{\mathbb{C}[f, \bar{f}]}^{wk*}$, then argue that $\mathbb{C}[f, \bar{f}]$ (i.e. the set of polynomials) is weak* dense in $L^\infty(\mathbb{T}, m) = L^1(\mathbb{T}, m)^*$.

2.2.4. Let Γ be a discrete abelian group and let $\widehat{\Gamma}$ be its Pontryagin dual group, which is a compact abelian group and hence has a finite Haar measure μ. Show that $L(\Gamma) \cong L^\infty(\widehat{\Gamma}, \mu)$.

Chapter 3

The Predual

Let (X, Ω, μ) be a σ-finite measure space. We saw in Exercise 1.1.4 that the weak operator topology on $L^\infty(X, \mu)$ as a von Neumann algebra in $B(L^2(X, \mu))$ is precisely the weak* topology induced by $L^\infty(X, \mu) = L^1(X, \mu)^*$. In this chapter, we will show that in fact for any von Neumann algebra $M \subset B(\mathcal{H})$, there exists a Banach space M_* whose dual is M: $M = (M_*)^*$. We call this Banach space M_* the *predual* of M. However, the weak* topology induced by this duality is not quite the weak operator topology but rather a finer topology called the *σ-weak operator topology* (σ-*WOT*). The duality relation $M = (M_*)^*$ ends up being so important to the structure of a von Neumann algebra that the σ-WOT is considered the *natural* topology on a von Neumann algebra and maps on a von Neumann algebra that are σ-WOT continuous are declared to be *normal* (see Definition 3.3.1).

In the next section, we will consider the case when $M = B(\mathcal{H})$. In this case, the role of the predual $B(\mathcal{H})_*$ is played by the trace class operators $L^1(B(\mathcal{H}))$. In fact, the analogy with $L^\infty(X, \mu) = L^1(X, \mu)^*$ was precisely the impetus for the notation "$L^1(B(\mathcal{H}))$." Moreover, the case of $B(\mathcal{H})$ is crucial to understand because from it stems the definitions of the σ-WOT and the predual for a general von Neumann algebra, which will appear in Sections 3.2 and 3.3. We conclude the chapter with an important result called the Kaplansky Density Theorem.

Lecture Preview: Section 3.1 contains detailed but technical proofs of the properties we stated for trace class operators on the first day of GOALS. For our purposes, we only need Theorem 3.1.10 (which says $K(\mathcal{H})^* = L^1(B(\mathcal{H}))$) and Theorem 3.1.12 (which says $L^1(B(\mathcal{H}))^* = B(\mathcal{H})$), and consequently we recommend that you skip the details in Section 3.1 for now. In the first lecture we will introduce the so-called σ-topologies $B(\mathcal{H})$ (see Definition 3.2.2) and compare them to the SOT and WOT as well as the weak* topology induced by $B(\mathcal{H}) = L^1(B(\mathcal{H}))^*$. In the second lecture, we will define the predual of a von Neumann algebra and explore the weak* topology induced by $M = (M_*)^*$. Time permitting, we will discuss the Kaplansky Density Theorem (see Theorem 3.4.6), however we also recommend that you skip the details for now.

3.1 Trace Class Operators

Before we can formally treat the trace class operatoras, we need to establish the *polar decomposition* of an operator. Recall that for $x \in B(\mathcal{H})$, we denote $|x| = (x^*x)^{\frac{1}{2}}$.

Theorem 3.1.1. *Let $x \in B(\mathcal{H})$. Then there exists a partial isometry $v \in W^*(x)$ so that $x = v|x|$. Also $\ker(v) = \ker(|x|)$, and v^*v and vv^* are the projections onto $\overline{ran(|x|)}$ and $\overline{ran(x)}$, respectively. This decomposition is unique in that if $x = wy$ for $y \geq 0$ and w a partial isometry with $\ker(w) = \ker(y)$, then $w = v$ and $y = |x|$.*

Proof. Define $v_0\colon \operatorname{ran}(|x|) \to \operatorname{ran}(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Exercise 3.1.1 implies v_0 is well-defined and can be extended to an isometry $v\colon \overline{\operatorname{ran}(|x|)} \to \overline{\operatorname{ran}(x)}$. Extend v to a bounded operator on $\mathcal{H}$ by defining v be zero on $\overline{\operatorname{ran}(|x|)}^\perp = \ker(|x|)$. Then v a partial isometry with $\ker(v) = \ker(|x|)$ and $\overline{\operatorname{ran}(v)} = \overline{\operatorname{ran}(x)}$.

This implies v^*v is the projection onto $\ker(v)^\perp = \ker(|x|)^\perp = \overline{\mathrm{ran}(|x|)}$, and vv^* is the projection onto $\overline{\mathrm{ran}(v)} = \overline{\mathrm{ran}(x)}$. By definition we have $v|x| = x$.

To see that $v \in W^*(x)$, suppose $y \in W^*(x)'$. Then for all $\xi \in \mathcal{H}$ we have

$$yv|x|\xi = yx\xi = xy\xi = v|x|y\xi = vy|x|\xi.$$

Consequently, $yv = vy$ on $\overline{\mathrm{ran}(|x|)}$. For $\xi \in \overline{\mathrm{ran}(|x|)}^\perp = \ker(|x|)$, note that $y\xi \in \ker(|x|)$ since y commutes with $|x|$ (by virtue of $|x| \in C^*(x) \subset W^*(x)$). Since $\ker(|x|) = \ker(v)$ we therefore have $yv\xi = 0 = vy\xi$. Thus $vy = vy$ on all of $\mathcal{H}$ and so $v \in W^*(x)'' = W^*(x)$.

Suppose $x = wy$ for some $y \geq 0$ and w a partial isometry with $\ker(w) = \ker(y)$. Then

$$\ker(w)^\perp = \ker(y)^\perp = \overline{\mathrm{ran}(y)}.$$

Thus $w^*wy = y$ since w^*w is the projection onto $\ker(w)^\perp$. Consequently,

$$|x|^2 = x^*x = yw^*wy = y^2,$$

which implies $|x| = y$ by the uniqueness of the square root. Thus

$$(w - v)|x|\xi = wyxi - v|x|\xi = x\xi - x\xi = 0$$

for all $\xi \in \mathcal{H}$. Since $\ker(w) = \ker(y) = \ker(|x|)$, this implies $w = v$. □

Corollary 3.1.2. *Let $x \in B(\mathcal{H})$ with polar decomposition $x = v|x|$. Then $|x^*| = v|x|v^*$ and $x^* = v^*|x^*|$.*

Proof. Observe that $v|x|v^*$ is positive with

$$(v|x|v^*)(v|x|v^*) = v|x|v^*v|x|v^* = v|x|^2v^* = vxx^*v^* = xx^*.$$

By the uniqueness of the square root, we have $v|x|v^* = (xx^*)^{1/2} = |x^*|$. From this we further deduce

$$v^*|x^*| = v^*v|x|v^* = |x|v^* = (v|x|)^* = x^*.$$ □

Now, recall that $x \in B(\mathcal{H})$ is a *trace class* operator if

$$\|x\|_1 = \mathrm{Tr}(|x|) = \sum_{\xi \in \mathcal{E}} \langle |x|\xi, \xi \rangle < \infty,$$

where $\mathcal{E} \subset \mathcal{H}$ is any orthornomal basis. In this section we will make good on our promise of proofs for results stated on Day 1 (see [Theorem 7.25, Day 1 Lectures]). We will make use of the following, which appeared as Exercises 7.52 and 7.54 in the Day 1 lectures.

Proposition 3.1.3. *Let $\mathcal{H}$ be a Hilbert space.*

(i) For positive $x \in B(\mathcal{H})$,

$$Tr(x) = \sum_{\xi \in \mathcal{E}} \langle x\xi, \xi \rangle$$

does not depend on the choice of orthonormal basis $\mathcal{E} \subset \mathcal{H}$.

(ii) For $x \in B(\mathcal{H})$, $Tr(x^*x) = Tr(xx^*)$.

In order to show that we can define $\mathrm{Tr}(x)$ for any $x \in L^1(B(\mathcal{H}))$, we need the following lemma.

Lemma 3.1.4. *For $x \in B(\mathcal{H})$,*

$$|\langle x\xi, \xi \rangle| \leq \left\langle \frac{1}{2}(|x| + |x^*|)\xi, \xi \right\rangle \qquad \forall \xi \in \mathcal{H}.$$

Proof. Let $x = v|x|$ be the polar decomposition and recall from Corollary 3.1.2 that $|x^*| = v|x|v^*$. For $\xi \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, observe that

$$\begin{aligned}
0 \leq \langle |x|(\xi - \lambda v^*\xi), (\xi - \lambda v^*\xi)\rangle &= \langle |x|\xi, \xi\rangle - \bar{\lambda}\langle |x|\xi, v^*\xi\rangle - \lambda \langle |x|v^*\xi, \xi\rangle + |\lambda|^2 \langle |x|v^*\xi, |x|v^*\xi\rangle \\
&= \langle |x|\xi, \xi\rangle - \bar{\lambda}\langle v|x|\xi, \xi\rangle - \lambda\langle \xi, v|x|\xi\rangle + |\lambda|^2\langle v|x|v^*\xi, \xi\rangle \\
&= \langle |x|\xi, \xi\rangle - 2\mathrm{Re}\,\bar{\lambda}\langle x\xi, \xi\rangle + |\lambda|^2 \langle |x^*|\xi, \xi\rangle .
\end{aligned}$$

Setting $\lambda := \frac{\langle x\xi,\xi\rangle}{|\langle x\xi,\xi\rangle|}$ yields

$$0 \leq \langle |x|\xi, \xi\rangle - 2|\langle x\xi, \xi\rangle| + \langle |x^*|\xi, \xi\rangle ,$$

and a bit of algebra produces the desired inequality. $\square$

Proposition 3.1.5. *Let $x \in L^1(B(\mathcal{H}))$. Then $x^* \in L^1(B(\mathcal{H}))$ with $\|x^*\|_1 = \|x\|_1$. Also, for any orthonormal basis $\mathcal{E} \subset \mathcal{H}$ the series*

$$\sum_{\xi\in\mathcal{E}} \langle x\xi, \xi\rangle$$

converges absolutely and its value is independent of the choice of orthonormal basis.

Proof. Let $x = v|x|$ be the polar decomposition. Using Corollary 3.1.2 and Proposition 3.1.3.(ii) we have

$$\|x^*\|_1 = \mathrm{Tr}(|x^*|) = \mathrm{Tr}(v|x|v^*) = \mathrm{Tr}((v|x|^{1/2})(v|x|^{1/2})^*) = \mathrm{Tr}((v|x|^{1/2})^*(v|x|^{1/2})) = \mathrm{Tr}(|x|^{1/2}v^*v|x|^{1/2}).$$

Recall from Theorem 3.1.1 that v^*v is the projection onto $\overline{\mathrm{ran}(|x|)}$. Since

$$\overline{\mathrm{ran}(|x|)} = \ker(|x|)^\perp = \ker(|x|^{1/2})^\perp = \overline{\mathrm{ran}(|x|^{1/2})},$$

we have therefore have $v^*v|x|^{1/2} = |x|^{1/2}$. Thus we can continue our previous computation with

$$\|x^*\|_1 = \mathrm{Tr}(|x|^{1/2}v^*v|x|^{1/2}) = \mathrm{Tr}(|x|) = \|x\|_1.$$

Now, let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal basis. Using Lemma 3.1.4 and the first part of the proof we have

$$\sum_{\xi\in\mathcal{E}} |\langle x\xi, \xi\rangle| \leq \sum_{\xi\in\mathcal{E}} \left\langle \frac{1}{2}(|x| + |x^*|)\xi, \xi \right\rangle = \frac{1}{2}\mathrm{Tr}(|x|) + \frac{1}{2}\mathrm{Tr}(|x^*|) = \frac{1}{2}\|x\|_1 + \frac{1}{2}\|x^*\|_1 = \|x\|_1 < \infty.$$

So the series converges absolutely. Towards showing that the value of the series does not depend on $\mathcal{E}$, we will express x as a linear combination of positive operators trace class operators. The identity

$$x = v|x| = \frac{1}{4}\sum_{k=0}^{3} i^k(v + i^k)|x|(v + i^k)^*$$

follows by expanding the right-hand side. Note that each

$$(v + i^k)|x|(v + i^k)^* = \left((v + i^k)|x|^{1/2}\right)\left((v + i^k)|x|^{1/2}\right)^*$$

is positive. Before we show they are trace class, let's see why this finishes the proof. If $(v + i^k)|x|(v + i^k)^* \in L^1(B(\mathcal{H}))$ for each k, then $\frac{1}{4}\sum i^k \mathrm{Tr}((v + i^k)|x|(v + i^k)^*)$ is defined and by Proposition 3.1.3.(i) this quantity is independent of the choice of orthonormal basis. Consequently,

$$\frac{1}{4}\sum_{k=0}^{3} i^k \mathrm{Tr}((v + i^k)|x|(v + i^k)^*) = \frac{1}{4}\sum_{k=0}^{3} i^k \sum_{\xi\in\mathcal{E}} \left\langle (v + i^k)|x|(v + i^k)^*)\xi, \xi\right\rangle = \sum_{\xi\in\mathcal{E}} \langle x\xi, \xi\rangle$$

is also independent of the choice of orthonormal basis.

Now, Proposition 3.1.3.(ii) tells us that $(v + i^k)|x|(v + i^k)^* = \left((v + i^k)|x|^{1/2}\right)\left((v + i^k)|x|^{1/2}\right)^*$ is trace class if and only if $\left((v + i^k)|x|^{1/2}\right)^*\left((v + i^k)|x|^{1/2}\right) = |x|^{1/2}(v + i^k)^*(v + i^k)|x|^{1/2}$ is trace class. Since

$$|x|^{1/2}(v + i^k)^*(v + i^k)|x|^{1/2} \leq \|(v + i^k)^*(v + i^k)\||x|^{1/2}1|x|^{1/2} = \|v + i^k\|^2|x| \leq 4|x|,$$

we have

$$\mathrm{Tr}(|x|^{1/2}(v+i^k)^*(v+i^k)|x|^{1/2}) \leq \mathrm{Tr}(4|x|) = 4\mathrm{Tr}(|x|) < \infty.$$

(If $0 \leq a \leq b$ are bounded operators, then $\langle a\xi, \xi\rangle \leq \langle b\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$ implies $\mathrm{Tr}(a) \leq \mathrm{Tr}(b)$.) Thus each $(v+i^k)|x|(v+i^k)^*$ is positive and trace class. □

The previous proposition enables us to make the following definition:

Definition 3.1.6. For $x \in L^1(B(\mathcal{H}))$, the **trace** of x is the quantity

$$\mathrm{Tr}(x) := \sum_{\xi \in \mathcal{E}} \langle x\xi, \xi\rangle$$

where $\mathcal{E} \subset \mathcal{H}$ is any orthonormal basis for $\mathcal{H}$.

Note that in the proof of Proposition 3.1.5, it was shown that $|\mathrm{Tr}(x)| \leq \|x\|_1$.

Theorem 3.1.7. *Let $x \in L^1(B(\mathcal{H}))$. Then for any $a, b \in B(\mathcal{H})$, $axb \in L^1(B(\mathcal{H}))$ with $\|axb\|_1 \leq \|a\|\|x\|_1\|b\|$ and $|Tr(axb)| \leq \|a\|\|x\|_1\|b\|$. Moreover, $Tr(ax) = Tr(xa)$ for all $a \in B(\mathcal{H})$.*

Proof. The second claim follows from the first claim and the observation preceding the statement of the theorem. Now, let $x = v|x|$ and $axb = w|axb|$ be the polar decompositions. For an orthonormal basis $\mathcal{E} \subset \mathcal{H}$ we estimate

$$\sum_{\xi \in \mathcal{E}} \langle |axb|\xi, \xi\rangle = \sum_{\xi \in \mathcal{E}} \langle w^*av|x|b\xi, \xi\rangle = \sum_{\xi \in \mathcal{E}} \left\langle |x|^{1/2}b\xi, |x|^{1/2}v^*a^*w\xi \right\rangle \leq \sum_{\xi \in \mathcal{E}} \||x|^{1/2}b\xi\|\||x|^{1/2}v^*a^*w\xi\|,$$

by the Cauchy–Schwarz inequality. Using the Cauchy–Schwarz inequality on $\ell^2(\mathcal{E})$ followed by Exercise 3.1.10 we can continue our estimate with

$$\sum_{\xi \in \mathcal{E}} \langle |axb|\xi, \xi\rangle \leq \left(\sum_{\xi \in \mathcal{E}} \||x|^{1/2}b\xi\|^2\right)^{1/2} \left(\sum_{\xi \in \mathcal{E}} \||x|^{1/2}v^*a^*w\xi\|^2\right)^{1/2} \leq \left(\|x\|_1\|v^*a^*w\|^2\right)^{1/2} \left(\|x\|_1\|b\|^2\right)^{1/2}.$$

Since v and w are partial isometries, we have $\|v^*a^*w\| \leq \|v^*\|\|a^*\|\|w\| \leq \|a^*\| = \|a\|$. Hence the above estimate yields $\|axb\|_1 \leq \|a\|\|x\|_1\|b\|$.

For the final claim, any $a \in B(\mathcal{H})$ can be written as a linear combination of four unitaries by Exercise 3.1.7. Thus it suffices show $\mathrm{Tr}(ux) = \mathrm{Tr}(xu)$ for a unitary $u \in B(\mathcal{H})$. In this case we have

$$\mathrm{Tr}(ux) = \sum_{\xi \in \mathcal{E}} \langle ux\xi, \xi\rangle = \sum_{\xi \in \mathcal{E}} \langle x\xi, u^*\xi\rangle = \sum_{\xi \in \mathcal{E}} \langle xuu^*\xi, u^*\xi\rangle = \sum_{\xi \in \mathcal{E}} \langle (xu)u^*\xi, u^*\xi\rangle.$$

Noting that $\{u^*\xi \colon \xi \in \mathcal{E}\}$ is an orthonormal basis, we see that the above equals $\mathrm{Tr}(xu)$ since the trace is independent of the choice of orthonormal basis. □

Theorem 3.1.8. *The map $\|\cdot\|_1$ is a norm on $L^1(B(\mathcal{H}))$ that makes it into a Banach space. The finite rank operators $FR(\mathcal{H})$ are dense in $L^1(B(\mathcal{H}))$ with respect this norm, and $\|x\| \leq \|x\|_1$ for all $x \in L^1(B(\mathcal{H}))$.*

Proof. To show that $\|\cdot\|_1$ is a norm, it suffices to show it satisfies the triangle inequality. For $x, y \in L^1(B(\mathcal{H}))$ let $x+y = w|x+y|$ be the polar decomposition. Then $|x+y| = w^*(x+y) = w^*x + w^*y$, and by Theorem 3.1.7 we have

$$\mathrm{Tr}(|x+y|) = \mathrm{Tr}(w^*x + w^*y) \leq |\mathrm{Tr}(w^*x)| + |\mathrm{Tr}(w^*y)| \leq \|w^*\|\|x\|_1 + \|w^*\|\|y\|_1 \leq \|x\|_1 + \|y\|_1,$$

where the last inequality follows from the fact that w is a partial isometry. This shows that $x+y \in L^1(B(\mathcal{H}))$ with $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$, and so $\|\cdot\|_1$ is a norm on $L^1(B(\mathcal{H}))$.

We next show that $\|\cdot\|_1$ dominates the operator norm. For $x \in L^1(B\mathcal{H}))$, since $\|x\xi\| = \||x|\xi\|$ for all $\xi \in \mathcal{H}$ and $|x|$ is positive, we have

$$\|x\| = \||x|\| = \sup_{\|\xi\|=1} \langle |x|\xi, \xi\rangle.$$

Let $\epsilon > 0$ and let $\xi_0 \in \mathcal{H}$ be a unit vector such that $\langle |x|\xi_0, \xi_0\rangle \geq \|x\| - \epsilon$. Let $\mathcal{E} \subset \mathcal{H}$ be an orthornomal basis with $\xi_0 \in \mathcal{E}$. Then

$$\|x\|_1 = \mathrm{Tr}(|x|) = \sum_{\xi \in \mathcal{E}} \langle |x|\xi, \xi\rangle \geq \langle |x|\xi_0, \xi_0\rangle \geq \|x\| - \epsilon.$$

Letting ϵ tend to zero yields $\|x\| \leq \|x\|_1$.

Now we will show that $FR(\mathcal{H})$ is dense in $L^1(B(\mathcal{H}))$. Fix $x \in L^1(B(\mathcal{H}))$ and an orthonormal basis $\mathcal{E} \subset \mathcal{H}$, and let $\epsilon > 0$. Since $\|x\|_1 < \infty$, we can find a finite subset $F \subset \mathcal{E}$ so that

$$\sum_{\xi \in \mathcal{E}\setminus F} \langle |x|\xi, \xi\rangle < \epsilon.$$

Let $p \in FR(\mathcal{H})$ be the projection onto $\operatorname{span} F$. Then $xp \in FR(\mathcal{H})$ and if $x = v|x|$ and $x - xp = w|x - xp|$ are the polar decompositions then we have

$$|x - xp| = w^*(x - xp) = w^*x(1 - p) = w^*v|x|(1 - p).$$

Thus we estimate

$$\begin{aligned}\|x - xp\|_1 = \sum_{\xi \in \mathcal{E}} \langle |x - xp|\xi, \xi\rangle &= \sum_{\xi \in \mathcal{E}\setminus F} \langle w^*v|x|\xi, \xi\rangle = \sum_{\xi \in \mathcal{E}\setminus F} \left\langle |x|^{1/2}\xi, |x|^{1/2}v^*w\xi \right\rangle \\ &\leq \sum_{\xi \in \mathcal{E}\setminus F} \||x|^{1/2}\xi\|\||x|^{1/2}v^*w\xi\| \leq \left(\sum_{\xi \in \mathcal{E}\setminus F} \||x|^{1/2}\xi\|^2\right)^{1/2} \left(\sum_{\xi \in \mathcal{E}\setminus F} \||x|^{1/2}v^*w\xi\|^2\right)^{1/2}.\end{aligned}$$

Observe that by our choice of F the first factor in the last expression satisfies

$$\left(\sum_{\xi \in \mathcal{E}\setminus F} \||x|^{1/2}\xi\|^2\right)^{1/2} = \left(\sum_{\xi \in \mathcal{E}\setminus F} \langle |x|\xi, \xi\rangle\right)^{1/2} < \epsilon^{1/2}.$$

While the second factor we can estimate using Exercise 3.1.10:

$$\left(\sum_{\xi \in \mathcal{E}\setminus F} \||x|^{1/2}v^*w\xi\|^2\right)^{1/2} \leq \left(\sum_{\xi \in \mathcal{E}} \||x|^{1/2}v^*w\xi\|^2\right)^{1/2} \leq (\|x\|_1\|v^*w\|)^{1/2} \leq \|x\|_1^{1/2}.$$

Putting this together yields $\|x - xp\|_1 \leq \epsilon^{1/2}\|x\|_1^{1/2}$. Hence $FR(\mathcal{H})$ is dense in $L^1(B(\mathcal{H}))$.

Finally, to see that $L^1(B(\mathcal{H}))$ is a Banach space (i.e. complete) suppose $(x_n)_{n\in\mathbb{N}} \subset L^1(B(\mathcal{H}))$ is a Cauchy sequence with respect to the $\|\cdot\|_1$ norm. Since $\|x_n - x_m\| \leq \|x_n - x_m\|_1$, this sequence is Cauchy with respect to the operator norm and so it converges in operator norm to some $x \in B(\mathcal{H})$. Let $x = v|x|$ be the polar decomposition and note that $|x| = v^*x$ is the norm limit of $(v^*x_n)_{n\in\mathbb{N}}$. This in conjunction with Lemma 3.1.4 yields for any $\xi \in \mathcal{H}$

$$\langle |x|\xi, \xi\rangle = \lim_{n\to\infty} |\langle v^*x_n\xi, \xi\rangle| \leq \limsup_{n\to\infty} \left\langle \frac{1}{2}(|v^*x_n| + |x_n^*v|)\xi, \xi \right\rangle.$$

So if $\mathcal{E} \subset \mathcal{H}$ is an orthonormal basis, then for any finite subset $F \subset \mathcal{E}$ we have

$$\sum_{\xi\in F} \langle |x|\xi, \xi\rangle \leq \sum_{\xi\in F} \limsup_{n\to\infty} \left\langle \frac{1}{2}(|v^*x_n|\xi + |x_n^*v|)\xi, \xi \right\rangle = \limsup_{n\to\infty} \frac{1}{2} \sum_{\xi\in F} \langle |v^*x_n|\xi, \xi\rangle + \frac{1}{2} \sum_{\xi\in F} \langle |x_n^*v|\xi, \xi\rangle$$

Now, Proposition 3.1.5 and Theorem 3.1.7 imply $v^*x_n, x_n^*v \in L^1(B(\mathcal{H}))$ with $\|v^*x_n\|_1 \leq \|x_n\|1$ and $\|x_n^*v\|_1 \leq \|x_n^*\|_1 = \|x_n\|_1$. This implies we can continue our above estimate with

$$\sum_{\xi\in F} \langle |x|\xi, \xi\rangle \leq \limsup_{n\to\infty} \|x_n\|_1,$$

and this last expression is finite since $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the $\|\cdot\|_1$ norm. Since $F \subset \mathcal{E}$ was an arbitrary finite subset, we obtain

$$\|x\|_1 \leq \limsup_{n\to\infty} \|x_n\|_1 < \infty.$$

Thus $x \in L^1(B(\mathcal{H}))$. Now, let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N$. Since $x - x_N \in L^1(B(\mathcal{H}))$ we can find a finite subset $F \subset \mathcal{E}$ so that

$$\|x - x_N\|_1 \leq \sum_{\xi\in F} \langle |x - x_N|\xi, \xi\rangle + \epsilon = \sum_{\xi\in F} |\langle (x - x_N)\xi, v\xi\rangle| + \epsilon,$$

where $x - x_N = v|x - x_N|$ is the polar decomposition. Recalling that x is the norm limit of $(x_n)_{n\in\mathbb{N}}$, let $n \geq N$ be large enough so that $\|x - x_n\| < \frac{\epsilon}{|F|}$. Note that this implies

$$\sum_{\xi\in F} |\langle (x - x_n)\xi, v\xi\rangle| \leq \sum_{\xi\in F} \|(x - x_n)\xi\|\|v\xi\| < \sum_{\xi\in F} \frac{\epsilon}{|F|}\|\xi^2 = \epsilon.$$

So we have

$$\begin{aligned}
\|x - x_N\|_1 &\leq \sum_{\xi\in F} |\langle (x - x_N)\xi, v\xi\rangle| + \epsilon \leq \sum_{\xi\in F} |\langle (x - x_n)\xi, v\xi\rangle| + |\langle (x_n - x_N)\xi, v\xi\rangle| + \epsilon \\
&\leq \sum_{\xi\in F} |\langle v^*(x_n - x_N)\xi, \xi\rangle| + 2\epsilon \leq \sum_{\xi\in F} \left\langle \frac{1}{2}(|v^*(x_n - x_N)| + |(x_n - x_N)^*v|)\xi, \xi \right\rangle + 2\epsilon \\
&\leq \frac{1}{2}\|v^*(x_n - x_N)\|_1 + \frac{1}{2}\|(x_n - x_N)^*v\|_1 + 2\epsilon \leq \|x_n - x_N\|_1 + 2\epsilon < 3\epsilon.
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ also converges to x in the $\|\cdot\|_1$ norm, and so $L^1(B(\mathcal{H}))$ is complete. □

For $x \in L^1(B(\mathcal{H}))$, the density of $FR(\mathcal{H})$ implies the existence of a sequence $(x_n)_{n\in\mathbb{N}} \subset FR(\mathcal{H})$ such that $\|x - x_n\|_1 \to 0$. Since $\|x - x_n\| \leq \|x - x_n\|_1$, this sequence also converges to x in the operator norm. Since operator norm limits of finite-rank operators are compact operators, we obtain $x \in K(\mathcal{H})$. This yields the following corollary:

Corollary 3.1.9. $L^1(B(\mathcal{H})) \subset K(\mathcal{H})$.

Theorem 3.1.7 implies that for any $x \in L^1(B(\mathcal{H}))$ and $y \in B(\mathcal{H})$ we have

$$|\mathrm{Tr}(xy)| \leq \|x\|_1\|y\|.$$

This shows simultaneously that $x \mapsto \mathrm{Tr}(xy)$ and $y \mapsto \mathrm{Tr}(yx)$ are bounded linear functionals on $L^1(B(\mathcal{H}))$ and $B(\mathcal{H})$, respectively, with norms at most $\|y\|$ and $\|x\|_1$, respectively. From this we will deduce two important facts. The first is that the dual of $L^1(B(\mathcal{H}))$ as a Banach space with norm $\|\cdot\|_1$ is $B(\mathcal{H})$. The second, which will show in the next theorem, is that $L^1(B(\mathcal{H}))$ is itself the dual of $K(\mathcal{H})$ as a Banach space with the operator norm. You should compare this with $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$. Since dual spaces of Banach spaces are always Banach spaces (see [Exercise 2.5, GOALS Prerequisite Notes]), this gives another proof—albeit an indirect one—that $L^1(B(\mathcal{H}))$ is a Banach space.

Theorem 3.1.10. *For $x \in L^1(B(\mathcal{H}))$, define $\psi_x\colon K(\mathcal{H}) \to \mathbb{C}$ by $\psi_x(y) = Tr(xy)$. Then the map*

$$\begin{aligned}
L^1(B(\mathcal{H})) &\to K(\mathcal{H})^* \\
x &\mapsto \psi_x
\end{aligned}$$

is an isometric isomorphism.

Proof. Fix $x \in L^1(B(\mathcal{H}))$. The discussion preceding the statement of the theorem implies $\psi_x \in K(\mathcal{H})^*$ with $\|\psi_x\| \leq \|x\|_1$. Conversely, let $x = v|x|$ be the polar decomposition and let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal basis. Let $\epsilon > 0$ and let $F \subset \mathcal{E}$ be a finite subset such that

$$\|x\|_1 \leq \sum_{\xi \in F} \langle |x|\xi, \xi\rangle + \epsilon.$$

Let $p \in FR(\mathcal{H})$ be the projection onto $\operatorname{span} F$. Then $pv^* \in FR(\mathcal{H}) \subset K(\mathcal{H})$ and

$$\|x\|_1 - \epsilon = \sum_{\xi \in F} \langle |x|\xi, \xi\rangle = \sum_{\xi \in \mathcal{E}} \langle v^* x\xi, p\xi\rangle = |\mathrm{Tr}(pv^*x)| = |\mathrm{Tr}(xpv^*)| \leq \|\psi_x\| \|pv^*\| \leq \|\psi_x\|.$$

Thus $\|x\|_1 = \|\psi_x\|$. It remains to show that $x \mapsto \psi_x$ is surjective.

Let $\psi \in K(\mathcal{H})^*$. Then for any $\xi, \eta \in \mathcal{H}$ we have

$$|\psi(\xi \otimes \bar{\eta})| \leq \|\psi\| \|\xi \otimes \bar{\eta}\| \leq \|\psi\| \|\xi\| \|\eta\|,$$

where the last inequality follows from [Exercise 7.49, Day 1 Lectures]. Thus $(\xi, \eta) \mapsto \psi(\xi \otimes \bar{\eta})$ is a bounded sesquilinear form, and so

$$\psi(\xi \otimes \bar{\eta}) = \langle x\xi, \eta\rangle \qquad \forall \xi, \eta \in \mathcal{H}.$$

for some $x \in B(\mathcal{H})$ by Lemma 2.1.2. If $x = v|x|$ is the polar decomposition, then for any finite subset $F \subset \mathcal{E}$ define

$$a_F := \sum_{\xi \in F} \xi \otimes \overline{v\xi} \in FR(\mathcal{H}).$$

Note that $a_F = pv^*$ where p is the projection onto $\operatorname{span} F$, and consequently $\|a_F\| \leq \|p\| \|v^*\| \leq 1$. We then have

$$\sum_{\xi \in F} \langle |x|\xi, \xi\rangle = \sum_{\xi \in F} \langle x\xi, v\xi\rangle = \sum_{\xi \in F} \psi(\xi \otimes \overline{v\xi}) = \psi(a_F) \leq \|\psi\| \|a_F\| \leq \|\psi\|.$$

Consequently, $x \in L^1(B(\mathcal{H}))$ with $\|x\|_1 \leq \|\psi\|$. We claim that $\psi_x = \psi$. Since they are both bounded linear functionals on $K(\mathcal{H})$, it suffices to show they agree on the dense subspace $FR(\mathcal{H})$. Given $a \in FR(\mathcal{H})$, [Theorem 7.10, Day 1 Lectures] we can find $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ so that

$$a = \sum_{j=1}^n \xi_j \otimes \overline{\eta_j}.$$

Then

$$\psi(a) = \sum_{j=1}^n \psi(\xi_j \otimes \overline{\eta_j}) = \sum_{j=1}^n \langle x\xi_j, \eta_j\rangle,$$

while using Exercise 3.1.8 we have

$$\psi_x(a) = \sum_{j=1}^n \mathrm{Tr}(x(\xi_j \otimes \overline{\eta_j})) \sum_{j=1}^n \mathrm{Tr}((x\xi_j) \otimes \overline{\eta_j}) = \sum_{j=1}^n \langle x\xi_j, \eta_j\rangle.$$

Thus $\psi = \psi_x$ and so $x \mapsto \psi_x$ is surjective. $\square$

Remark 3.1.11. As indicated by the discussion preceding Theorem 3.1.10, for $x \in L^1(B(\mathcal{H}))$ the linear functional ψ_x naturally extends to a linear functional $\Psi_x : B(\mathcal{H}) \to \mathbb{C}$. Moreover, the same argument as in Theorem 3.1.10 shows $\|\Psi_x\| = \|x\|_1$. Hence $x \mapsto \Psi_x$ is an isometric embedding of $L^1(B(\mathcal{H}))$ in $B(\mathcal{H})^*$. However, this mapping is not surjective. Morally, the reason is because $FR(\mathcal{H})$ is not operator norm dense in $B(\mathcal{H})$ like it is in $K(\mathcal{H})$, and we showed surjectivity by analyzing these functionals only on $FR(\mathcal{H})$. The technical reason for the lack of surjectivity follows from some facts about Banach spaces: since $K(\mathcal{H})$ is not dense in $B(\mathcal{H})$ (assuming $\mathcal{H}$ is infinite dimensional) the Hahn–Banach Theorem implies the existence of a non-trivial $\phi \in B(\mathcal{H})^*$ with $\phi|_{K(\mathcal{H})} \equiv 0$. We cannot have $\phi = \Psi_x$ for any $x \in L^1(B(\mathcal{H}))$ because $\Psi_x|_{K(H)} = \psi_x$ being identically zero implies $x = 0$ which in turn implies $\Psi_x \equiv 0$.

We conclude this section by showing that $B(\mathcal{H})$ is the dual of $L^1(B(\mathcal{H}))$.

Theorem 3.1.12. *For $y \in B(\mathcal{H})$, define $\phi_y \colon L^1(B(\mathcal{H})) \to \mathbb{C}$ by $\phi_y(x) = Tr(xy)$. Then the map*

$$\begin{aligned} B(\mathcal{H}) &\mapsto L^1(B(\mathcal{H}))^* \\ y &\mapsto \phi_y \end{aligned}$$

is an isometric isomorphism.

Proof. Fix $y \in B(\mathcal{H})$. The discussion preceding Theorem 3.1.10 implies $\phi_y \in L^1(B(\mathcal{H}))^*$ with $\|\phi_y\| \leq \|y\|$. Conversely, recall that

$$\|y\| = \||y|\| = \sup_{\|\xi\|=1} \langle |y|\xi, \xi\rangle .$$

Let $\epsilon > 0$ and let $\xi \in \mathcal{H}$ be a unit vector satisfying

$$\|y\| \leq \langle |y|\xi, \xi\rangle + \epsilon.$$

Note that $\xi \otimes \bar{\xi}$ is a rank 1 projection and so $\|\xi \otimes \bar{\xi}\|_1 = 1$ by [Example 7.21, Day 1 Lectures]. We therefore have

$$\begin{aligned} \|y\| - \epsilon \leq \langle |y|\xi, \xi\rangle = |\langle y\xi, v\xi\rangle| &= |\mathrm{Tr}((y\xi) \otimes \overline{v\xi})| = |\mathrm{Tr}(y(\xi \otimes \bar{\xi})v^*)| = |\mathrm{Tr}((\xi \otimes \bar{\xi})v^* y)| \\ &= |\phi_y((\xi \otimes \bar{\xi})v^*)| \leq \|\phi_y\| \|(\xi \otimes \bar{\xi})v^*\|_1 \leq \|\phi_y\| \|\xi \otimes \bar{\xi}\|_1 \|v^*\| \leq \|\phi_y\|. \end{aligned}$$

Thus $\|y\| = \|\phi_y\|$. It remains to show that $y \mapsto \phi_y$ is surjective.

Let $\phi \in L^1(B(\mathcal{H}))$. Then for any $\xi, \eta \in \mathcal{H}$ we have

$$|\phi(\xi \otimes \bar{\eta})| \leq \|\phi\| \|\xi \otimes \bar{\eta}\|_1 \leq \|\phi\| \|\xi\| \|\eta\|,$$

where the last inequality follows from Exercise 3.1.8. Thus $(\xi, \eta) \mapsto \phi(\xi \otimes \bar{\eta})$ is a bounded sesquilinear form, and so

$$\phi(\xi \otimes \bar{\eta}) = \langle y\xi, \eta\rangle \qquad \forall \xi, \eta \in \mathcal{H}$$

for some $y \in B(\mathcal{H})$ by Lemma 2.1.2. Proceeding exactly as in Theorem 3.1.10 we can show ϕ and ϕ_y agree on finite-rank operators, which are dense in $L^1(B(\mathcal{H}))$. Thus $\phi = \phi_y$ and the map $y \mapsto \phi_y$ is surjective. $\square$

Exercises

3.1.1. Let $x \in B(\mathcal{H})$.

(a) Show that

$$\||x|\xi\| = \|x\xi\| \qquad \forall \xi \in \mathcal{H}.$$

(b) Show that $\ker(|x|) = \ker(x)$ and $\overline{\operatorname{ran}(|x|)} = \overline{\operatorname{ran}(x^*)}$.

3.1.2. Find the polar decomposition for $x \in B(\mathcal{H})$ when:

(a) x is positive;

(b) x is a partial isometry;

(c) x is a projection.

3.1.3. Find the polar decomposition for

$$\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} \in M_n(\mathbb{C}).$$

3.1.4. Let $\{p_i : i \in I\} \subset B(\mathcal{H})$ be an orthogonal family of projections and let $\{z_i : i \in I\} \subset \mathbb{C}$ be a bounded subset. Show $\sum_i z_i p_i$ defines a bounded operator on $\mathcal{H}$ and find its polar decomposition.

3.1.5. Show that if $x \in B(\mathcal{H})$ is self-adjoint with polar decomposition $x = v|x|$, then v is self-adjoint.

3.1.6. Let $x \in L^1(B(\mathcal{H}))$ with polar decomposition $x = v|x|$. Verify the identity

$$x = v|x| = \frac{1}{4} \sum_{k=0}^{3} i^k (v + i^k)|x|(v + i^k)^*,$$

by expanding the right-hand side.

3.1.7. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra.

(a) For $x \in M$ self-adjoint with $\|x\| \leq 1$, show that

$$u := x + i\sqrt{1 - x^2}$$

is a unitary operator in M.

(b) Show that $x = \frac{1}{2}(u + u^*)$.

(c) Show that element of M can be written as a linear combination of four unitaries in M.

3.1.8. Show that $FR(\mathcal{H}) \subset L^1(B(\mathcal{H}))$ and that $\operatorname{Tr}(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle$ and $\|\xi \otimes \eta\|_1 \leq \|\xi\|\|\eta\|$ for $\xi, \eta \in \mathcal{H}$.

3.1.9. Let $p \in FR(\mathcal{H})$ be a rank n projection. Show that $\operatorname{Tr}(p) = n$.

3.1.10. For $x \in L^1(B(\mathcal{H}))$, $a \in B(\mathcal{H})$, and any orthonormal basis $\mathcal{E} \subset \mathcal{H}$ we have

$$\sum_{\xi \in \mathcal{E}} \||x|^{1/2} a\xi\|^2 \leq \|x\|_1 \|a\|^2.$$

3.1.11. Let $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be sequences such that $\sum_n \|\xi_n\|^2, \sum_n \|\eta_n\|^2 < \infty$. Show that the series

$$\sum_{n\in\mathbb{N}} \xi_n \otimes \overline{\eta}_n$$

converges (in the $\|\cdot\|_1$ norm) to an element of $L^1(B(\mathcal{H}))$ with

$$\operatorname{Tr}\left(\sum_{n\in\mathbb{N}} \xi_n \otimes \overline{\eta}_n\right) = \sum_{n\in\mathbb{N}} \langle \xi_n, \eta_n \rangle .$$

3.2 The σ-Topologies

Unless explicitly stated otherwise, the weak* topology on $B(\mathcal{H})$ will from now on mean the topology induced by the duality $B(\mathcal{H}) = L^1(B(\mathcal{H}))^*$ established in Theorem 3.1.12. In this section we will analyze the weak* topology on $B(\mathcal{H})$. In particular, we will produce alternate characterizations of the weak* topology that are more readily generalized to arbitrary von Neumann algebras. Let us begin by showing that weak* convergence implies WOT convergence.

Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a net converging weak* to some $x \in B(\mathcal{H})$. For $\xi, \eta \in \mathcal{H}$, we have $\xi \otimes \bar{\eta} \in FR(\mathcal{H}) \subset L^1(B(\mathcal{H}))$ and using Exercise 3.1.8 we have

$$\langle x_i\xi, \eta\rangle = \mathrm{Tr}((x_i\xi)\otimes\bar{\eta}) = \mathrm{Tr}(x_i(\xi\otimes\bar{\eta})) \to \mathrm{Tr}(x(\xi\otimes\bar{\eta})) = \langle x\xi, \eta\rangle .$$

Thus $(x_i)_{i\in I}$ also converges to x in the WOT. However, it is not true that WOT convergence implies weak* convergence as the following example demonstrates.

Example 3.2.1. Let $\{e_n : n \in \mathbb{N}\} \subset \mathcal{H}$ be an orthonormal set, and for $m \leq n$ define

$$x_{m,n} := e_m \otimes \bar{e}_m + m^2 e_n \otimes \bar{e}_n.$$

Observe that $\{(m, n) : m, n \in \mathbb{N},\ m \leq n\}$ is a directed set under the following ordering: $(m, n) \leq (m', n')$ if and only if $m \leq m'$ and $n \leq n'$ (**Exercise:** check this), and so $(x_{m,n})_{m\leq n}$ gives a net in $B(\mathcal{H})$. This net converges to zero in the WOT (Exercise 3.2.1), but does not converge to zero in the weak* topology. To see this we will construct a particular trace class operator which witnesses this lack of convergence. Since

$$\sum_{n\in\mathbb{N}} \|\frac{1}{n}e_n\|^2 = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty,$$

Exercise 3.1.11 implies

$$a := \sum_{n\in\mathbb{N}} (\frac{1}{n}e_n) \otimes \overline{(\frac{1}{n}e_n)} = \sum_{n\in\mathbb{N}} \frac{1}{n^2} e_n \otimes \bar{e}_n \in L^1(B(\mathcal{H})).$$

Moreover, the same exercise tells us that for any bounded operator $x \in B(\mathcal{H})$ we have $\sum \frac{1}{n^2}(xe_n) \otimes \bar{e}_n \in L^1(B(\mathcal{H}))$ with

$$\mathrm{Tr}\left(\sum_{n\in\mathbb{N}} \frac{1}{n^2}(xe_n)\otimes\bar{e}_n\right) = \sum_{n\in\mathbb{N}} \frac{1}{n^2}\langle xe_n, e_n\rangle .$$

In particular, we compute

$$\mathrm{Tr}(x_{m,n}a) = \sum_{k\in\mathbb{N}} \frac{1}{k^2}\langle x_{m,n}e_k, e_k\rangle = \sum_{k\in\mathbb{N}} \frac{1}{k^2}\langle \langle e_k, e_m\rangle e_m + m^2\langle e_k, e_n\rangle e_n, e_k\rangle = \frac{1}{m^2} + \frac{m^2}{n^2}.$$

Thus $\mathrm{Tr}(x_{m,n}a)$ cannot converge to zero because for any $m \leq n$, we have $(m, n) \leq (n, n)$ and $\mathrm{Tr}(x_{n,n}a) = \frac{1}{n^2} + 1 \geq 1$. ■

The key idea in the previous example was to use the fact that weak* convergence implies a stronger variation of WOT convergence: if a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges weak* to some $x \in B(\mathcal{H})$ then for any sequences $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ satisfying $\sum_n \|\xi_n\|^2, \sum_n \|\eta_n\|^2 < \infty$ we have (using Exercise 3.1.11)

$$\sum_{n\in\mathbb{N}} \langle x_i\xi_n, \eta_n\rangle = \mathrm{Tr}\left(\sum_{n\in\mathbb{N}}(x_i\xi_n)\otimes\bar{\eta}_n\right) = \mathrm{Tr}\left(x_i\sum_{n\in\mathbb{N}}\xi_n\otimes\bar{\eta}_n\right) \to \mathrm{Tr}\left(x\sum_{n\in\mathbb{N}}\xi_n\otimes\bar{\eta}_n\right) = \sum_{n\in\mathbb{N}}\langle x\xi_n, \eta_n\rangle .$$

This compels us to define another topology on $B(\mathcal{H})$ with this mode of convergence. There is also a similarly strengthened version of SOT convergence and a corresponding topology which we will define at the same. Note that the sequences of vectors $(xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ are really elements of the Hilbert space $\ell^2(\mathbb{N}, \mathcal{H})$ (see [Section 1.3, GOALS Prerequisite Notes]).

Definition 3.2.2. The **σ-strong operator topology** (**σ-SOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U\left(x; (\xi_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\xi_n^{(d)})_{n\in\mathbb{N}}; \epsilon\right) := \left\{ y \in B(\mathcal{H}) \colon \left(\sum_{n\in\mathbb{N}} \|(x-y)\xi_n^{(j)}\|^2 \right)^{\frac{1}{2}} < \epsilon,\ j=1,\ldots,d \right\},$$

for $x \in B(\mathcal{H})$, $(\xi_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\xi_n^{(d)})_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathcal{H})$, and $\epsilon > 0$.

The **σ-weak operator topology** (**σ-WOT**) on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$\begin{aligned} &U\left(x; (\xi_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\xi_n^{(d)})_{n\in\mathbb{N}};\ (\eta_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\eta_n^{(d)})_{n\in\mathbb{N}}; \epsilon\right) \\ &\qquad := \left\{ y \in B(\mathcal{H}) \colon \left| \sum_{n\in\mathbb{N}} \left\langle (x-y)\xi_n^{(j)}, \eta_n^{(j)} \right\rangle \right| < \epsilon,\ j=1,\ldots,d \right\}, \end{aligned}$$

for $x \in B(\mathcal{H})$, $(\xi_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\xi_n^{(d)})_{n\in\mathbb{N}}, (\eta_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\eta_n^{(d)})_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathcal{H})$, and $\epsilon > 0$.

As with the SOT and WOT, the formal definitions above are not as important to understand as what it means for nets of operators to converge in these topologies: a net $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the σ-SOT if and only if

$$\lim_{i\to\infty} \sum_{n\to\infty} \|(x-x_i)\xi_n\|^2 = 0 \qquad \forall (\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathcal{H}),$$

and $(x_i)_{i\in I}$ converges to x in the σ-WOT if and only if

$$\lim_{i\to\infty} \sum_{n\to\infty} \langle (x-x_i)\xi, \eta\rangle = 0 \qquad \forall (\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathcal{H}).$$

Operator norm convergence implies σ-SOT convergence, which in turn implies σ-WOT convergence, and both of the σ-convergences imply their non-σ counterparts (see Exercise 3.2.2). In general it is **not** true that SOT (resp. WOT) convergence implies σ-SOT (resp. σ-WOT) convergence. Indeed, the net $(x_{m,n})_{m\leq n}$ in Example 3.2.1 converges in the WOT but not the σ-WOT. In fact, the same net converges to zero in the SOT but not the σ-SOT. However, if a net is uniformly bounded then SOT (resp. WOT) convergence is equivalent to σ-SOT (resp. σ-WOT) convergence (see Exercise 3.2.4).

Also note that because σ-SOT convergence and σ-WOT convergence implies SOT and WOT convergence, respectively, any SOT (resp. WOT) closed subset of $B(\mathcal{H})$ is σ-SOT (resp. σ-WOT). closed. In particular, von Neumann algebras are both σ-SOT and σ-WOT closed.

Remark 3.2.3. Equipping $B(\mathcal{H})$ with the SOT, WOT, σ-SOT, or σ-WOT makes it into a *locally convex space*. This means the topology is determined by a family of seminorms $p \colon B(\mathcal{H}) \to [0,\infty)$, rather than a single norm like the operator norm topology. The σ-WOT, for example, is determined by seminorms of the form

$$p(x) = \left| \sum_{n\in\mathbb{N}} \langle (x-x_0)\xi_n, \eta_n\rangle \right|$$

for $x_0 \in B(\mathcal{H})$, $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathcal{H})$. Note that in the notation of Definition 3.2.2 we have

$$U(x_0; (\xi_n)_{n\in\mathbb{N}}; (\eta_n)_{n\in\mathbb{N}}; \epsilon) := \{x \in B(\mathcal{H}) \colon p(x) < \epsilon\},$$

and this set is convex since p is a seminorm. As you might guess, locally convex spaces are named for having an abundance of open convex subsets, and have a robust theory in functional analysis. We will not require the full force of this theory in our mini-courses, but we will note that the Hahn–Banach Separation Theorem (see [Theorem 4.7, GOALS Prerequisite Notes]) is applies in this greater generality: if $B(\mathcal{H})$ is equipped with a topology $\mathcal{T} \in \{\text{SOT, WOT, } \sigma\text{-SOT, } \sigma\text{-WOT}\}$ and $X, Y \subset B(\mathcal{H})$ are disjoint closed convex sets with Y compact, then there exists a continuous linear functional $\varphi \colon B(\mathcal{H}) \to \mathbb{C}$, $t \in \mathbb{R}$, and $\epsilon > 0$ such that

$$\text{Re } \varphi(x) < t < t+\epsilon < \text{Re } \varphi(y) \qquad \forall x \in X,\ y \in Y.$$

(Note that closed, compact, and continuous in the above statement all mean with respect to the chosen topology $\mathcal{T}$.)

From the discussion preceding Definition 3.2.2, we see that weak* convergence implies σ-WOT convergence. It turns out, as we shall see below, that the converse holds; that is, the weak* topology equals the σ-WOT. We first require a lemma.

Lemma 3.2.4. *Let $a \in L^1(B(\mathcal{H}))$. Then there exists $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H})$ satisfying*

$$Tr(xa) = \sum_{n\in\mathbb{N}} \langle x\xi_n, \eta_n\rangle \qquad \forall x \in B(\mathcal{H}).$$

Proof. Let $a = v|a|$ be the polar decomposition of a and let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$. Then Exercise 3.1.10 implies

$$\sum_{\zeta\in\mathcal{E}} \|v|a|^{1/2}\zeta\|^2, \sum_{\zeta\in\mathcal{E}} \||a|^{1/2}\zeta\|^2 \leq \|a\|_1.$$

Consequently, there are at most countably many $\zeta \in \mathcal{E}$ such that both $\|v|a|^{1/2}\zeta\| \neq 0$ and $\||a|^{1/2}\zeta\| \neq 0$. Enumerate them as $\{\zeta_n : n \in \mathbb{N}\}$ and set $\xi_n := v|a|^{1/2}\xi_n$ and $\eta_n := |a|^{1/2}\zeta_n$. The above inequalities imply $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H})$, and for any $x \in B(\mathcal{H})$ we have

$$\begin{aligned}\mathrm{Tr}(xa) &= \mathrm{Tr}(xv|a|) = \mathrm{Tr}(|a|^{1/2}xv|a|^{1/2})\\ &= \sum_{\zeta\in\mathcal{E}} \left\langle |a|^{1/2}xv|a|^{1/2}\zeta, \zeta\right\rangle = \sum_{n\in\mathbb{N}} \left\langle xv|a|^{1/2}\zeta_n, |a|^{1/2}\zeta_n\right\rangle = \sum_{n\in\mathbb{N}} \langle x\xi_n, \eta_n\rangle,\end{aligned}$$

as claimed. □

Theorem 3.2.5. *The weak* topology on $B(\mathcal{H}) \cong L^1(B(\mathcal{H}))^*$ equals the σ-weak operator topology.*

Proof. To show two topologies are equal, we need to show they have the same open sets. By taking complements, this is equivalent to showing they have the same closed sets. Since being closed is characterized by containing the limits of convergent nets, it suffices to show weak* convergence is equivalent to σ-WOT convergence. Let $(x_i)_{i\in I} \subset B(\mathcal{H})$ be a net and let $x \in B(\mathcal{H})$. We have already seen that $(x_i)_{i\in I}$ converging weak* to x implies it converges in the σ-WOT. Conversely, suppose the net conveges to x in the σ-WOT. Then previous lemma implies $\mathrm{Tr}(x_i a) \to \mathrm{Tr}(xa)$ for all $a \in L^1(B(\mathcal{H}))$. Hence the net also converges weak* to x. □

Recall that the Banach–Alaoglu Theorem says the closed unit ball of the dual of a Banach space is weak* compact. Thus we obtain the following as an immediate corollary.

Corollary 3.2.6. *The closed unit ball of $B(\mathcal{H})$ is σ-WOT compact.*

We will conclude this section by analyzing σ-SOT and σ-WOT continuous linear functionals. It will be helpful to first understand SOT and WOT continuous linear functionals.

Lemma 3.2.7. *Let $\varphi : B(\mathcal{H}) \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

(i) *φ is SOT continuous.*

(ii) *φ is WOT continuous.*

(iii) *There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ so that $\varphi(x) = \sum_{i=1}^n \langle x\xi_i, \eta_i\rangle$.*

(iv) *There exists $a \in FR(\mathcal{H})$ so that $\varphi(x) = Tr(xa)$.*

Proof. $(iii) \Leftrightarrow (iv)$ follows from Exercise 3.1.8 and [Theorem 7.10, Day 1 Lectures]. $(iii) \Rightarrow (ii)$ is immediate, and $(ii) \Rightarrow (i)$ follows from the fact that SOT convergence implies WOT convergence. So it suffices to prove $(i) \Rightarrow (iii)$.

Suppose φ is SOT continuous. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Then $\varphi^{-1}(\mathbb{D})$ contains an SOT open neighborhood of the zero operator. Consequently there exists $\epsilon > 0$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$ so that $\varphi(x) \in \mathbb{D}$ whenever $\|x\xi_i\| \leq \epsilon$ for each $j = 1, \ldots, n$. Consequently, $\varphi(x) \in \mathbb{D}$ holds whenever

$$\left(\sum_{i=1}^{n} \|x\xi_i\|^2\right)^{1/2} \leq \epsilon.$$

Thus

$$\left|\varphi\left(\frac{\epsilon x}{\left(\sum_{i=1}^{n} \|x\xi_i\|^2\right)^{1/2}}\right)\right| < 1 \qquad \forall x \in B(\mathcal{H}).$$

The linearity of φ then implies

$$|\varphi(x)| \leq \frac{1}{\epsilon}\left(\sum_{i=1}^{n} \|x\xi_i\|^2\right)^{1/2} \qquad \forall x \in B(\mathcal{H}).$$

It follows that

$$(x\xi_1, \ldots, x\xi_n) \mapsto \varphi(x).$$

is a well-defined, bounded map on the closure of $\{(x\xi_1, \ldots, x\xi_n) \in \mathcal{H}^{\oplus n} \colon x \in B(\mathcal{H})\}$. Thus there exists $(\eta_1, \ldots, \eta_n) \in \mathcal{H}^{\oplus n}$ such that

$$\varphi(x) = \langle (x\xi_1, \ldots, x\xi_n), (\eta_1, \ldots, \eta_n)\rangle = \sum_{i=1}^{n} \langle x\xi_i, \eta_i\rangle$$

for all $x \in B(\mathcal{H})$ (see [Theorem 1.35, GOALS Prerequisite Notes]). □

The previous lemma implies that the finite-rank operators are the dual of $B(\mathcal{H})$ equipped with either the SOT or WOT. This is one reason why the SOT and WOT are not sufficient for a comprehensive study of von Neumann algebras. We also have:

Corollary 3.2.8. *For $K \subset B(\mathcal{H})$ convex, the SOT and WOT closures coincide.*

Proof. $\overline{K}^{SOT} \subset \overline{K}^{WOT}$ holds for any set, not to mention convex ones, since SOT convergence implies WOT convergence. Suppose, towards a contradiction, that there exists $y \in \overline{K}^{WOT} \setminus \overline{K}^{SOT}$. In the SOT, $\overline{K}^{SOT}$ and $\{y\}$ are disjoint closed convex sets with $\{y\}$ compact. Remark 3.2.3 implies that there is SOT continuous linear functional $\varphi \colon B(\mathcal{H}) \to \mathbb{C}$, $t \in \mathbb{R}$, and $\epsilon > 0$ such that

$$\mathrm{Re}\ \varphi(x) < t < t + \epsilon < \mathrm{Re}\ \varphi(y) \qquad \forall x \in \overline{K}^{SOT}.$$

Lemma 3.2.7 tells us that φ is also WOT continuous and so $S := \{x \in B(\mathcal{H}) \colon \mathrm{Re}\ \varphi(x) \leq t\}$ is a WOT closed subset. Since $K \subset S$, we must have $\overline{K}^{WOT} \subset S$, but this contradicts $\mathrm{Re}\ \varphi(y) > t$. Thus we must have $\overline{K}^{SOT} = \overline{K}^{WOT}$. □

Theorem 3.2.9. *Let $\varphi \colon B(\mathcal{H}) \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

(i) φ is σ-SOT continuous.

(ii) φ is σ-WOT continuous.

(iii) There exists $(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H})$ so that $\varphi(x) = \sum_n \langle x\xi_n, \eta_n\rangle$.

(iv) There exists $a \in L^1(B(\mathcal{H}))$ so that $\varphi(x) = Tr(xa)$.

Proof. $(iii) \Rightarrow (ii)$ is immediate, and $(ii) \Rightarrow (i)$ follows from the fact that σ-SOT convergence implies σ-WOT convergence. $(iv) \Rightarrow (iii)$ follows from Lemma 3.2.4, and $(iii) \Rightarrow (iv)$ follows from Exercise 3.1.11. So it suffices to prove $(i) \Rightarrow (iii)$.

Suppose φ is σ-SOT continuous. The map $\pi\colon B(\mathcal{H}) \to B(\ell^2(\mathbb{N}, \mathcal{H}))$ given by defining $\pi(x)$ for $x \in B(\mathcal{H})$ by

$$\pi(x)(\xi_n)_{n\in\mathbb{N}} = (x\xi_n)_{n\in\mathbb{N}} \qquad (\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H}),$$

is an isometric $*$-homomorphism, and $(x_i)_{i\in I} \subset B(\mathcal{H})$ is σ-SOT convergent if and only if $(\pi(x))_{i\in I} \subset B(\ell^2(\mathbb{N}, \mathcal{H}))$ is SOT convergent (see Exercise 3.2.6). Thus if we define $\Phi\colon \pi(B(\mathcal{H})) \to \mathbb{C}$ by $\Phi := \varphi \circ \pi^{-1}$, then Φ is SOT continuous. Using the Hahn–Banach Theorem, we can find an SOT continuous extension of Φ to all of $B(\ell^2(\mathbb{N}, \mathcal{H}))$, which we also denote by Φ. Lemma 3.2.7 then yields $(\xi_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\xi_n^{(d)})_{n\in\mathbb{N}}$, $(\eta_n^{(1)})_{n\in\mathbb{N}}, \ldots, (\eta_n^{(d)})_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H})$ so that

$$\Phi(X) = \sum_{j=1}^{d} \left\langle X(\xi_n^{(j)})_{n\in\mathbb{N}}, (\eta_n^{(j)})_{n\in\mathbb{N}} \right\rangle \qquad X \in B(\ell^2(\mathbb{N}, \mathcal{H})).$$

In particular, for $x \in B(\mathcal{H})$ we have

$$\varphi(x) = \Phi(\pi(x)) = \sum_{j=1}^{d} \left\langle \pi(x)(\xi_n^{(j)})_{n\in\mathbb{N}}, (\eta_n^{(j)})_{n\in\mathbb{N}} \right\rangle = \sum_{j=1}^{d} \left\langle (x\xi_n^{(j)})_{n\in\mathbb{N}}, (\eta_n^{(j)})_{n\in\mathbb{N}} \right\rangle = \sum_{j=1}^{d} \sum_{n\in\mathbb{N}} \left\langle x\xi_n^{(j)}, \eta_n^{(j)} \right\rangle.$$

Re-indexing $\{(\xi_n^{(j)}, \eta_n^{(j)})\colon j = 1, \ldots, d,\ n \in \mathbb{N}\}$ yields the desired sequences. □

Note that a linear functional $\varphi\colon B(\mathcal{H}) \to \mathbb{C}$ satisfying any (hence all) of the above conditions is also norm continuous and hence bounded. The corollary below follows *mutatis mutandis* from the proof of Corollary 3.2.8.

Corollary 3.2.10. *For $K \subset B(\mathcal{H})$ convex, the σ-SOT and σ-WOT closures coincide.*

With σ-topologies in hand, we conclude with the following important definition.

Definition 3.2.11. Let $M \subset B(\mathcal{H})$ and $N \subset B(\mathcal{K})$ be von Neumann algebras. We say a map $\pi\colon M \to N$ is **normal** if it is σ-WOT continuous.

If $\varphi\colon M \to \mathbb{C}$ is a normal linear functional, then it satisifies all of the equivalent conditions in Theorem 3.2.9. In particular, it is σ-SOT continuous. If $\pi\colon M \to N$ is a normal $*$-homomorphism, then it is also σ-SOT continuous. Indeed, if $(x_i)_{\in I} \subset M$ converges to zero in the σ-SOT, then $(x_i^* x_i)_{i\in I}$ converges to zero in the σ-WOT by Exercise 3.2.3. Consequently $(\pi(x_i^* x_i))_{i\in I} = (\pi(x_i)^*\pi(x_i))_{i\in I}$ converges to zero in the σ-WOT by normality and invoking Exercise 3.2.3 again implies $(\pi(x_i))_{i\in I}$ converges to zero in the σ-SOT.

We will see a partial justification for why we call such maps normal in the next section, where it will be shown that the collection of normal linear functionals on a von Neumann algebra plays the role of the predual. The full justification (whose proof we must delay a little longer, see Theorem 4.2.8) is that if $\pi\colon M \to N$ is a normal unital $*$-homomorphism, then $\pi(M)$ is a von Neumann subalgebra of N. This tells us that in the category of von Neumann algebras, the correct morphisms to use are unital $*$-homomorphisms.

Exercises

3.2.1. Show that the net $(x_{m,n})_{m\leq n}$ in Example 3.2.1 converges to zero in the WOT. [**Hint:** use Theorem 1.22 in the Prerequisite Notes.]

3.2.2. Let $(x_i)_{i\in I} \subset B(\mathcal{H})$ be a net and let $x \in B(\mathcal{H})$. Consider the following statements:

(i) $(x_i)_{i\in I}$ converges to x in operator norm.

(ii) $(x_i)_{i\in I}$ converges to x in the σ-strong operator topology.

(iii) $(x_i)_{i\in I}$ converges to x in the σ-weak operator topology.

(iv) $(x_i)_{i\in I}$ converges to x in the strong operator topology.

(v) $(x_i)_{i\in I}$ converges to x in the weak operator topology.

Verify implications and non-implications in the following diagram:

$$\begin{tikzcd}
 & & \text{(iii)} \arrow[dl, shift left, "/" marking] \arrow[dr, shift left] \arrow[d, shift right, "/" marking] & \\
\text{(i)} \arrow[r, shift left] & \text{(ii)} \arrow[l, shift left, "/" marking] \arrow[ur, shift left] \arrow[dr, shift left] & & \text{(v)} \arrow[ul, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \\
 & & \text{(iv)} \arrow[ul, shift left, "/" marking] \arrow[ur, shift left] \arrow[u, shift right, "/" marking] &
\end{tikzcd}$$

3.2.3. Show that $(x_i)_{\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the σ-strong operator topology if and only if $((x - x_i)^*(x - x_i))_{i\in I}$ converges to zero in the σ-weak operator topology.

3.2.4. Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a uniformly bounded net: $\sup_i \|x_i\| < \infty$.

(a) Show that $(x_i)_{i\in I}$ converges in the σ-SOT if and only if it converges in the SOT.

(b) Show that $(x_i)_{i\in I}$ converges in the σ-WOT if and only if it converges in the WOT.

3.2.5. Show that the net $(x_{m,n})_{m\leq n}$ in Example 3.2.1 is not uniformly bounded.

3.2.6. For $x \in B(\mathcal{H})$, define a linear operator $\pi(x)$ on $\ell^2(\mathbb{N}, \mathcal{H})$ by

$$\pi(x)(\xi_n)_{n\in\mathbb{N}} = (x\xi_n)_{n\in\mathbb{N}}$$

(a) Show that $\pi(x) \in B(\ell^2(\mathbb{N}, \mathcal{H}))$ with $\|\pi(x)\| = \|x\|$ for all $x \in B(\mathcal{H})$.

(b) Show that $\pi\colon B(\mathcal{H}) \to B(\ell^2(\mathbb{N}, \mathcal{H}))$ is a $*$-homomorphism.

(c) Show that $U \subset B(\mathcal{H})$ is open in the σ-SOT (resp. σ-WOT) if and only if $\pi(U) \subset B(\ell^2(\mathbb{N}, \mathcal{H}))$ is open in the SOT (resp. WOT).

3.2.7. Show that a spatial isomorphism $\pi\colon M \to N$ is normal.

3.3 The Predual of a von Neumann Algebra

Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Our goal in this section is to find a Banach space M_* satisfying $(M_*)^* = M$ (i.e. the *predual* of M). Thanks to Theorem 3.1.12, we already know the answer for one case: $B(\mathcal{H})_* = L^1(B(\mathcal{H}))$. Starting with this fact, one is virtually guaranteed to stumble across the general answer after citing enough Banach space facts. But in order to avoid feeling like we are wandering around in the dark, we will first give a definition for the predual and check the desired properties afterwards. We can still partially motivate our definition.

The pairing

$$B(\mathcal{H}) \times L^1(\mathcal{H}) \ni (x, a) \mapsto \operatorname{Tr}(xa)$$

allows us to identify $B(\mathcal{H})$ with the dual of $L^1(B(\mathcal{H}))$. It also allows us to identify $L^1(B(\mathcal{H}))$ with a subspace of the dual of $B(\mathcal{H})$: $L^1(B(\mathcal{H})) \ni a \mapsto \operatorname{Tr}(\,\cdot\, a)$. Indeed, from the proof of Theorem 3.1.10 we know this is an isometric embedding. Moreover, Theorem 3.2.9 tells us the image of this embedding is precisely the σ-WOT continuous linear functionals. Thus we make the following definition.

Definition 3.3.1. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. We denote the set of all normal linear functionals by M_* and call this set the **predual** of M.

Note that a normal linear functional is also norm continuous and hence bounded; that is, $M_* \subset M^*$ and moreover M_* is a subspace (Exercise 3.3.1). We should also remark that it is not at all obvious from this definition that the dual of M_* is M. It is not even clear that M_* is a Banach space, though it is a normed space as a subspace of M^*. We will check all of these details below and begin with a lemma.

Lemma 3.3.2. *Let $X \subset B(\mathcal{H})$ be a subspace. Then any σ-WOT continuous linear functional $\varphi\colon X \to \mathbb{C}$ has a σ-WOT continuous extension to $B(\mathcal{H})$.*

Proof. The set
$$U := \{x \in X : |\varphi(x)| < 1\}$$
is a σ-WOT open neighborhood of zero. Consequently, there are $(\xi_n^{(j)})_{n\in\mathbb{N}}, (\eta_n^{(j)})_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathcal{H})$, $j = 1\ldots, d$, and $\epsilon > 0$ so that
$$\left\{x \in X : \left|\sum_{n\in\mathbb{N}} \left\langle x\xi_n^{(j)}, \eta_n^{(j)}\right\rangle\right| < \epsilon,\ j = 1, \ldots, d\right\} \subset U.$$
Define a seminorm on $B(\mathcal{H})$ by
$$p(x) := \frac{2}{\epsilon}\sum_{j=1}^{d}\left|\sum_{n\in\mathbb{N}} \left\langle x\xi_n^{(j)}, \eta_n^{(j)}\right\rangle\right|.$$
Observe that $\{x \in X : p(x) \leq 1\} \subset U$. It follows that $|\varphi(x/p(x))| < 1$ for all $x \in X$, or equivalently $|\varphi(x)| < p(x)$ for all $x \in X$. Thus we can apply the Hahn–Banach theorem to find an extension of φ to $B(\mathcal{H})$ (which we will continue to denote by φ) that satisfies $|\varphi(x)| \leq p(x)$ for all $x \in B(\mathcal{H})$. This extension is σ-WOT continuuous because if $(x_i)_{i\in I} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the σ-WOT, then $|\varphi(x) - \varphi(x_i)| = |\varphi(x_i - x)| \leq p(x_i - x) \to 0$. $\square$

The above lemma in conjunction with Theorem 3.2.9 tells us that for any $\varphi \in M_*$, there exists $a \in L^1(B(\mathcal{H}))$ with $\phi = \psi_a|_M$, where $\psi_a(x) = \mathrm{Tr}(xa)$ for $x \in M$. Thus $L^1(B(\mathcal{H})) \ni a \mapsto \psi_a|_M$ is a surjection onto M_*, and so if we can understand its kernel then we can identity M_* with a quotient of $L^1(B(\mathcal{H}))$. Suppose $\psi_a|_M \equiv 0$ for some $a \in L^1(B(\mathcal{H}))$. This is equivalent to $\mathrm{Tr}(xa) = 0$ for all $x \in M$. Thus the kernel of the map $a \mapsto \psi_a|_M$ is the set
$$M_\perp := \{a \in L^1(B(\mathcal{H})) : \mathrm{Tr}(xa) = 0\ \forall x \in M.\}$$
This is a closed subspace of $L^1(B(\mathcal{H}))$ (Exercise 3.3.2), so we can consider the quotient Banach space $L^1(B(\mathcal{H}))/M_\perp$ with the norm given by:
$$\|a + M_\perp\|_1 := \inf_{b\in M_\perp} \|a + b\|_1.$$
Observe that if $a + M_\perp = b + M_\perp$ in $L^1(B(\mathcal{H}))/M_\perp$, then $a - b \in M_\perp$ and so $\mathrm{Tr}(x(a - b)) = 0$ or $\mathrm{Tr}(xa) = \mathrm{Tr}(xb)$ for all $x \in M$. That is, $\psi_a|_M = \psi_b|_M$. Hence $L^1(B(\mathcal{H}))/M_\perp \ni a + M_\perp \mapsto \psi_a|_M$ is a well-defined map, and from the above discussion we know it is a surjection onto M_*. We will see below that it is in fact an isometric isomorphism, but we first require a lemma.

Lemma 3.3.3. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and consider the set*
$$(M_\perp)^\perp := \{y \in B(\mathcal{H}) : Tr(ya) = 0\ \forall a \in M_\perp\}.$$
Then $(M_\perp)^\perp = M$.

Proof. We first note that $M \subset (M_\perp)^\perp$ by definition of $M_\perp$. Suppose, towards a contradiction, that there exists $y \in (M_\perp)^\perp \setminus M$. Then M and $\{y\}$ are disjoint σ-WOT closed convex sets with $\{y\}$ compact, and so by Remark 3.2.3 there is a σ-WOT continuous linear functional $\varphi\colon B(\mathcal{H}) \to \mathbb{C}$, $t \in \mathbb{R}$, and $\epsilon > 0$ such that
$$\mathrm{Re}\ \varphi(x) < t < t + \epsilon < \mathrm{Re}\ \varphi(y) \qquad \forall x \in X.$$
Because φ is σ-WOT continuous, we know from Theorem 3.2.9 that $\varphi = \mathrm{Tr}(\cdot\, b)$ for some $b \in L^1(B(\mathcal{H}))$. If $b \in M_\perp$, then $\mathrm{Re}\ \varphi(x) = 0$ for all $x \in M$, but also $\mathrm{Re}\ \varphi(y) = 0$ by definition of $(M_\perp)^\perp$, which contradicts the above inequalities. If $b \notin M_\perp$, then we can find $x_0 \in M$ so that $\varphi(x_0) \neq 0$. Letting $x_n := \frac{n}{\varphi(x_0)} x_0 \in M$ for each $n \in \mathbb{N}$, we have $\varphi(x_n) = n$. But then $t > \mathrm{Re}\ \varphi(x_n) = n$ for all $n \in \mathbb{N}$, another contradiction. So in either case we have a contradiction, and so we must have $(M_\perp)^\perp = M$. $\square$

Theorem 3.3.4. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $a \in L^1(B(\mathcal{H}))$, define $\psi_a \colon B(\mathcal{H}) \to \mathbb{C}$ by $\psi_a(x) = Tr(xa)$. Then the map*

$$\begin{aligned} L^1(B(\mathcal{H}))/M_\perp &\to M_* \\ a + M_\perp &\mapsto \psi_a|_M \end{aligned}$$

is an isometric isomorphism.

Proof. The discussion preceding the statement of the theorem implies this map is well-defined, valued in M_*, and is a surjection. Also note that it is linear by virtue of the trace being linear. Thus it suffices to show that this is an isometry.

Fix $a \in L^1(B(\mathcal{H}))$. For $x \in M$ we have

$$|\psi_a|_M(x)| = |\mathrm{Tr}(xa)| = \inf_{b \in M_\perp} |\mathrm{Tr}(x(a+b)| \leq \inf_{b \in M_\perp} \|x\|\|a+b\|_1 = \|x\|\|a + M_\perp\|_1.$$

Thus $\|\psi_a|_M\| \leq \|a + M_\perp\|_1$. Showing the reverse inequality will take a bit more work. First note that if $\|a + M_\perp\|_1 = 0$ then the previous inequality is automatically an equality, so we will assume $\|a + M_\perp\|_1 > 0$. In particular, this implies $a \notin M_\perp$. Now, since $L^1(B(\mathcal{H}))^* \cong B(\mathcal{H})$, we can use the Hahn–Banach Theorem to find a $y \in B(\mathcal{H})$ with $\|y\| = 1$ satisfying $\mathrm{Tr}(ya) = \|\psi_a\|$ and $\mathrm{Tr}(yb) = 0$ for all $b \in M_\perp$ (see Exercise 3.3.3). This means $y \in (M_\perp)^\perp$ and so $y \in M$ by Lemma 3.3.3. Thus

$$\psi_a|_M(y) = \psi_a(y) = \mathrm{Tr}(ya) = \|\psi_a\| = \|a\|_1,$$

where the last equality follows from Theorem 3.1.10. Since $\|y\| = 1$, this shows that $\|\psi_a|_M\| \geq |\psi_a(y)| = \|a\|_1 \geq \|a + M_\perp\|_1$ and the proof is complete. $\square$

Observe that one consequence of the previous theorem is that M_* is indeed a Banach space when equipped with the norm it inherits from $M_* \subset M^*$. Our final task of this section is to show prove that M is the dual of M_*. We will use the previous theorem to identify M_* with $L^1(B(\mathcal{H}))/M_\perp$ and establish $(L^1(B(\mathcal{H}))/M_\perp)^* \cong M$. This is actually a more generic result in Banach space theory (see Exercise 3.3.4), but we will only present the details for our situation (and they do not differ greatly from the generic ones anyways).

Theorem 3.3.5. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $x \in M$, define $\hat{x} \colon M_* \to \mathbb{C}$ by $\hat{x}(\varphi) = \varphi(x)$. Then the map*

$$\begin{aligned} M &\to (M_*)^* \\ x &\mapsto \hat{x} \end{aligned}$$

is an isometric isomorphism. Moreover, the weak topology on M induced by $M \cong (M_*)^*$ is the σ-WOT.*

Proof. Note that $x \mapsto \hat{x}$ is linear by virtue of the linearity of each $\varphi \in M_*$. For $x \in M$ and $\varphi \in M_*$ we have

$$|\hat{x}(\varphi)| = |\varphi(x)| \leq \|\varphi\|\|x\|,$$

and so $\|\hat{x}\| \leq \|x\|$. In particular, the map $x \mapsto \hat{x}$ is indeed valued in the dual of M_*. Fix $x \in M$ and let ϕ_x be as in Theorem 3.1.12. Recall that $\|\phi_x\| = \|x\|$, and so given $\epsilon > 0$ we can find $a \in L^1(B(\mathcal{H}))$ with $\|a\|_1 = 1$ so that $|\phi_x(a)| \geq \|x\| - \epsilon$. If we let ψ_a be as in Theorem 3.3.4, then $\psi_a|_M \in M_*$ with $\|\psi_a|_M\| = \|a + M_\perp\|_1 \leq \|a\|_1 = 1$. Thus

$$\frac{|\hat{x}(\psi_a|_M)|}{\|\psi_a|_M\|} \geq |\hat{x}(\psi_a|_M)| = |\psi_a(x)| = |\mathrm{Tr}(xa)| = |\phi_x(a)| \geq \|x\| - \epsilon.$$

So $\|\hat{x}\| \geq \|x\| - \epsilon$, and letting ϵ tend to zero yields $\|\hat{x}\| = \|x\|$.

Next we will show $x \mapsto \hat{x}$ is a surjection. Fix $\mu \in (M_*)^*$. By identifying $M_* \cong L^1(B(\mathcal{H}))/M_\perp$ as in Theorem 3.3.4, we can view μ as a linear functional on the quotient Banach space $L^1(B(\mathcal{H}))/M_\perp$. Let $Q \colon L^1(B(\mathcal{H})) \to L^1(B(\mathcal{H}))/M_\perp$ be the quotient map. Then $\mu \circ Q \in L^1(B(\mathcal{H}))^*$ and so $\mu \circ Q = \phi_y$ for some $y \in B(\mathcal{H})$ by Theorem 3.1.12. Note that for all $a \in M_\perp$ we have

$$\mathrm{Tr}(ya) = \phi_y(a) = \mu \circ Q(a) = \mu(0) = 0.$$

Hence $y \in (M_\perp)^\perp = M$ by Lemma 3.3.3. Consequently, for all $a \in L^1(B(\mathcal{H}))$ we have

$$\mu(a + M_\perp) = \mu \circ Q(a) = \phi_y(a) = \operatorname{Tr}(ya) = \psi_a|_M(y) = \hat{y}(\psi_a|_M).$$

That is, $\mu = \hat{y}$ (up to the identification $M_* \cong L^1(B(\mathcal{H}))/M_\perp$), and the map $x \mapsto \hat{x}$ is surjective.

Finally, we show that the weak* topology on M induced by $M \cong (M_*)^*$ is the σ-WOT. Suppose a net $(x_i)_{i\in I} \subset M$ converges to $x \in M$ in this weak* topology. Then $\varphi(x_i) \to \varphi(x)$ for all $\varphi \in M_*$. But Lemma 3.3.2 implies each $\varphi \in M_*$ is the restriction to M of a σ-WOT continuous linear functional on $B(\mathcal{H})$, and hence is of the form $\varphi = \operatorname{Tr}(\,\cdot\, a)$ for some $a \in L^1(B(\mathcal{H}))$ by Theorem 3.2.9. Consequently, $(x_i)_{i\in I}$ converges to x in the weak* topology on $B(\mathcal{H})$, which is the σ-WOT by Theorem 3.2.5. Conversely, if $(x_i)_{i\in I} \subset M$ converges to $x \in M$ in the σ-WOT, then $\varphi(x_i) \to \varphi(x)$ for all $\varphi \in M_*$ by definition of normality. Hence the net converges in the weak* topology on M. □

Since the weak* topology on M is just the σ-WOT, we obtain the following corollary from the Banach–Alaoglu Theorem.

Corollary 3.3.6. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Then the norm closed unit ball of M is σ-WOT compact.*

Remark 3.3.7. You might have noticed that we did not actually use the algebra structure of M at any point in this section, just that it was a σ-WOT closed (equivalently weak* closed) subspace. Thus all of the theorems we proved in this section hold when M is replaced by an arbitrary weak* closed subspace. In fact, as mentioned at the beginning of the section, analogous theorems hold when $L^1(B(\mathcal{H}))$ and $B(\mathcal{H})$ are replaced with an arbitrary Banach space and its dual. You will explore this greater generality in the exercises.

Exercises

3.3.1. Show that a linear combination of normal linear functionals is normal.

3.3.2. Let $\mathcal{X}$ be a Banach space with dual space $\mathcal{X}^*$. For $Y \subset \mathcal{X}$ define

$$Y^\perp := \{x^* \in \mathcal{X}^* \colon x^*(y) = 0 \ \forall y \in Y\},$$

and for $Y^* \subset \mathcal{X}^*$ define

$$Y^*_\perp := \{x \in \mathcal{X} \colon y^*(x) = 0 \ \forall y^* \in Y^*\}.$$

(a) For $Y \subset \mathcal{X}$, show that $Y^\perp$ is a weak* closed subspace.

(b) For $Y^* \subset \mathcal{X}^*$, show that $Y^*_\perp$ is weakly closed subspace.

(c) For a subspace $Y \subset \mathcal{X}$, show that $(Y^\perp)_\perp$ is the weak closure of Y.

(d) For a subspace $Y^* \subset \mathcal{X}^*$, show that $(Y^*_\perp)^\perp$ is the weak* closure of Y^*.

3.3.3. Let $\mathcal{X}$ be a Banach space, $Y \subset \mathcal{X}$ a closed subspace, and $x_0 \in \mathcal{X} \setminus Y$.

(a) Show that $Z := \operatorname{span}(Y \cup \{x_0\})$ is closed.

(b) Show that $Z/Y \cong \mathbb{C}$.

(c) Find a $\varphi \in Z^*$ with $\|\varphi\| = 1$ satisfying $\varphi(x_0) = \|x_0\|$ and $\varphi|_Y \equiv 0$.

(d) Find a $\varphi \in \mathcal{X}^*$ with $\|\varphi\| = 1$ satisfying $\varphi(x_0) = \|x_0\|$ and $\varphi|_Y \equiv 0$.

3.3.4. Let $\mathcal{X}$ be a Banach space with $Y \subset \mathcal{X}$ a closed subspace.

(a) Show that $\mathcal{X}^*/Y^\perp \ni x^* + Y^\perp \mapsto x^*|_Y \in Y^*$ is an isometric isomorphism.

(b) Let $Q \colon \mathcal{X} \to \mathcal{X}/Y$ be the quotient map. Show that $(\mathcal{X}/Y)^* \ni x^* \mapsto x^* \circ Q \in Y^\perp$ is an isometric isomorphism.

3.3.5. Show that a map $\pi \colon M \to N$ between two von Neumann algebras is normal if and only if $\varphi \circ \pi \in M_*$ for all $\varphi \in N_*$.

3.4 The Kaplansky Density Theorem

Suppose $A \subset B(\mathcal{H})$ is a unital $*$-algebra and $M := A''$ is the von Neumann algebra generated by A. By the Bicommutant Theorem, we know M is also the SOT closure of A and thus for any $x \in M$ there is a net $(x_i)_{i\in I} \subset A$ which converges to x in the SOT. In this section, we show that the net $(x_i)_{i\in I}$ can be chosen so that $\|x_i\| \leq \|x\|$ for all $i \in I$. This is a result due to Kaplansky, and although it may not seem like it at first, it is the kind of result one uses every day and twice on Sundays. We begin with a pro.

Lemma 3.4.1. *Let $(x_i)_{i\in I}, (y_i)_{i\in I} \subset B(\mathcal{H})$ be nets (indexed by the same directed set I) that converge in then SOT. If $\sup_{i\in I} \|x_i\| < \infty$, then $(x_i y_i)_{i\in I}$ is SOT convergent.*

Proof. Let $x, y \in B(\mathcal{H})$ be the respective SOT limits of $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$, and set $R := \sup_i \|x_i\|$. For $\xi \in \mathcal{H}$ we have

$$\|xy\xi - x_i y_i \xi\| \leq \|(x - x_i)y\xi\| + \|x_i(y - y_i)\xi\| \leq \|(x - x_i)y\xi\| + R\|(y - y_i)\xi\| \to 0.$$

Thus $(x_i y_i)_{i\in I}$ converges to xy in the SOT. □

Proposition 3.4.2. *If $f \in C(\mathbb{C})$, then the map $x \mapsto f(x)$ on normal operators in $B(\mathcal{H})$ is SOT continuous on bounded subsets.*

Proof. Let $(x_i)_{i\in I} \subset B(\mathcal{H})$ be a net of uniformly bounded normal operators converging to $x \in B(\mathcal{H})$ in the SOT. Let $R = \sup_i \|x_i\|$, and note that $\|x\| \leq \limsup_i \|x_i\| \leq R$ (see Exercise 1.1.2). The Stone–Weierstrass theorem allows us to approximate f uniformly on $\{z \in \mathbb{C} \colon |z| \leq R\}$ by a sequence polynomials $(p_n(z, \bar{z}))_{n\in\mathbb{N}}$. Note that $(x_i^*)_{i\in I}$ converges to x^* in the SOT by Exercise 1.1.8, and since multiplication is SOT continuous on bounded sets (see Exercise 1.1.10) it follows that for each $n \in \mathbb{N}$ the net $(p_n(x_i, x_i^*))_{i\in I}$ converges to $p_n(x, x^*)$ in the SOT.

Now, fix $\xi \in \mathcal{H}$ and let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$\sup_{|z|\leq R} |f(z) - p_N(z, \bar{z})| < \frac{\epsilon}{3\|\xi\|}.$$

Then let $i_0 \in I$ be such that for all $i \geq i_0$

$$\|(p_N(x, x^*) - p_N(x_i, x_i^*))\xi\| < \frac{\epsilon}{3}.$$

We can then estimate for $i \geq i_0$

$$\begin{aligned}\|(f(x) - f(x_i))\xi\| &\leq \|(f(x) - p_N(x, x^*))\xi\| + \|(p_N(x, x^*) - p_N(x_i, xi_i^*))\xi\| + \|(p_N(x_i, x_i^*) - f(x_i))\xi\| \\ &< \|f(x) - p_N(x, x^*)\|\|\xi\| + \frac{\epsilon}{3} + \|f(x_i) - p_N(x_i, x_i^*)\|\|\xi\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus $(f(x_i))_{i\in I}$ converges to $f(x)$ in the SOT. □

Recall that if $x \in B(\mathcal{H})$ is self-adjoint, then $\sigma(x) \subset \mathbb{R}$. Consequently, $x + z$ is invertible for any $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$.

Definition 3.4.3. For a self-adjoint operator $x \in B(\mathcal{H})$, the operator

$$(x - i)(x + i)^{-1} \in B(\mathcal{H})$$

is called the **Cayley transform** of x.

Note that the Cayley transform is given by the continuous functional calculus $f(x)$ for $f(t) = \frac{t-i}{t+i}$. Using this one can show that the Cayley transform of x is a unitary operator and that $(x - i)(x + i)^{-1} = (x + i)^{-1}(x - i)$.

Proposition 3.4.4. *The Cayley transform is SOT continuous on self-adjoint operators.*

Proof. Let $(x_j)_{j\in J} \subset B(\mathcal{H})$ be a net of self-adjoint operators converging to a necessarily self-adjoint operator $x \in B(\mathcal{H})$. Note that by the continuous functional calculus $\|(x_j+i)^{-1}\| \leq 1$ for all $j \in J$. For $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x-i)(x+i)^{-1}\xi - (x_j-i)(x_j+i)^{-1}\xi\| &= \|(x-i)(x+i)^{-1}\xi - (x_j+i)^{-1}(x_j-i)\xi\| \\ &= \|(x_j+i)^{-1}\left[(x_j+i)(x-i) - (x_j-i)(x+i)\right](x-i)^{-1}\xi\| \\ &= \|2i(x_j+i)^{-1}\left[x-x_j\right](x-i)^{-1}\xi\| \\ &\leq 2\|[x-x_j](x-i)^{-1}\xi\|.\end{aligned}$$

Thus SOT convergence of $(x_j)_{j\in J}$ to x implies the SOT convergence of the Cayley transforms. □

Corollary 3.4.5. *If $f \in C_0(\mathbb{R})$, then the map $x \mapsto f(x)$ on self-adjoint operators is SOT continuous.*

Proof. Since f vanishes at infinity,

$$g(z) := \begin{cases} 0 & \text{if } z = 1 \\ f\left(i\frac{1+z}{1-z}\right) & \text{otherwise} \end{cases}$$

defines a continuous function on $\mathbb{T} \subset \mathbb{C}$. By Proposition 3.4.2, $x \mapsto g(x)$ is SOT continuous on the set of unitary operators. Then using Proposition 3.4.4, we see that $x \mapsto g((x-i)(x+i)^{-1}) = f(x)$ is SOT continuous as the composition of two SOT continuous maps. □

For $S \subset B(\mathcal{H})$, we adopt the following notation:

$$\begin{aligned} S_{s.a.} &:= \{x \in S \colon x = x^*\} \\ (S)_R &= \{x \in S \colon \|x\| \leq R\}. \end{aligned}$$

Theorem 3.4.6 (The Kaplansky Density Theorem)**.** *For a $*$-subalgebra $A \subset B(\mathcal{H})$,*

$$\begin{aligned} \overline{A_{s.a.}}^{SOT} &= \left(\overline{A}^{SOT}\right)_{s.a.} \\ \overline{(A)_1}^{SOT} &= \left(\overline{A}^{SOT}\right)_1 \end{aligned}$$

Proof. Denote $B := \overline{A}^{SOT}$. We first show that it suffices to assume A is operator norm closed. If C is the operator norm closure of A, then

$$A \subset C \subset \overline{A}^{SOT} (= B)$$

since operator norm convergence implies SOT convergence. Consequently, $\overline{C}^{SOT} = \overline{A}^{SOT}$, and the same argument holds when the pair A, C is replaced with $A_{s.a.}, C_{s.a.}$ or $(A)_1, (C)_1$. So replacing A with C if necessary, we may assume that A is operator norm closed and hence a C^*-algebra.

Now, using that SOT convergence implies WOT convergence, it follows that $\overline{A_{s.a.}}^{SOT} \subset B_{s.a.}$. Let $x \in B_{s.a.} \subset B$, then there exists a net $(x_i)_{i\in I} \subset A$ converging strongly to x. Since taking adjoints is WOT continuous (see Exercise 1.1.6), we have that $\left(\frac{x_i+x_i^*}{2}\right)_{i\in I} \subset A_{s.a.}$ converges to x in the WOT. So $x \in \overline{A_{s.a.}}^{WOT}$, but $A_{s.a.}$ is a convex subset of $B(\mathcal{H})$ and so $\overline{A_{s.a.}}^{WOT} = \overline{A_{s.a.}}^{SOT}$ by Corollary 3.2.8. Thus $B_{s.a.} = \overline{A_{s.a.}}^{SOT}$.

In order to show $\overline{(A)_1}^{SOT} = (B)_1$, we need a pair of claims:

Claim 1: $\overline{(A_{s.a.})_1} = (B_{s.a.})_1$.

Indeed, let $x \in (B_{s.a.})_1$ and let $(x_i)_{i\in I} \subset A_{s.a.}$ be net converging to x in the SOT (which exists by the previous argument). Letting $f \in C_0(\mathbb{R})$ be a function with $\|f\|_\infty = 1$ satisfying $f(t) = t$ for $|t| \leq 1$, we have that $(f(x_i))_{i\in I} \subset (A_{s.a.})_1$ converges to $f(x) = x$ in the SOT by Corollary 3.4.5. Thus $(A_{s.a.})_1$ is SOT dense in $(B_{s.a.})_1$. ■

Claim 2: $\overline{M_2(A)}^{SOT} = M_2(B)$.

Fix

$$\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2}, \end{pmatrix} \in M_2(B).$$

Let $(\xi_1, \xi_2)^T \in \mathcal{H}^2$ and $\epsilon > 0$. Using $\overline{A}^{SOT} = B$, for each $i, j = 1, 2$ we can find $a_{i,j} \in A$ satisfying $\|(a_{i,j} - b_{i,j})\xi_j\| < \epsilon$. Then

$$\left\| \left[\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}, \end{pmatrix} - \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2}, \end{pmatrix} \right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\|^2 = \sum_{i=1}^{2} \|(a_{i,1} - b_{i,1})\xi_1 + (a_{i,2} - b_{i,2})\xi_2\|^2 < 8\epsilon^2.$$

Thus $M_2(A)$ is SOT dense in $M_2(B)$. ■

Now, the inclusion $\overline{(A)_1}^{SOT} \subset (B)_1$ follows from Exercise 1.1.2. Conversely, let $x \in (B)_1$ and consider

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1.$$

Note that $\tilde{x}$ is self-adjoint. By Claim 2, we can apply Claim 1 with A and B replaced by $M_2(A)$ and $M_2(B)$, respectively. Thus there exists a net $(\tilde{x}_i)_{i \in I} \subset (M_2(A)_{s.a.})_1$ converging to $\tilde{x}$ in the SOT. For each $i \in I$, define x_i to be the $(1,2)$-entry of $\tilde{x}_i$. Applying $\tilde{x}_i$ to vectors of the form $(0, \xi)^T \in \mathcal{H}^2$ shows that $\|x_i\| \leq 1$, so $(x_i)_{i \in I} \subset (A)_1$. Finally, applying the net $(\tilde{x}_i)_{i \in I}$ to the same type of vectors shows that $(x_i)_{i \in I}$ converges to x in the SOT. □

The choice of 1 for our operator norm bounds in the Kaplansky Density Theorem is essentially arbitrary. Indeed, for any $R > 0$ if $x \in (B)_R$ then $\frac{1}{R}x \in (B)_1$. So we can find a net $(x_i)_{i \in I} \subset (A)_1$ converging to $\frac{1}{R}x$ in the SOT, which implies $(Rx_i)_{i \in I}$ converges to x is the SOT and satisfies $\|Rx_i\| \leq R$. Thus $\overline{(A)_R} = (B)_R$ for all $R > 0$. In particular, for any $x \in B$, by taking $R = \|x\|$, we can find a net $(x_i)_{i \in I} \subset A$ converging to x in the SOT and satisfying $\|x_i\| \leq \|x\|$ for all $i \in I$.

Corollary 3.4.7. *A unital $*$-algebra $M \subset B(\mathcal{H})$, one has*

$$\overline{M}^{\sigma-SOT} = \overline{M}^{\sigma-WOT} = \overline{M}^{SOT} = \overline{M}^{WOT} = M''$$

Consequently, M is a von Neumann algebra if and only if it is σ-SOT or σ-WOT closed.

Proof. We already know the last two equalities hold by the Bicommutant Theorem. Since σ-SOT convergence implies σ-WOT convergence implies WOT-convergence (see Exercise 3.2.2), we have $\overline{M}^{\sigma-SOT} \subset \overline{M}^{\sigma-WOT} \subset \overline{M}^{WOT} = \overline{M}^{SOT}$. So it suffices to show $\overline{M}^{SOT} \subset \overline{M}^{\sigma-SOT}$. If $x \in \overline{M}^{SOT}$, then the Kaplansky Density theorem implies we can find a net $(x_i)_{i \in I} \subset (M)_{\|x\|}$ converging to x in the SOT. Since the net is uniformly bounded, Exercise 3.2.4 implies it also converges to x in the σ-SOT. Hence $\overline{M}^{SOT} \subset \overline{M}^{\sigma-SOT}$. □

If $A = M$ is a von Neumann algebra, then the Kaplansky Density Theorem implies that $(M)_1$ is SOT closed. Conversely, for a unital $*$-subalgebra $A \subset B(\mathcal{H})$, if $(A)_1$ is SOT closed, then the Krein–Smulian theorem from functional analysis implies A is SOT closed and therefore is a von Neumann algebra. We can make the same assertion for the WOT, σ-SOT, and σ-WOT using Corollaries 3.2.8 and 3.2.10 and Exercise 3.2.4. This yields the following corollary.

Corollary 3.4.8. *Let $A \subset B(\mathcal{H})$ be a unital $*$-subalgebra. The following are equivalent:*

(i) A is a von Neumann algebra.

(ii) $(A)_1$ is SOT closed.

(iii) $(A)_1$ is WOT closed.

(iv) $(A)_1$ is σ-SOT closed.

(v) $(A)_1$ is σ-WOT closed.

Chapter 4

Types of von Neumann Algebras

We saw back in Corollary 2.1.4 that von Neumann algebras are equal to the C^*-algebra generated by its projections. So it is perhaps unsurprising that much of the structure of a von Neumann algebras is determined by its projections. More precisely, there is an equivalence relation on the projections in a von Neumann algebra, and one can classify von Neumann algebras into three types according to the behavior of this equivalence relation

In the first section we will define and study this equivalence relation on projections. In the second section we study certain subalgebras related to projections called *compressions*. In the third section we will define the three types of von Neumann algebras and show how any von Neumann algebra decomposes into a direct sum of the three types. We will also consider a few examples.

Lecture Preview: The content of this lecture will covered over two days: Wednesday, July 8th (p. 44–54) and Friday, July 10th (p. 55–63). The first lecture on July 8th will cover equivalence of projections (Definition 4.1.1), central supports (Definition 4.1.7), and the Comparison Theorem. We will likely forego most proofs in favor of concrete examples. Regardless, it is recommended that you skip the proof of Proposition 4.1.5. The second lecture on July 8th will cover compressions of von Neumann algebras (Definition 4.2.1) and various properties of projections (Definitions 4.2.5 and 4.3.1), and emphasis will be put on concrete examples.

For the first lecture on July 10th, we will state the type decomposition (see Theorem 4.3.7) and its refinements (see Definitions 4.3.10 and 4.3.13), though we will not prove them. Instead we will focus on the examples at the end of Section 4.3 (Examples 4.3.14, 4.3.15, and 4.3.16).

4.1 Equivalence of Projections

Throughout this section, let $M \subset B(\mathcal{H})$ be a von Neumann algebra. We will write $\mathcal{P}(M)$ for the collection of projections in M. Also, for a subset $\mathcal{S} \subset \mathcal{H}$ we write $[\mathcal{S}]$ for the projection onto the closed span of $\mathcal{S}$; that is, $[\mathcal{S}] = P_{\overline{\text{span}}\mathcal{S}}$.

Recall that, viewing $B(\mathcal{H})$ as C^*-algebra, positivity gives us a partial ordering on projections: $p \leq q$ if and only if $q - p \geq 0$. In fact, $(\mathcal{P}(M), \leq)$ is a complete lattice for any von Neumann algebra $M \subset B(\mathcal{H})$ (see Exercise 4.1.1). For $\mathcal{P} \subset \mathcal{P}(M)$ a set of projections (not assumed to be pairwise orthogonal) the *infimum* and *supremum* of $\mathcal{P}$ are defined by

$$\bigwedge \mathcal{P} := \left[\bigcap_{p\in\mathcal{P}} p\mathcal{H}\right] \qquad\qquad \bigvee \mathcal{P} := \left[\bigcup_{p\in\mathcal{P}} p\mathcal{H}\right].$$

If $\mathcal{P} = \{p_1, \ldots, p_n\}$ is a finite subset, we also write $p_1 \wedge \cdots \wedge p_n := \bigwedge \mathcal{P}$ and $p_1 \vee \cdots \vee p_n := \bigvee \mathcal{P}$. Note that $\mathcal{P} \subset M$ implies that the subspaces used to define $\bigwedge \mathcal{P}$ and $\bigvee \mathcal{P}$ are reducing for M', and consequently $\bigwedge \mathcal{P}, \bigvee \mathcal{P} \in M$ by Lemma 1.2.5.

Unfortunately, this lattice structure tends to be too rigid for our purposes. For example, in $M_2(\mathbb{C})$ the projections $E_{1,1}$ and $E_{2,2}$ have the same rank but are not comparable via $\leq$. The underlying issue is that this partial ordering it too dependent on the Hilbert space: $p \leq q$ if and only $p\mathcal{H} \subset q\mathcal{H}$. Partial isometries will be the key ingredient for loosening this dependence.

Definition 4.1.1. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}$, we say that p is **equivalent** to q in M and write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. We say that p is **subequivalent** to q in M and write $p \preceq q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. If $p \preceq q$ but $p \not\sim q$, we write $p \prec q$.

Note that if $p, q \in \mathcal{P}(M)$ are such that $p \leq q$, then by taking $v = p$ we see that $p \preceq q$. Thus $p \preceq q$ is a coarser relation than $p \leq q$.

Example 4.1.2. Consider the following projections in $M_3(\mathbb{C})$:

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we set

$$V := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then $V^*V = P$ and $VV^* = Q$, so $P \sim Q$. We also have $E_{1,1} \preceq P$ (since $E_{1,1} \leq P$) and $E_{1,1} \preceq Q$ (using either of the partial isometries $E_{2,1}$ or $E_{3,1}$). Actually, we have $E_{1,1} \prec P, Q$. To see this note that for any partial isometry $V \in M_3(\mathbb{C})$ with $V^*V = E_{1,1}$ we have

$$\mathrm{Tr}(VV^*) = \mathrm{Tr}(V^*V) = \mathrm{Tr}(E_{1,1}) = 1 < 2 = \mathrm{Tr}(P), \mathrm{Tr}(Q).$$

So VV^* can never equal P or Q. In general, a projection in $M_3(\mathbb{C})$ is equivalent to another projection if and only if they have the same trace (see Exercise 4.1.4). ■

Remark 4.1.3. A subtle aspect of Definition 4.1.1 is that we can only say p is subequivalent to q <u>in M</u> if we can find a partial isometry v <u>in M</u> that satisfies $v^*v = p$ and $vv^* \leq q$. To emphasize this, we may write $p \preceq_M q$ or $p \sim_M q$. If $M \subset N \subset B(\mathcal{H})$ is a larger von Neumann algebra, it may be that $p \sim_N q$ but $p \not\sim_M q$. For example, $E_{1,1}$ and $E_{2,2}$ are equivalent in $M_2(\mathbb{C})$, but not in the von Neumann algebra $\mathbb{C}E_{1,1} \oplus \mathbb{C}E_{2,2}$.

Proposition 4.1.4. *For a von Neumann algebra $M \subset B(\mathcal{H})$, $\sim$ is an equivalence relation on $\mathcal{P}(M)$, and the relation $\preceq$ is reflexive and transitive (a preorder).*

Proof. The reflexivity of $\sim$ and $\preceq$ follows form the fact that a projection is also a partial isometry. The symmetry of $\sim$ is evident from the definition. The transitivity of $\sim$ will follow as a special case of the transitivity of $\preceq$, which we now show. Let $p, q, r \in \mathcal{P}(M)$ with $p \preceq q$ and $q \preceq r$. Then there exists partial isometries $u, v \in M$ so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. It follows that

$$qu = quu^*u = uu^*u = u,$$

so that

$$(vu)^*(vu) = u^*v^*vu = u^*qu = u^*u = p$$

and

$$(vu)(vu)^* = vuu^*v \leq vqv^* = v(v^*v)v^* = vv^* \leq r.$$

Thus $p \preceq r$, and $\preceq$ is transitive. □

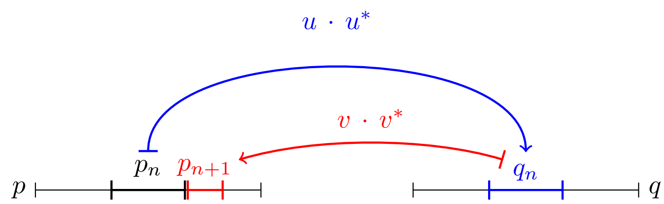
The relation $\preceq$ is **not** a partial order because $p \preceq q$ and $q \preceq p$ does not imply $p = q$. For example, in $M_n(\mathbb{C})$ we have $E_{1,1} \preceq E_{2,2}$ and $E_{2,2} \preceq E_{1,1}$, but $E_{1,1} \neq E_{2,2}$. Instead, we have $E_{1,1} \sim E_{2,2}$. We will see in the next proposition that this actually holds in general: $p \preceq q$ and $q \preceq p$ imply $p \sim q$ (it would be a crime against notation for this not to hold). Although the proof appears to be rather complicated, it more or less follows the same argument used to prove the Schröder–Berstein Theorem.

Proposition 4.1.5. *For a von Neumann algebra $M \subset B(\mathcal{H})$ and $p, q \in \mathcal{P}(M)$, $p \preceq q$ and $q \preceq p$ imply $p \sim q$.*

Proof. Let $u, v \in M$ be partial isometries so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Set $p_1 = p - vv^*$, $q_1 = up_1u^*$, and inductively define sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ by

$$p_n = vq_{n-1}v^* \qquad \text{and} \qquad q_n = up_nu^*.$$

By Exercise 4.1.8, $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ are each pairwise orthogonal families of projections, with $p_n \leq p$ and $q_n \leq q$ for each $n \in \mathbb{N}$. In particular, $p_n \leq vv^*$ for all $n \geq 2$ and $q_n \leq uu^*$ for all $n \geq 1$. The following diagram provides a rough but helpful mental picture of how these sequences are defined:



Using Proposition 1.1.5, we define projections

$$p_\infty := p - \sum_{n=1}^\infty p_n \qquad \text{and} \qquad q_\infty := q - \sum_{n=1}^\infty q_n.$$

We also define

$$w := v^*p_\infty + u\sum_{n=1}^\infty p_n = v^*p_\infty + \sum_{n=1}^\infty up_n.$$

We claim $w^*w = p$ and $ww^* = q$. The argument will be broken up into the following smaller claims:

(I) $(p_nu^*)(up_m) = \delta_{n=m}p_n$ and $(up_n)(up_m)^* = \delta_{n=m}q_n$ for all $m, n \in \mathbb{N}$.

(II) $(p_\infty v)(v^*p_\infty) = p_\infty$ and $(v^*p_\infty)(v^*p_\infty)^* = q_\infty$.

(III) $(p_nu^*)(v^*p_\infty) = 0$, $(p_\infty v)(up_n) = 0$, $(v^*p_\infty)(p_nu^*) = 0$, and $(up_n)(p_\infty v^*) = 0$ for all $n \in \mathbb{N}$.

Before proving these claims, observe that they are simply the multiplication rules needed to expand the products w^*w and ww^*:

$$\begin{aligned} w^*w &= \left(p_\infty v + \sum_{m=1}^\infty p_mu^*\right)\left(v^*p_\infty + \sum_{n=1}^\infty up_n\right) \\ &= (p_\infty v)(v^*p_\infty) + \sum_{n=1}^\infty (p_\infty v)(up_n) + \sum_{m=1}^\infty (p_mu^*)(v^*p_\infty) + \sum_{m,n=1}^\infty (p_mu^*)(up_n) = p_\infty + \sum_{n=1}^\infty p_n = p \end{aligned}$$

and similarly $ww^* = q$. Thus proving these claims with complete the proof.

(I): We compute

$$(up_n)^*(up_m) = p_nu^*up_m = p_npp_m = p_np_m = \delta_{n=m}p_n.$$

Also

$$(up_n)(up_m)^* = up_np_mu^* = \delta_{n=m}up_nu^* = \delta_{n=m}q_n.$$

(II): Let $v_k = v^*\left(p - \sum_{n=1}^k p_n\right)$. Then

$$v_kv_k^* = v^*\left(p - \sum_{n=1}^k p_n\right)v = v^*pv - \sum_{n=1}^k v^*p_nv = q - \sum_{n=2}^k q_{n-1} = q - \sum_{n=1}^{k-1} q_n,$$

where we we have used $v^*pv = q$, $v^*p_1v = 0$, and $v^*p_nv = q_{n-1}$ for $n \geq 2$. Also

$$v_k^*v_k = \left(p - \sum_{n=1}^k p_n\right) vv^* \left(p - \sum_{n=1}^k p_n\right) = vv^* - \sum_{n=2}^k p_n = p - p_1 - \sum_{n=2}^k p_n = p - \sum_{n=1}^k p_n.$$

where we have used $vv^* \leq p$, $p_1vv^* = 0$, and $p_n \leq vv^*$ for $n \geq 2$. Taking limits in the SOT we obtain

$$(p_\infty v)(v^*p_\infty) = \lim_{k\to\infty} v_k^*v_k = \lim_{k\to\infty} \left(p - \sum_{n=1}^k p_n\right) = p_\infty,$$

and

$$(v^*p_\infty)(p_\infty v) = \lim_{k\to\infty} v_kv_k^* = \lim_{k\to\infty} \left(q - \sum_{n=1}^{k-1} q_n\right) = q_\infty.$$

(III): First note that by taking adjoints, the second equality follows from the first and the fourth from the third. The third equality is simply a consequence of $p_\infty p_n = 0$. To see the first equality, note that $v^*p = v^* = v^*q$ and $v^*p_n = q_{n-1}v^*$, while $v^*p_1 = 0$. It follows that $v^*p_\infty = q_\infty v^*$, which along with $p_nu^* = u^*q_n$ imply $(p_nu^*)(v^*p_\infty) = u^*q_nq_\infty v^* = 0$. □

The next lemma is an important example of equivalence, and a nice application of the polar decomposition. Recall that for a subset $\mathcal{S} \subset \mathcal{H}$, $[\mathcal{S}]$ denotes the projection onto $\overline{\operatorname{span}}\,\mathcal{S}$.

Lemma 4.1.6. *For a von Neumann algebra $M \subset B(\mathcal{H})$ and $x \in M$, $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ and $[x\mathcal{H}] \sim_M [x^*\mathcal{H}]$.*

Proof. Let $x = v|x|$ be the polar decomposition and recall that $v \in M$. From Theorem 3.1.1 we know that vv^* is the projection onto $\overline{\operatorname{ran}(x)} = \overline{x\mathcal{H}}$ and v^*v is the projection onto

$$\overline{\operatorname{ran}(|x|)} = \ker(|x|)^\perp = \ker(x)^\perp = \overline{\operatorname{ran}(x^*)} = \overline{x^*\mathcal{H}}.$$

Thus $vv^* = [x\mathcal{H}]$ and $v^*v = [x^*\mathcal{H}]$, which shows the projections are equivalent and in M. □

Another way to see that $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ is to observe that the subspaces $\overline{x\mathcal{H}}$ and $\overline{x^*\mathcal{H}}$ are reducing for M' and use Lemma 1.2.5.

Definition 4.1.7. For $x \in M$, the **central support** of x in M is the projection

$$\mathbf{z}(x) := \bigwedge \{z \in \mathcal{P}(\mathcal{Z}(M)) \colon xz = zx = x\}.$$

We amy also write $\mathbf{z}_M(x) := \mathbf{z}(x)$ to emphasize the role of M in the above. We say $p, q \in \mathcal{P}(M)$ are **centrally orthogonal** if their central supports are orthogonal: $\mathbf{z}(p)\mathbf{z}(q) = 0$.

Note that for $p \in \mathcal{P}(M)$, $zp = p$ for $z \in \mathcal{P}(\mathcal{Z}(M))$ implies $p \leq z$, and therefore $p \leq \mathbf{z}(p)$. So in this case we can think of $\mathbf{z}(p)$ as the smallest central projection that is larger than p (*central* being the key word here). Also, if $p, q \in \mathcal{P}(M)$ are centrally orthogonal, then this shows p and q are also orthogonal. The next lemma provides another way to think of the central support.

Lemma 4.1.8. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. The central support of $p \in \mathcal{P}(M)$ is*

$$\mathbf{z}(p) = \bigvee_{x\in M} [xp\mathcal{H}] = [Mp\mathcal{H}].$$

Proof. The second equality above follows from the definition of the supremum. Let $z = [Mp\mathcal{H}]$. Since M is unital, we have $p \leq z$. Because $\overline{Mp\mathcal{H}}$ is reducing for M and M', we have that $z \in M \cap M' = \mathcal{Z}(M)$. Thus $\mathbf{z}(p) \leq z$. Conversely, for any $x \in M$ we have

$$xp\mathcal{H} = x\mathbf{z}(p)p\mathcal{H} = \mathbf{z}(p)xp\mathcal{H},$$

which implies $[xp\mathcal{H}] \leq \mathbf{z}(p)$. Since this holds for all $x \in M$, we have $z \leq \mathbf{z}(p)$. □

Proposition 4.1.9. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, the following are equivalent:*

(i) p and q are centrally orthogonal.

(ii) $pMq = \{0\}$.

(iii) There does not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ such that $p_0 \sim q_0$.

Proof. We first show (i) and (ii) are equivalent. If p and q are centrally orthogonal, then for any $x \in M$ we have

$$pxq = p\mathbf{z}(p)x\mathbf{z}(q)q = px\mathbf{z}(p)\mathbf{z}(q)q = 0.$$

Thus $pMq = \{0\}$. Conversely, if $pMq = \{0\}$, then by Lemma 4.1.8 $p\mathbf{z}(q) = p[Mq\mathcal{H}] = 0$. This implies $p \leq 1 - \mathbf{z}(q)$, and since $1 - \mathbf{z}(q) \in \mathcal{Z}(M)$ we have $\mathbf{z}(p) \leq 1 - \mathbf{z}(q)$. That is, $\mathbf{z}(p)\mathbf{z}(q) = 0$. Thus (i) and (ii) are equivalent.

Next we show (ii) and (iii) are equivalent. Suppose (ii) does not hold and let $x \in M$ be such that $pxq \neq 0$. Then $qx^*p \neq 0$ and consequently, $p_0 := [pxq\mathcal{H}]$ and $q_0 := [qx^*p\mathcal{H}]$ are non-zero projections. Clearly $p_0 \leq p$ and $q_0 \leq q$, and by Lemma 4.1.6 $p_0 \sim q_0$. Conversely, suppose (iii) does not hold and $p_0 \leq p$ and $q_0 \leq q$ are non-zero projections such that $p_0 \sim q_0$. Let $v \in M$ be a partial isometry so that $v^*v = p_0$ and $vv^* = q_0$. Then $v^* = p_0 v^* q_0$ so that

$$pv^*q = pp_0v^*q_0q = p_0v^*q_0 = v^* \neq 0.$$

Thus $pMq \neq \{0\}$, and we see that (ii) and (iii) are equivalent. □

Our next objective in this section is to prove the Comparison Theorem (see Theorem 4.1.11), which says that—modulo multiplying by a central projection—all projections are comparable via $\preceq$. We must first prove a lemma that will also be useful in our forthcoming classification of von Neumann algebras.

Lemma 4.1.10. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If $\{p_i : i \in I\}, \{q_i : i \in I\} \subset \mathcal{P}(M)$ are two pairwise orthogonal families such that $p_i \preceq q_i$ for each $i \in I$, then $\sum_{i \in I} p_i \preceq \sum_{i\in I} q_i$. In particular, if $p_i \sim q_i$ for each $i \in I$, then $\sum_{i\in I} p_i \sim \sum_{i \in I} q_i$.*

Proof. Let $u_i \in M$ be a partial isometry such that $u_i^*u_i = p_i$ and $u_iu_i^* \leq q_i$. Write $r_i = u_iu_i^*$ and note that $\{r_i : i \in I\}$ is pairwise orthogonal because $\{q_i : i \in I\}$ is. We have for $i \neq j$

$$u_i^*u_j = u_i^*u_iu_i^*u_ju_j^*u_j = u_i^*r_ir_ju_j = 0,$$

and

$$u_iu_j^* = u_iu_i^*u_iu_j^*u_ju_j^* = u_ip_ip_ju_j^* = 0.$$

Consequently,

$$\left(\sum_{i\in I} u_i\right)^* \left(\sum_{j \in I} u_j\right) = \sum_{i\in I} u_i^*u_i = \sum_{i \in I} p_i$$

and

$$\left(\sum_{i\in I} u_i\right)\left(\sum_{j\in I} u_j\right)^* = \sum_{i\in} u_iu_i^* = \sum_{i\in I} r_i \leq \sum_{i \in I} q_i.$$

Thus $\sum p_i \preceq \sum q_i$. The last assertion follows from the above and Proposition 4.1.5. □

Theorem 4.1.11 (Comparison theorem)**.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that*

$$pz \preceq qz \qquad \text{and} \qquad q(1-z) \preceq p(1-z).$$

Proof. By Zorn's Lemma there exists maximal families $\{p_i : i \in I\}, \{q_i : i \in I\} \subset \mathcal{P}(M)$ of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$ and

$$p_0 := \sum_{i \in I} p_i \leq p$$
$$q_0 := \sum_{i \in I} q_i \leq q.$$

Note that $p_0 \sim q_0$ by Lemma 4.1.10. Choose $z := \mathbf{z}(q - q_0)$. By maximality of the families, Proposition 4.1.9 yields $\mathbf{z}(p - p_0)z = 0$. Consequently, $(p - p_0)z = 0$, or $pz = p_0 z$. Now, if $v \in M$ is such that $v^*v = p_0$ and $vv^* = q_0$, then one easily checks that $p_0 z \sim q_0 z$ via the partial isometry vz. Thus

$$pz = p_0 z \sim q_0 z \leq qz.$$

Similarly, $p_0(1 - z) \sim q_0(1 - z)$ and since $q - q_0 \leq z$ we have

$$q(1 - z) = q_0(1 - z) \sim p_0(1 - z) \leq p(1 - z).$$ □

Corollary 4.1.12. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If M is a factor, then for $p, q \in \mathcal{P}(M)$ exactly one of the following holds:*

$$p \prec q \qquad p \sim q \qquad q \prec p.$$

Proof. By the Comparison Theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ so that $pz \preceq qz$ and $q(1 - z) \preceq p(1 - z)$. Since $\mathcal{Z}(M) = \mathbb{C}$, we have either $z = 0$ or $z = 1$ and the result follows. □

Exercises

4.1.1. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. In this exercise you will show that $(\mathcal{P}(M), \leq)$ is a complete lattice.

(a) Show that $\bigwedge \mathcal{P}, \bigvee \mathcal{P} \in M$.

(b) Show that $\bigwedge \mathcal{P} \leq p \leq \bigvee \mathcal{P}$ for all $p \in \mathcal{P}$.

(c) Show $\bigwedge \mathcal{P}$ is a greatest lower bound for $\mathcal{P}$ and that $\bigvee \mathcal{P}$ is a least upper bound for $\mathcal{P}$.

4.1.2. Let $\mathcal{P} \subset B(\mathcal{H})$ be a set of projections

$$\bigvee \mathcal{P} = 1 - \bigwedge \mathcal{P}^\perp \qquad\qquad \bigwedge \mathcal{P} = 1 - \bigvee \mathcal{P}^\perp.$$

4.1.3. Let $\{\xi_1, \ldots, \xi_n\}, \{\eta_1, \ldots, \eta_n\} \subset \mathcal{H}$ be two orthornomal subsets. Show that $\sum_{i=1}^n \xi_i \otimes \bar{\eta}_i$ is a partial isometry that implements the equivalence $(\sum_{i=1}^n \eta_i \otimes \bar{\eta}_i) \sim \left(\sum_{i=1}^n \xi_i \otimes \bar{\xi}\right)$.

4.1.4. Let $p, q \in (B(\mathcal{H}))$ be finite-rank projections. Show that $p \sim q$ if and only if $\mathrm{Tr}(p) = \mathrm{Tr}(q)$.

4.1.5. Let $\mathcal{E}, \mathcal{F} \subset \mathcal{H}$ be two orthonormal subsets with the same cardinality. Show that $[\mathcal{E}] \sim [\mathcal{F}]$. [**Hint:** start with a bijection from $\mathcal{E}$ to $\mathcal{F}$ (as sets).]

4.1.6. Let $A \subset B(\mathcal{H})$ be an abelian von Neumann algebra. For $p, q \in \mathcal{P}(A)$, show that $p \sim_A q$ if and only if $p = q$.

4.1.7. For $p \preceq q$, let v be a partial isometry satisfying $v^*v = p$ and $vv^* \leq q$. Show that $qvp = v$.

4.1.8. Let p, q be projections, and let u, v be partial isometries so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Set $p_1 := p - vv^*$, $q_1 = up_1u^*$, and inductively define sequences $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ by

$$p_n = vq_{n-1}v^* \qquad \text{and} \qquad q_n = up_nu^*.$$

(a) For each $n \in \mathbb{N}$, show that $p_n = (vu)^{n-1}p_1((vu)^*)^{n-1}$ and $q_n = (uv)^{n-1}q_1((uv)^*)^{n-1}$.

(b) For each $n \in \mathbb{N}$, show that $(vu)^n$ and $(uv)^n$ are partial isometries. In particular, show

$$((vu)^*)^n(vu)^n = p \qquad (vu)^n((vu)^*)^n \leq vv^*$$
$$((uv)^*)^n(uv)^n = q \qquad (uv)^n((uv)^*)^b \leq uu^*.$$

(c) For each $n \in \mathbb{N}$, show that p_n and q_n are projections satisfying $p_n \leq p$ and $q \leq q$.

(d) For $m < n$, show that

$$((vu)^*)^m(vu)^n = (vu)^{n-m} \qquad \text{and} \qquad ((uv)^*)^m(uv)^n = (uv)^{n-m}.$$

(e) For $m < n$, show that $p_m p_n = 0$ and $q_m q_n = 0$. [**Hint:** first check that $p_1 v = 0$ and $q_1 uv = 0$.]

4.1.9. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $p, q \in \mathcal{P}(M)$ satisfy $p \preceq q$. Show that $\mathbf{z}(p) \leq \mathbf{z}(q)$. [**Hint:** use Lemma 4.1.8.]

4.1.10. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $p, q \in \mathcal{P}(M)$. In this exercise you will prove **Kaplansky's formula**:

$$(p \vee q - p) \sim (q - p \wedge q).$$

(a) For $x := (1-p)q$, show that $[x^*\mathcal{H}] = q - p \wedge q$.
[**Hint:** first compute $[\ker(x)]$.]

(b) For x as above, show that $[x\mathcal{H}] = p \vee q - p$.
[**Hint:** use the previous part and Exercise 4.1.2.]

(c) Use Lemma 4.1.6 to deduce the desired equivalence.

4.2 Compressions

Before we can continue our study of projections, it is necessary to understand an important operation on von Neumann algebras.

Definition 4.2.1. For a von Neumann algebra $M \subset B(\mathcal{H})$ and $p \in B(\mathcal{H})$ a projection,

$$pMp := \{pxp \colon x \in M\}$$

is called a **compression** (or **corner**) of M.

The terminology comes from the fact that under the identification $\mathcal{H} \cong p\mathcal{H} \oplus (1-p)\mathcal{H}$, pxp for $x \in M$ is identified with

$$\begin{pmatrix} pxp & 0 \\ 0 & 0 \end{pmatrix} \in B(p\mathcal{H} \oplus (1-p)\mathcal{H}),$$

where we view pxp as an operator on $p\mathcal{H}$. In fact, for $M = B(\mathcal{H})$ we have $pB(\mathcal{H})p \cong B(p\mathcal{H})$

Note that pMp is a subspace and is closed under taking adjoints. There are two cases where pMp is actually a $*$-algebra. The first is if $p \in M$, in which case pMp is actually a $*$-subalgebra of M. The second is if $p \in M'$, where $pxp = xp$ for all $x \in M$ implies $pMp = Mp$. In both cases p is the unit of the $*$-algebra, so if $p < 1$ then they cannot be von Neumann algebras in $B(\mathcal{H})$. However, p is the identity operator on $B(p\mathcal{H})$, and by the above identification we can view pMp as operators on $p\mathcal{H}$.

Theorem 4.2.2. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$. Then pMp and $M'p$ are von Neumann algebras in $B(p\mathcal{H})$ and are commutants of one another.*

Proof. From the discussion preceding the theorem, we see that pMp and $M'p$ are both unital $*$-subalgebras of $B(p\mathcal{H})$. So it suffices to show $(pMp)'' = pMp$ and $(M'p)'' = M'p$, where the commutants here are taken in $B(p\mathcal{H})$ (rather than $B(\mathcal{H})$). Toward this end we will show the following equalities:

$$(M'p)' \cap B(p\mathcal{H}) = pMp$$
$$(pMp)' \cap B(p\mathcal{H}) = M'p.$$

The inclusion $pMp \subset (M'p)' \cap B(p\mathcal{H})$ is immediate. Conversely, suppose $x \in (M'p)' \cap B(p\mathcal{H})$. Define $\tilde{x} \in B(\mathcal{H})$ by

$$\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, $\tilde{x} = p\tilde{x}p$, and for $p\xi \in p\mathcal{H}$ we have $\tilde{x}p\xi = xp\xi$. If $y \in M'$, then for $\xi \in \mathcal{H}$ we have

$$y\tilde{x}\xi = yp\tilde{x}p\xi = ypxp\xi = xyp\xi = xpy\xi = \tilde{x}py\xi = \tilde{x}y\xi.$$

So $y\tilde{x} = \tilde{x}y$ and hence $\tilde{x} \in M'' = M$. As operators on $p\mathcal{H}$ we have $x = p\tilde{x}p \in pMp$.

The inclusion $M'p \subset (pMp)' \cap B(p\mathcal{H})$ is immediate. Suppose $y \in (pMp)' \cap B(p\mathcal{H})$. Using the functional calculus to write y as a linear combination of four unitaries, we may assume $y = u$ is a unitary. We will extend u to an element $\tilde{u} \in B(\mathcal{H})$. Define $\tilde{u}$ on $Mp\mathcal{H}$ by

$$\tilde{u}\left(\sum_{i=1}^{n} x_i p\xi_i\right) = \sum_i x_i up\xi_i,$$

for $x_1, \ldots, x_n \in M$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$. Observe that

$$\begin{aligned}
\left\| \tilde{u} \sum_{i=1}^{n} x_i p\xi_i \right\|^2 &= \sum_{i,j=1}^{n} \langle x_i up\xi_i, x_j up\xi_j \rangle \\
&= \sum_{i,j=1}^{n} \left\langle px_j^* x_i pu\xi_i, up\xi_j \right\rangle \\
&= \sum_{i,j=1}^{n} \left\langle upx_j^* x_i p\xi_i, up\xi_j \right\rangle \\
&= \sum_{i,j=1}^{n} \left\langle px_j^* x_i p\xi_i, p\xi_j \right\rangle = \left\| \sum_{i=1}^{n} x_i p\xi_i \right\|^2.
\end{aligned}$$

Thus $\tilde{u}$ is well-defined and an isometry, which we extend to $\overline{Mp\mathcal{H}}$. Observe that $\tilde{u}$ commutes with M on $Mp\mathcal{H}$ by definition of $\tilde{u}$, and consequently they commute on $\overline{Mp\mathcal{H}}$. Recall that $\mathbf{z}(p) = [Mp\mathcal{H}]$ by Lemma 4.1.8. So if we extend $\tilde{u}$ to $\mathcal{H}$ by setting $\tilde{u}|_{(Mp\mathcal{H})^\perp} \equiv 0$, then $\tilde{u} = \tilde{u}\mathbf{z}(p)$. It follows that for $x \in M$ and $\xi \in \mathcal{H}$ we have

$$x\tilde{u}\xi = x\tilde{u}\mathbf{z}(p)\xi = \tilde{u}\mathbf{z}(p)x\xi = \tilde{u}x\xi.$$

That is, $\tilde{u} = M' \cap \mathcal{B}(\mathcal{H})$. By definition $\tilde{u}$, we have $\tilde{u}p = u$ and so $u \in M'p$. □

Corollary 4.2.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$. If M is a factor then pMp and $M'p$ are factors.*

Proof. Since pMp and $M'p$ are each commutants of one another in $B(p\mathcal{H})$ by Theorem 4.2.2, they have the same center and so it suffices to show $M'p$ is a factor. First note that for $y \in M'$, if $yp = 0$ then for all $x \in M$ and $\xi \in \mathcal{H}$ we have

$$yxp\xi = xyp\xi = 0.$$

Since M is a factor, we have $[Mp\mathcal{H}] = \mathbf{z}(p) = 1$ by Lemma 4.1.8. This means $Mp\mathcal{H}$ is dense in $\mathcal{H}$ and consequently the above implies $y = 0$. Now, if $zp \in \mathcal{Z}(M'p)$ for $z \in M'$, then for all $y \in M'$ we have $[z, y]p = [zp, yp] = 0$. By what we just argued, $[z, y] = 0$ and so $z \in \mathcal{Z}(M')$. Since M' is a factor (by virtue of M being a factor), we have $z \in \mathbb{C}$ and $zp \in \mathbb{C}p$. Thus $\mathcal{Z}(M'p) = \mathbb{C}p$ and $M'p$ is a factor. □

The next proposition shows that a compression depends only on the equivalence class of p in M.

Proposition 4.2.4. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ are equivalent in M, then pMp and qMq are spatially isomorphic.*

Proof. Let $v \in M$ be a partial isometry satisfying $v^*v = p$ and $vv^* = q$. We will show that $v|_{p\mathcal{H}}$ is a unitary from $p\mathcal{H}$ to $q\mathcal{H}$ that implements the spatial isomorphism. Note that $v = qvp$. This implies $v|_{p\mathcal{H}}$ is indeed valued in $q\mathcal{H}$, and is surjective since $q\xi = vv^*\xi = vpv^*\xi$ for any $x \in \mathcal{H}$. For $p\xi, p\eta \in p\mathcal{H}$, we have

$$\langle vp\xi, vp\eta\rangle = \langle v^*vp\xi, p\eta\rangle = \langle p\xi, p\eta\rangle .$$

Thus $v|_{p\mathcal{H}} \colon p\mathcal{H} \to q\mathcal{H}$ is a unitary. Using $v = qvp$ again, we have for any $x \in M$

$$vpxpv^* = vxv^* = q(vxv^*)q.$$

and

$$qxq = vv^*xvv^* = v(pv^*xvp)v^*.$$

Thus $v(pMp)v^* = qMq$. □

Note that in the above proof, we used $v \in M$ to guarantee $vxv^* \in M$ and $v^*xv \in M$ for all $x \in M$. Also note that if $y \in M'$, then $vypv^* = yvpv^* = vq$, which shows the spatial isomorphism sends $M'p$ to $M'q$.

Definition 4.2.5. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. We say $p \in \mathcal{P}(M)$ is **minimal** in M if $p \neq 0$ and $pMp = \mathbb{C}p$. We say p is **abelian** in M if pMp is abelian.

Note that a minimal projection is also abelian.

Example 4.2.6.

(1) Let $p \in B(\mathcal{H})$. Then $pB(\mathcal{H})p \cong B(p\mathcal{H})$. Since $B(p\mathcal{H})$ is always a factor, it can only be abelian if $B(p\mathcal{H}) \cong \mathbb{C}$. This holds off and only if $p\mathcal{H} \cong \mathbb{C}$; that is, if and only if p is a rank 1 projection.

(2) Let (X, μ) be a σ-finite measure space. Recall $f \in \mathcal{P}(L^\infty(X, \mu))$ if and only if $f = 1_E$ for some measurable $E \subset X$ (see Exercise 1.3.3). Consequently, all compressions of $L^\infty(X, \mu)$ are of the form $L^\infty(E, \mu|_E)$ for some measurable $E \subset X$, and so all projections in $L^\infty(X, \mu)$ are abelian. If 1_E is minimal, then $1_E \neq 0$ and $L^\infty(E, \mu|_E) = \mathbb{C}1_E$. The former holds if and only if $\mu(E) \neq 0$ and the latter holds if and only if for all measurable subsets $F \subset E$ we have $\mu(F) \in \{0, \mu(E)\}$ (see Exercise 4.2.3). We call such a subset E an *atom* of μ. Thus $L^\infty(X, \mu)$ has minimal projections if and only if μ has atoms. ■

If $p \in \mathcal{P}(M)$ is minimal, then whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$ we must have $q \in \{0, p\}$ since $q = pqp \in pMp = \mathbb{C}p$. Conversely, if $p \in \mathcal{P}(M)$ is such that $q \in \{0, p\}$ whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$, then p and 0 are the only projections in pMp. Since von Neumann algebras are equal to the C^*-algebras generated by their projections (see Corollary 2.1.4), we must have $pMp = \mathbb{C}p$ and so p is minimal. Thus, "$q \in \{0, p\}$ whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$" is an equivalent definition of being minimal, and this is non-commutative analogue of an atom for a measure.

Proposition 4.2.4 implies that if p is minimal (resp. abelian) and $q \in \mathcal{P}(M)$ satisfies $q \sim_M p$, then q is also minimal (resp. abelian). In fact, if $q \neq 0$ and $q \preceq p$ then it is minimal (resp. abelian). For p minimal, this is simply because $q \preceq p$ implies $q \sim p$ by the above characterization of minimality. For p abelian, suppose $v \in M$ is a partial isometry satisfying $v^*v = q$ and $vv^* \leq p$. Then $(vv^*)M(vv^*)$ is abelian as a subalgebra of pMp, and hence $qMq (\cong (vv^*)M(vv^*))$ is abelian. We record these observations in the following proposition.

Proposition 4.2.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $q \preceq p$. If p is minimal (resp. abelian), then q is minimal (resp. abelian).*

We conclude this section by using compressions to prove that the image of a von Neumann algebra under a normal unital $*$-homomorphism is again a von Neumann algebra.

Theorem 4.2.8. *Let $M \subset B(\mathcal{H})$ and $N \subset B(\mathcal{K})$ be von Neumann algebras. If $\pi \colon M \to N$ is a normal unital $*$-homomorphism, then $\pi(M) \subset B(\mathcal{K})$ is a von Neumann algebra.*

Proof. We first prove a special case: assume π is injective. Because π is a unital $*$-homorphism, $\pi(M)$ is a unital $*$-subalgebra of $B(\mathcal{K})$ and so by Corollary 3.4.8 we just need to check that $(\pi(M))_1$ is σ-WOT closed. Because $*$-homomorphisms preserve positivity, for $x \in M$ we have

$$\pi(x)^*\pi(x) = \pi(x^*x) \leq \pi(\|x^*x\|1) = \|x^*x\|\pi(1) = \|x^*x\|,$$

and hence $\|\pi(x)\| = \|\pi(x)^*\pi(x)\|^{1/2} \leq \|x^*x\|^{1/2} = \|x\|$. The same argument applied to $\pi^{-1}\colon \pi(M) \to M$ gives $\|\pi(x)\| = \|x\|$ for all $x \in M$. Thus $(\pi(M))_1 = \pi((M)_1)$. The duality $M \cong (M_*)^*$ and the Banach–Alaoglue theorem imply $(M)_1$ is σ-WOT compact, and consequently so is its σ-WOT continuous image $\pi((M)_1) = (\pi(M))_1$. In particular, $(\pi(M))_1$ is σ-SOT closed and therefore $\pi(M)$ is a von Neumann algebra.

Now suppose π is not injective. Consider $p := [\ker(\pi)M]$ and note that $\overline{\ker(\pi)\mathcal{H}}$ is reducing for M since $\ker(\pi)$ is an ideal, and is reducing for M' since $\ker(\pi) \subset M$. Thus $p \in M \cap M' = \mathcal{Z}(M)$ by Lemma 1.2.5. We will show that $\pi(M)$ is the injective image of $(1-p)M(1-p) = M(1-p)$, which is a von Neumann algebra by Theorem 4.2.2, and hence $\pi(M)$ is a von Neumann algebra by the first part of the proof. Our first step, is to show that $p \in \ker(\pi)$.

Since π is $*$-homomorphism, $\ker(\pi)$ is a $*$-subalgebra of M, and it is norm closed by virtue of being σ-WOT closed. Consequently, $\ker(\pi)$ is a C^*-algebra and therefore has an approximate identity $(e_i)_{i\in I}$ by [Theorem 4.2, C^*-Algebras Mini-course]. We claim that $(e_i)_{i\in I}$ converges to p in the σ-WOT, and consequently $p \in \ker(\pi)$ since $\ker(\pi)$ is σ-WOT closed. Note that $x = pxp$ for all $x \in \ker(\pi)$, and so it suffices to check σ-WOT convergence on $p\mathcal{H}$. Moreover, because $(e_i)_{i\in I}$ is uniformly bounded, it not only suffices to show WOT convergence on $p\mathcal{H}$, it suffices to show this on the dense subset $\ker(\pi)\mathcal{H}$. For $x, y \in \ker(\pi)$ and $\xi, \eta \in \mathcal{H}$ we have

$$|\langle (e_i - p)x\xi, y\eta\rangle| = |\langle (e_i x - px)\xi, y\eta\rangle| = |\langle (e_i x - x)\xi, y\eta\rangle| \leq \|e_i x - x\|\|\xi\|\|y\eta\| \to 0$$

by definition of the approximate identity. Thus p is the σ-WOT limit of $(e_i)_{i\in I}$.

Since $p \in \ker(\pi)$, for $x \in M$ we have

$$\pi(x(1-p)) = \pi(x)(\pi(1) - \pi(p)) = \pi(x)(1-0) = \pi(x).$$

Thus $\pi(M)$ is the image of $M(1-p)$ under π. This also shows $x(1-p) \in \ker(\pi)$ if and only if $x \in \ker(\pi)$, but in this case $x(1-p) = x - xp = x - x = 0$. Thus $\pi|_{M(1-p)}$ is injective and so $\pi(M)$ is a von Neumann algebra by the first part of the proof. □

Remark 4.2.9. There is a partial converse to the above theorem: if $\pi\colon M \to B(\mathcal{K})$ is an injective $*$-homomorphism such that $\pi(M)$ is a von Neumann algebra, then π is normal. That is, $*$-isomorphisms between von Neumann algebras are automatically normal (compare this to how $*$-isomorphisms between C^*-algebras are automatically isometric). This follows from a characterization of normality in terms of the increasing but uniformly bounded nets of positive operators (see Section III.2.2 in *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras* by Bruce Blackadar).

Exercises

4.2.1. Let $p \in B(\mathcal{H})$ be a rank n projection for $n \in \mathbb{N}$. Show that $pB(\mathcal{H})p \cong M_n(\mathbb{C})$.

4.2.2. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $z \in \mathcal{P}(\mathcal{Z}(M))$. Show that M is spatially isomorphic to the direct sum of compressions $Mz \oplus M(1-z)$ (see Exercise 1.2.8).

4.2.3. Let (X, μ) be a positive σ-finite measure space. We call a measurable subset $A \subset X$ an **atom** of if $\mu(A) > 0$ and for all measurable subsets $E \subset A$ one has $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

(a) If $A_1, A_2 \subset X$ are atoms, show that either $1_{A_1 \cap A_2} = 0$ or $1_{A_1 \cap A_2} = 1_{A_1} = 1_{A_2}$.

(b) If $A \subset X$ is an atom, show that $f|_A$ is constant for all $f \in L^\infty(X, mu)$.

4.2.4. Let (X, μ) be a positive σ-finite measure space. Show that $L^\infty(X, \mu)$ is finite dimensional (as a vector space) if and only if X can be partitioned into a finite union of atoms. Also show that in this case the dimension is given by the number of distinct atoms.

4.2.5. Let $M \subset B(\mathcal{H})$ be a factor. Show that any abelian projection in M is either zero or minimal. [**Hint:** use Corollary 4.2.3.]

4.2.6. Let $M \subset B(\mathcal{H})$ be a factor. Show any two minimal projections are equivalent. [**Hint:** use the Comparison Theorem.]

4.2.7. Let $\pi\colon M \to N$ be a $*$-homomorphism between von Neumann algebras.

(a) Show that $\pi(\mathcal{P}(M)) \subset \mathcal{P}(N)$.

(b) For $p, q \in \mathcal{P}(M)$, show that $p \preceq q$ implies $\pi(p) \preceq \pi(q)$.

(c) Show that if p is minimal (resp. abelian) in M, then $\pi(p)$ is minimal (resp. abelian) in $\pi(M)$. Show that $\pi(p)$ need not be minimal (resp. abelian) in N.

4.2.8. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{I} \subset M$ be a σ-WOT closed subspace.

(a) Show that if $\mathcal{I}$ is a left ideal then there exists $p \in \mathcal{P}(M)$ so that $\mathcal{I} = Mp$.

(b) Show that if $\mathcal{I}$ is a right ideal then there exists $p \in \mathcal{P}(M)$ so that $\mathcal{I} = pM$.

(c) Show that if $\mathcal{I}$ is a (two-sided) ideal then there exists $p \in \mathcal{P}(\mathcal{Z}(M))$ so that $\mathcal{I} = Mp$.

4.3 The Type Decomposition

The following definition highlights some additional important properties of projections, which will be needed in the statement of the type decomposition of von Neumann algebras.

Definition 4.3.1. For $M \subset B(\mathcal{H})$ be a von Neumann algebra, $p \in \mathcal{P}(M)$ is said to be

- **finite** in M if $q \leq p$ and $q \sim_M p$ implies $p = q$ for $q \in \mathcal{P}(M)$.
- **semi-finite** in M if there exists a family $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ of pairwise orthogonal, finite projections such that $p = \sum_{i\in I} p_i$.
- **purely infinite** in M if $p \neq 0$ and there does not exist any non-zero finite projections $q \in \mathcal{P}(M)$ with $q \leq p$.
- **properly infinite** in M if $p \neq 0$ and for all non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$ the projection zp is not finite.

Furthermore, M is said to be **finite, semi-finite, purely infinite**, or **properly infinite** if $1 \in M$ has the corresponding property in M.

Recall that in an abelian von Neumann algebra, projections are equivalent if and only if they are equal (see Exercise 4.1.6). This implies abelian projections (and consequently minimal ones) are necessarily finite, and all abelian von Neumann algebras are finite. We also have a number of implications that follow from the above definitions:

$$\text{finite} \Longrightarrow \text{semi-finite} \Longrightarrow \text{not purely infinite},$$

and

$$\text{purely infinite} \Longrightarrow \text{properly infinite}.$$

Also note that a factor is either finite or properly infinite.

Example 4.3.2. In each of the examples below, we consider $M = B(\mathcal{H})$ and $p \in \mathcal{P}(B(\mathcal{H}))$.

(1) If p is finite-rank then it is finite in the above sense. Suppose $q \leq p$. Then $q\mathcal{H} \subset p\mathcal{H}$ and so q is finite-rank. Suppose $q \sim p$ and let v be partial isometry satisfying $v^*v = q$ and $vv^* = p$. Then by Exercise 3.1.9 we have

$$\dim(q\mathcal{H}) = \mathrm{Tr}(q) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = \mathrm{Tr}(p) = \dim(p\mathcal{H}).$$

Thus $q\mathcal{H} = p\mathcal{H}$ and $q = p$. If $\dim(H) < \infty$, then $1 \in B(\mathcal{H})$ is a finite-rank projection and hence finite, so $B(\mathcal{H})$ is finite.

(2) If $\dim(p\mathcal{H})$ is infinite, then p is **not** finite. Let $\mathcal{E} \subset p\mathcal{H}$ be an orthonormal basis. Since it is an infinite set by assumption, we can partition it into disjoint subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ so that $|\mathcal{E}| = |\mathcal{E}_1| = |\mathcal{E}_2|$. If $q := [\mathcal{E}_1]$, then $q \sim p$ (see Exercise 4.1.5), but $q < p$ since $p - q = [\mathcal{E}_2] \neq 0$. Since $B(\mathcal{H})$ is a factor, these projections are also properly infinite.

(3) p is always semi-finite, and consequently never purely infinite. Let $\mathcal{E} \subset p\mathcal{H}$ be an orthonormal basis. Then

$$p = \sum_{\xi \in \mathcal{E}} \xi \otimes \bar{\xi}$$

and each $\xi \otimes \bar{\xi}$ is finite by part (1). In particular, $1 \in B(\mathcal{H})$ is semi-finite and so $B(\mathcal{H})$ is semi-finite. ■

You probably learned in linear algebra that a matrix $A \in M_n(\mathbb{C})$ is left (or right) invertible if and only if it is invertible. In particular, any isometry in $M_n(\mathbb{C})$ is necessarily a unitary. Not only does this latter fact hold in *any* finite von Neumann algebra (which $M_n(\mathbb{C})$ is by Example 4.3.2.(1)), it actually characterizes them.

Proposition 4.3.3. *A von Neumann algebra $M \subset B(\mathcal{H})$ is finite if and only if all isometries are unitaries.*

Proof. Suppose M is finite and let $v \in M$ be an isometry: $v^*v = 1$. Then $vv^* \leq 1$ and so by finiteness $vv^* = 1$. That is, v is a unitary. Conversely, assume every isometry is a unitary, and suppose $p \leq 1$ satisfies $p \sim 1$. Let $v \in M$ satisfy $v^*v = 1$ and $vv^* = p$. Then v is an isometry and hence a unitary, and therefore $p = vv^* = 1$. Thus 1 is finite in M. □

We will need the next two propositions in proving the type decomposition.

Proposition 4.3.4. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $p \preceq q$. If q is finite (resp. purely infinite), then p is also finite (resp. purely infinite).*

Proof. Suppose q is finite, and further suppose $p \sim q$. Let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. If $u \in M$ satisfies $u^*u = p$ and $uu^* \leq p$, then

$$(vuv^*)^*(vuv^*) = vu^*v^*vuv^* = vu^*puv^* = vu^*uv^* = vpv^* = vv^* = q$$

and

$$(vuv^*)(vuv^*)^* = vuv^*vu^*v^* = vupu^*v^* = vuu^*v^* \leq vpv^* = q.$$

Since q is finite, we must have $(vuv^*)(vuv^*)^* = q$. But then

$$uu^* = pupu^*p = v^*(vuv^*)(vuv^*)^*v = v^*qv = p.$$

Thus p is finite.

Now assume $p \leq q$. If $u \in M$ is such that $u^*u = p$ and $uu^* \leq p$, then for $w = u + (q - p)$ we have

$$w^*w = u^*u + u^*(q - p) + (q - p)u + (q - p) = p + (q - p) = q,$$

and

$$ww^* = uu^* + u(q - p) + (q - p)u^* + (q - p) = uu^* + (q - p) \leq q.$$

Since q is finite, we have $uu^* + (q - p) = ww^* = q$ or $uu^* = p$. Thus p is finite. In general, if $p \preceq q$, then there exists $q_0 \in \mathcal{P}(M)$ such that $p \sim q_0 \leq q$. By the two previous arguments we see that p is finite.

Finally, if q is purely infinite then it has no finite subprojections. If $p \preceq q$ had a finite subprojection $p_0 \leq p$, then $p_0 \preceq q$. In particular, $p_0 \sim q_0 \leq q$, which is finite by the above arguments, a contradiction. □

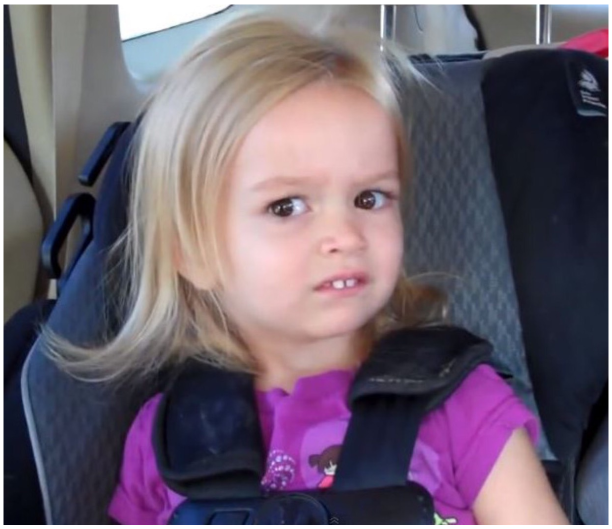
Proposition 4.3.5. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. A projection $p \in \mathcal{P}(M)$ is semi-finite if and only if it is a supremum of finite projections. In particular, the supremum of semi-finite projections is again semi-finite. Moreover, any subprojection of a semi-finite projection is also semi-finite.*

Proof. If $p \in \mathcal{P}(M)$ is semi-finite, then by definition it is the sum (hence supremum) of pairwise orthogonal finite projections. Conversely, suppose $p = \bigvee_i p_i$ for $\{p_i\}_{i \in I} \subset \mathcal{P}(M)$ finite projections. Let $\{q_j\}_{j\in J}$ be a maximal family of pairwise orthogonal finite subprojections of p. Suppose, towards a contradiction, that $q := p - \sum_{j\in J} q_j \neq 0$. Then, by definition of the supremum, there exists $i \in I$ so that q and p_i are not orthogonal. In particular, they are not centrally orthogonal and so by Proposition 4.1.9 there exists non-zero $q_0 \leq q$ so that $q_0 \preceq p_i$. Thus q_0 is finite by Proposition 4.3.4, which contradicts the maximality of $\{q_j\}_{j\in J}$. The final observation follows from the fact that the above argument also works if $p \leq \bigvee_i p_i$. □

Definition 4.3.6. A von Neumann algebra $M \subset B(\mathcal{H})$ is said to be

- **type** I if every non-zero projection has a non-zero abelian subprojection.
- **type** II if it is semi-finite and has no non-zero abelian projections.
- **type** III if it is purely infinite.

We can see immediately from the definition that any abelian von Neumann algebra is type I. We also have $B(\mathcal{H})$ is it type I, because a non-zero projection p has minimal (and hence abelian) subprojections of the form $\xi \otimes \bar{\xi}$ for any unit vector $\xi \in p\mathcal{H}$. On the other hand, group von Neumann algebras for i.c.c. groups give type II von Neumann algebras (see Example 4.3.14). Unfortunately, type III von Neumann algebras are beyond the scope of these notes. But Brent is a big fan and would love to tell you about them!



A von Neumann algebra need not be of any type. For example, if M_1 is type I and M_2 is type II, then their direct sum $M_1 \oplus M_2$ (see Exercise 1.2.8) has no type. Indeed, it is not type I because any non-zero projection $p \in \mathcal{P}(M_2)$ yields a non-zero projection $0 \oplus p \in \mathcal{P}(M_1 \oplus M_2)$ lacking non-zero abelian subprojections. It is not type II since any non-zero abelian projection $p \in M_1$ yields a non-zero abelian projection $p \oplus 0 \in \mathcal{P}(M_1 \oplus M_2)$. Since $p \oplus 0$ is finite by virtue of being abelian, we see that $M_1 \oplus M_2$ also not type III. However, note that $z_1 := 1 \oplus 0$ and $z_2 := 0 \oplus 1$ are central projections and the compressions $(M_1 \oplus M_2)z_1 = M_1 \oplus 0$ and $(M_1 \oplus M_2)z_2 = 0 \oplus M_2$ are type I and type II, respectively. The Type Decomposition tells us that this can always be done.

Theorem 4.3.7 (Type Decomposition)**.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Then there exists unique pairwise orthogonal central projections $\mathbf{z}_{\mathrm{I}}, \mathbf{z}_{\mathrm{II}}, \mathbf{z}_{\mathrm{III}} \in \mathcal{P}(\mathcal{Z}(M))$ such that $\mathbf{z}_{\mathrm{I}} + \mathbf{z}_{\mathrm{II}} + \mathbf{z}_{\mathrm{III}} = 1$ and the compression $M\mathbf{z}_T$ is type T for each $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.*

Proof. Let $\mathbf{z}_{\mathrm{I}}$ be the supremum of all abelian projections in M. Conjugating an abelian projection in M by a unitary in M yields another abelian projection in M. It follows that $u\mathbf{z}_{\mathrm{I}}u^* = \mathbf{z}_{\mathrm{I}}$ or $u\mathbf{z}_{\mathrm{I}} = \mathbf{z}_{\mathrm{I}}u$ for all unitaries $u \in M$. Since every element in M can be written as a linear combination of four unitaries, this implies $\mathbf{z}_{\mathrm{I}} \in M \cap M' = \mathcal{Z}(M)$. To see that $M\mathbf{z}_{\mathrm{I}}$ is type I, suppose $p \leq \mathbf{z}_{\mathrm{I}}$ is non-zero. Then by definition of the supremum there exists an abelian projection $r \in M$ so that $pr \neq 0$. Consequently, $pMr \neq \{0\}$ and

Proposition 4.1.9 tells us there exists non-zero $p \geq p_0 \sim r_0 \leq r$. Proposition 4.2.7 implies that p_0 is abelian and so $M\mathbf{z}_{\mathrm{I}}$ is type I.

Next, let $\mathbf{z}_{\mathrm{II}}$ be the supremum of all finite $p \in \mathcal{P}(M)$ such that $p \leq 1 - \mathbf{z}_{\mathrm{I}}$. By the same argument as above, we have $\mathbf{z}_{\mathrm{II}} \in \mathcal{Z}(M)$. Also, $\mathbf{z}_{\mathrm{II}}$ is semi-finite by Proposition 4.3.5. Since $\mathbf{z}_{\mathrm{II}} \leq 1 - \mathbf{z}_{\mathrm{I}}$, it has no non-zero abelian subprojections. Thus $M\mathbf{z}_{\mathrm{II}}$ is type II.

Finally, we let $\mathbf{z}_{\mathrm{III}} = 1 - \mathbf{z}_{\mathrm{I}} - \mathbf{z}_{\mathrm{II}}$. Note that any finite projection in M lies under $\mathbf{z}_{\mathrm{I}}$ if it is also abelian and otherwise lies under $\mathbf{z}_{\mathrm{II}}$. Consequently, $\mathbf{z}_{\mathrm{III}}$ has no finite subprojections and so $M\mathbf{z}_{\mathrm{III}}$ is type III.

Towards showing this decomposition is unique, suppose $p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}} \in \mathcal{P}(\mathcal{Z}(M))$ are pairwise orthogonal projections summing to one and satisfy Mp_R is type R for each $R \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then $p_{\mathrm{III}}\mathbf{z}_{\mathrm{I}}$ and $p_{\mathrm{III}}\mathbf{z}_{\mathrm{II}}$ are both finite and purely infinite by Proposition 4.3.4. That is, $p_{\mathrm{III}}\mathbf{z}_{\mathrm{I}} = p_{\mathrm{III}}\mathbf{z}_{\mathrm{II}} = 0$, and consequently $p_{\mathrm{III}} \leq \mathbf{z}_{\mathrm{III}}$. Reversing the roles of $\mathbf{z}$ and p yields $p_{\mathrm{III}} = \mathbf{z}_{\mathrm{III}}$. Next, $p_{\mathrm{II}}\mathbf{z}_{\mathrm{I}}$ is an abelian subprojection of p_{II} by Proposition 4.2.7. Since Mp_{II} is type II, we must therefore have $p_{\mathrm{II}}\mathbf{z}_{\mathrm{I}} = 0$. Thus $p_{\mathrm{II}} \leq \mathbf{z}_{\mathrm{II}}$ and by symmetry we obtain $p_{\mathrm{II}} = \mathbf{z}_{\mathrm{II}}$. Finally

$$p_{\mathrm{I}} = 1 - p_{\mathrm{II}} - p_{\mathrm{III}} = 1 - \mathbf{z}_{\mathrm{II}} - \mathbf{z}_{\mathrm{III}} = \mathbf{z}_{\mathrm{I}}.$$

So the decomposition is unique. □

Since $\mathbf{z}_{\mathrm{I}}, \mathbf{z}_{\mathrm{II}}, \mathbf{z}_{\mathrm{III}}$ are all central projections, Exercise 4.2.2 tells that $M \cong M\mathbf{z}_{\mathrm{I}} \oplus M\mathbf{z}_{\mathrm{II}} \oplus M\mathbf{z}_{\mathrm{III}}$. So even thoughh all von Neumann algebras need not have a type, they can all be written as a direct sums of type I, type II, and type III von Neumann algebras.

If M is a factor, then the only central projections are 0 and 1. Consequently, in the type decomposition for a factor the summation condition $\mathbf{z}_{\mathrm{I}} + \mathbf{z}_{\mathrm{II}} + \mathbf{z}_{\mathrm{III}} = 1$ implies $\mathbf{z}_T = 1$ for some $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and the rest are zero. This yields the following corollary.

Corollary 4.3.8. *A factor is either type* I*, type* II*, or type* III.

Remark 4.3.9. We remark here on some important (but non-trivial) facts whose proofs we have omitted from these notes. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then M is type T if and only if M' is type T. Additionally, if M is type T then pMp is type T for any $p \in \mathcal{P}(M)$. If $\mathbf{z}(p) = 1$, then the converse also holds.

Each of the three types can be further refined. We begin with type I.

Definition 4.3.10. Let $M \subset B(\mathcal{H})$ be a type I von Neumann algebra. For $n \in \mathbb{N}$, we say M is **type** I_n if there exists non-zero pairwise orthogonal and equivalent abelian projections $p_1, \ldots, p_n \in \mathcal{P}(M)$ satisfying $p_1 + \cdots p_n = 1$. We say M is **type** I_∞ if there is an infinite family of non-zero pairwise orthogonal and equivalent abelian projections that sum to 1.

A von Neumann algebra can only be type I_n for one $n \in \mathbb{N} \cup \{\infty\}$. Each type I von Neumann algebra uniquely decomposes into a direct sum of type I_1, type I_2,. . ., and type I_∞ von Neumann algebras, and consequently a type I factor is type I_n for exactly one $n \in \mathbb{N} \cup \{\infty\}$. The proofs of these facts are not terribly difficult, but we have omitted them from these notes.

Example 4.3.11.

(1) An abelian von Neumann algebra $A \subset B(\mathcal{H})$ is type I_1. Indeed, $1 \in A$ is an abelian projection and this cannot be further decomposed into a sum of pairwise orthogonal and equivalent projections, because in an abelian von Neumann algebra projections are eqiuvalent if and only if they are equal (see Exercise 4.1.6).

(2) $M_n(\mathbb{C})$ is type I_n. The projections $E_{1,1}, \ldots, E_{n,n} \in M_n(\mathbb{C})$ are non-zero pairwise orthogonal projections that sum to one. They are pairwise equivalent via the partial isometries $E_{i,j}$, and they are minimal (hence abelian) projections.

(3) $B(\mathcal{H})$ for $\dim(\mathcal{H}) = \infty$ is type I_∞. Let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal basis. Then the projections $\{\xi \otimes \bar{\xi} \colon \xi \in \mathcal{E}\}$ are non-zero pairwise orthogonal projections that sum to one. They are pairwise equivalent via the partial isometries $\xi \otimes \bar{\eta}$ for $\xi, \eta \in \mathcal{E}$, and they are minimal projections. ■

Theorem 4.3.12. *If $M \subset B(\mathcal{H})$ is a finite type* I *factor, then $M \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Proof. Since M is type I, $1 \in M$ has a non-zero abelian subprojection, and since M is a factor this abelian projection is minimal by Exercise 4.2.5. Thus M has non-zero minimal projections.

Let $\{p_i : i \in I\} \subset \mathcal{P}(M)$ be a maximal family of pairwise orthogonal minimal projections (note $I \neq \emptyset$ by the above). Consider

$$q := 1 - \sum_{i \in I} p_i.$$

Suppose $q \neq 0$. Then the Comparison Theorem and the factoriality of M imply either $q \preceq p_i$ or $p_i \preceq q$ for $i \in I$. The former implies $q \sim p_i$ since p_i is minimal, but then q is minimal by Proposition 4.2.7 and this contradicts the maximality of the $\{p_i : i \in I\}$. The latter implies $p_i \sim q_0 \leq q$ and the same argument shows q_0 contradicts the maximality of $\{p_i : i \in I\}$. So we must have $q = 0$, and therefore

$$\sum_{\in I} p_i = 1$$

Now, the factoriality of M implies $p_i \sim p_j$ for all $i, j \in I$ by Exercise 4.2.6. We claim that I is finite. If not, then let $I = I_1 \sqcup I_2$ be a partition of I satisfying $|I| = |I_1| = |I_2|$, which implies there is a bijection $\sigma : I \to I_1$. Setting $q_i := p_{\sigma(i)}$, we have $p_i \sim q_i$ for all $i \in I$ and so by Lemma 4.1.10

$$\sum_{i \in I} q_i \sim \sum_{i \in I} p_i = 1.$$

But

$$1 = \sum_{i \in I} p_i = \left(\sum_{i \in I_1} p_i \right) + \left(\sum_{i \in I_2} p_i \right) = \left(\sum_{i \in I} q_i \right) + \left(\sum_{i \in I_2} p_i \right) > \sum_{i \in I} q_i,$$

and so we have contradicted 1 being finite. Thus $n := |I| < \infty$, and so we can relabel $\{p_i : i \in I\} =: \{p_1, p_2, \ldots, p_n\}$. Since $p_1 \sim p_i$ for each $i = 1, \ldots, n$, we can find $v_i \in M$ satisfying $v_i^* v_i = p_i$ and $v_i v_i^* = p_1$. Using $v_i = p_1 v_i$ for each $i = 1, \ldots, n$ we have for any $x \in M$

$$x = \left(\sum_{i=1}^n p_i \right) x \left(\sum_{j=1}^n p_j \right) = \sum_{i,j=1}^n p_i x p_j = \sum_{i,j=1}^n v_i^* v_i x v_j^* v_j = \sum_{i,j=1}^n v_i^* p_i v_i x v_j^* p_1 v_j = \sum_{i,j=1}^n v_i^* (p_1 v_i x v_j^* p_1) v_j.$$

Because p_1 is minimal there exists a scalar $x_{i,j} \in \mathbb{C}$ so that $p_1 v_i x v_j^* p_1 = x_{i,j} p_1$. Thus we have

$$x = \sum_{i,j=1}^n v_i^* x_{i,j} p_1 v_j = \sum_{i,j=1}^n x_{i,j} v_i^* p_1 v_j = \sum_{i,j=1}^n x_{i,j}.$$

This computation shows that the map

$$\pi : M \ni x \mapsto \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \in M_n(\mathbb{C})$$

is injective. Since

$$p_1 v_i (v_k^* v_\ell) v_j p_1 = \delta_{i=k} \delta_{j=\ell} p_1 v_i v_i^* v_j v_j^* p_1 = \delta_{i=k} \delta_{j=\ell} p_1 p_1 p_1 p_1 = \delta_{i=k} \delta_{j=\ell} p_1,$$

we see that $\pi(v_k^* v_\ell) = E_{k,\ell} \in M_n(\mathbb{C})$. Thus π is a bijection, and we leave it for Exercise 4.3.5 to check that it is also a $*$-homomorphism. □

While we only considered *finite* type I factors in the above theorem, a similar proof (see Exercise 4.3.6) shows that properly infinite (i.e. non-finite) type I factors are of the form $B(\mathcal{H})$ for $\mathcal{H}$ infinite dimensional. Moreover, the form of *any* type I von Neumann algebra $M \subset B(\mathcal{H})$ can be given by a *tensor product* (see Exercise 4.3.7): $M \cong \mathcal{Z}(M) \bar{\otimes} B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Thus the theory of type I von Neumann algebras reduces to measure theory and functional analysis, and consequently researchers today focus their efforts on type II or type III von Neumann algebras.

We move on to the refinement of type II von Neumann algebras.

Definition 4.3.13. A type II von Neumann algebra $M \subset B(\mathcal{H})$ is said to be **type** II_1 if it is finite, and is said to be **type** II_∞ if M is properly infinite.

Equivalently, a von Neumann algebra if type II_1 if it is finite but has no non-zero abelian projections, and a von Neumann algebra is type II_∞ if it is properly infinite but semi-finite and has no non-zero abelian projections. Each type II von Neumann algebra uniquely decomposes into a direct sum of type II_1 and type II_∞ von Neumann algebras, and consequently each type II factor is either type II_1 or type II_∞.

Example 4.3.14. $L(\Gamma)$ for a countable i.c.c. group Γ is type II_1 factor. First note that $L(\Gamma)$ is a factor by Exercise 1.3.7. It is also finite by Exercise 4.3.1. So it remains to show it has no non-zero abelian projections. Suppose, towards a contradiction, that $p \in \mathcal{P}(L(\Gamma))$ is non-zero and abelian. Then p is actually minimal by Exercise 4.2.5. Let $\{p_i\}_{i\in I} \subset \mathcal{P}(L(\Gamma))$ is a maximal family of pairwise orthogonal minimal projections. Then $I \neq \emptyset$ by the above and the exact same argument as in the proof of Theorem 4.3.12 shows $n := |I| < \infty$ and $L(\Gamma) \cong M_n(\mathbb{C})$. Note that $M_n(\mathbb{C})$ is finite dimensional as a vector space. On the other hand, Γ is necessarily infinite as an i.c.c. group and so $\{\lambda(g) : g \in \Gamma\}$ is an infinite linearly independent set (just apply any linear combination to the vector δ_e). So $L(\Gamma) \cong M_n(\mathbb{C})$ yields a contradiction and hence $L(\Gamma)$ has no non-zero abelian projections. ■

Example 4.3.15. In this example we will construct an important type II_1 factor $\mathcal{R}$ called the **hyperfinite** II_1 **factor**. Observe that for any $n \in \mathbb{N}$ we can embed $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{C})$ via

$$M_n(\mathbb{C}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

These inclusions preserve the norm (since they are injective $*$-homomorphisms) and the normalized trace:

$$\frac{1}{2n}\mathrm{Tr}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \frac{1}{n}\mathrm{Tr}(A) \qquad A \in M_n(\mathbb{C}).$$

Thus if we consider the sequence of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow M_{2^n}(\mathbb{C}) \hookrightarrow \cdots .$$

and define $\mathcal{R}_0 := \bigcup_{n\geq 1} M_{2^n}(\mathbb{C})$, then $\mathcal{R}_0$ is a $*$-algebra with a norm (although it is not complete) and a linear functional $\tau_0 : \mathcal{R}_0 \to \mathbb{C}$ defined by $\tau_0(x) = \frac{1}{2^n}\mathrm{Tr}(x)$ when $x \in M_{2^n}(\mathbb{C})$. From the properties of the trace, it follows that τ_0 is

- **unital**: $\tau_0(1) = 1$;
- **positive**: $\tau_0(x^*x) \geq 0$ for all $x \in \mathcal{R}_0$;
- **faithful**: $\tau_0(x^*x) = 0$ if and only if $x = 0$;
- **tracial**: $\tau_0(xy) = \tau_0(yx)$ for all $x, y \in \mathcal{R}_0$.

We can therefore consider the GNS representation $(\mathcal{H}, \pi)$ for $(\mathcal{R}_0, \tau_0)$, and $\mathcal{R}_0$ gives a dense subspace of $\mathcal{H}$. Define

$$\mathcal{R} := \pi(\mathcal{R}_0)'' \subset B(\mathcal{H}).$$

We will show that $\mathcal{R}$ is a II_1 factor. We must first show it admits a WOT continuous faithful tracial state.

Viewing $1 \in \mathcal{R}_0$ as a vector in $\mathcal{H}$, we see that it is cyclic for $\mathcal{R}$ by construction. It is also separating for $\mathcal{R}$: it is separating for $\pi(\mathcal{R}_0)$ since τ_0 is faithful, so it is cyclic for $\pi(\mathcal{R}_0)'$ and hence separating for $\pi(\mathcal{R}_0)'' = \mathcal{R}$ by Proposition 2.2.4. Thus the linear functional $\tau : \mathcal{R} \to \mathbb{C}$ defined by $\tau(x) = \langle x1, 1\rangle$ is faithful, and as a vector state it WOT continuous. Using $\tau(\pi(x)) = \tau_0(x)$ for $x \in \mathcal{R}_0$, it can be shown that τ also tracial (see Exercise 4.3.10).

Now, suppose $z \in \mathcal{Z}(\mathcal{R})$). Define $\varphi : \mathcal{R} \to \mathbb{C}$ by $\varphi(x) := \tau(xz)$, which is still tracial since z commutes with everything in $\mathcal{R}$. Consequently, restricting $\varphi \circ \pi$ to $M_{2^n}(\mathbb{C})$ gives a tracial linear functional, and thus Exercise 1.3.2 implies

$$\varphi \circ \pi(x) = \varphi \circ \pi(1)\frac{1}{2^n}\mathrm{Tr}(x) = \tau(z)\tau(\pi(x))$$

for all $x \in M_{2^n}(\mathbb{C})$. Since this holds for all $n \in \mathbb{N}$, we have $\varphi(x) = \tau(z)\tau(x)$ for all $x \in \pi(\mathcal{R}_0)$. The WOT density of $\pi(\mathcal{R}_0)$ along with the WOT conintuity of τ implies this holds for all $x \in \mathcal{R}$. Thus $\tau(xz) = \tau(z)\tau(x)$, or equivalently $\tau(x(z - \tau(z))) = 0$ for all $x \in \mathcal{R}$. In particular, letting $x = (z - \tau(z))^*$ we see that the faithfulness of τ implies $z - \tau(z) = 0$ or $z = \tau(z) \in \mathbb{C}$. Thus $\mathcal{R}$ is a factor.

To see that it is finite, suppose $v \in \mathcal{R}$ is a partial isometry satisfying $v^*v = 1$ and $vv^* \leq 1$, then

$$\tau((1 - vv^*)^*(1 - vv^*)) = \tau(1 - vv^*) = \tau(1) - \tau(vv^*) = \tau(1) - \tau(v^*v) = \tau(1) - \tau(1) = 0.$$

Since τ is faithful, we must have $vv^* = 1$ and so $\mathcal{R}$ is finite.

It remains to show that $\mathcal{R}$ has no non-zero abelian projections. Proceeding exactly as in Example 4.3.14, we see that if this is not the case then $\mathcal{R} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. This is a contradiction because $M_n(\mathbb{C})$ is finite dimensional while $\mathcal{R}$ is infinite dimensional since $\pi(\mathcal{R}_0)$ is infinite dimensional. Thus $\mathcal{R}$ is a type II_1 factor. ∎

The term *hyperfinite* refers to the fact that $\mathcal{R}$ is generated by the finite dimensional algebras $\pi_\tau(M_{2^n}(\mathbb{C}))$. Alain Connes showed in 1976 that $\mathcal{R}$ is the *unique* II_1 factor with this property. Moreover, this same work, as mentioned back in Section 1.3.3, shows that the two previous examples coincide when Γ is an amenable i.c.c. group.

Example 4.3.16. Let (X, Ω, μ) be a probability space and let Γ be a countable discrete group. Suppose there is a homomorphism $\alpha\colon \Gamma \to \mathrm{Aut}(L^\infty(X, \mu))$, where $\mathrm{Aut}(L^\infty(X, \mu))$ is the set of (normal) $*$-isomorphisms. In this case we call α an **action** of Γ on $L^\infty(X, \mu)$ and write $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$. We say the action is

- **probability measure preserving (p.m.p.)** if $\int_X \alpha_g(f)\, d\mu = \int_X f\, d\mu$ for all $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.
- **free** if for $f \in L^\infty(X, \mu)$ and $g \in \Gamma$, we have $f = 0$ whenever $f\alpha_g(h) = fh$ for all $h \in L^\infty(X, \mu)$.
- **ergodic** if $f \in L^\infty(X, \mu)$ is such that $\alpha_g(f) = f$ for all $g \in \Gamma$ then $f = \mathbb{C}1$.

For $f \in L^\infty(X, \mu)$, define a linear operator $\pi_\alpha(f)$ on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ by

$$\pi_\alpha(f)\left(\sum_{g \in \Gamma} \delta_g \otimes f_g\right) = \sum_{g \in \Gamma} \delta_g \otimes [\alpha_{g^{-1}}(f) f_g] \qquad f_g \in L^2(X, \mu).$$

Then one can show that $\pi_\alpha(f) \in B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ and $\pi_\alpha\colon L^\infty(X, \mu) \to B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ is a normal unital injective $*$-homomorphism (Exercise 4.3.11). For $g \in \Gamma$, we define

$$\lambda(g)\left(\sum_{h \in \Gamma} \delta_h \otimes f_h\right) = \sum_{h \in \Gamma} \delta_{gh} \otimes f_h \qquad f_h \in L^2(X, \mu).$$

Note that $\lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha(f))$ (Exercise 4.3.11.(c)). This implies the $*$-algebra generated by $\pi_\alpha(L^\infty(X, \mu))$ and $\lambda(\Gamma)$ is the set

$$\mathbb{C}\left\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma) \right\rangle := \left\{ \sum_{j=1}^d \pi_\alpha(f_j)\lambda(g_j) \colon d \in \mathbb{N},\ f_1, \ldots, f_d \in L^\infty(X, \mu),\ g_1, \ldots, g_d \in \Gamma \right\}.$$

Note that $\mathbb{C}\left\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma) \right\rangle$ is unital. The von Neumann algebra

$$L^\infty(X, \mu) \rtimes_\alpha \Gamma := \mathbb{C}\left\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma) \right\rangle''$$

is called the **crossed product** of $L^\infty(X, \mu)$ by Γ. You should think of it as a von Neumann algebra containing both $L^\infty(X, \mu)$ and $L(\Gamma)$ with the action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ encoded via commutation relations. Consider the normal linear functional $\tau\colon L^\infty(X, \mu) \rtimes_\alpha \Gamma \to \mathbb{C}$ defined by $\tau(x) = \langle x(\delta_e \otimes 1), \delta_e \otimes 1 \rangle$. Since $\delta_e \otimes 1$ is a unit vector and separating for $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ (see Exercise 4.3.13), τ is a unital and faithful.

Assume $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ is a free ergodic p.m.p. action and that Γ is an infinite group. The freeness and ergodicity imply $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is a factor by Exercise 4.3.15, while the action being p.m.p implies τ

is tracial by Exercise 4.3.16. Consequently, by the same arguement as in the previous two examples we see that $L^\infty(X,\mu)\rtimes_\alpha\Gamma$ is finite and therefore either a type I_n or type II_1 factor. Since $L(\Gamma)\subset L^\infty(X,\mu)\rtimes_\alpha\Gamma$ and Γ is infinite, we see that the crossed product is not finite dimensional. Thus $L^\infty(X,\mu)\rtimes_\alpha\Gamma$ is a type II_1 factor. ■

We only considered type II_1 factors in the examples above, but for any type II_1 von Neuman algebra M the tensor product $M\bar{\otimes}B(\mathcal{H})$ for $\mathcal{H}$ infinite dimensional yields a type II_∞ von Neumann algebra. In fact, all type II_∞ factors are of this form.

The class of type III factors can also be further decomposed into types III_λ for $\lambda\in[0,1]$. This classification is achieved via some very beautiful mathematics known as *Tomita–Takesaki theory*. Essentially, von Neumann algebras of this type have intrinsic dynamical systems which determine the parameter $\lambda\in[0,1]$.

We conclude this chapter with a summary of types for factors. Recall that factor is either finite or properly infinite. We will also say a factor is **atomic** if it contains a minimal projection, and otherwise say it is **diffuse**.

	atomic	diffuse	
finite	type I_n, $n\in\mathbb{N}$	type II_1	
properly infinite	type I_∞	type II_∞	type III
	semi-finite		purely infinite

Exercises

4.3.1. Let Γ be a countable discrete group. Show that all projections in $L(\Gamma)$ are finite. [Hint: use the trace.]

4.3.2. Let $M\subset B(\mathcal{H})$ be a von Neumann algebra and let $p,q\in\mathcal{P}(M)$ satisfy $p\preceq q$. Show that if q is semi-finite then p is semi-finite.

4.3.3. Let $\pi\colon M\to N$ be a $*$-isomorphism between von Neumann algebras and let $p\in\mathcal{P}(M)$.

(a) Show p is finite in M if and only if $\pi(p)$ is finite in N.

(b) Assuming π is normal, show p is semi-finite in M if and only if $\pi(p)$ is finite in N.

(c) Show p is purely infinite in M if and only if $\pi(p)$ is purely infinite in N.

(d) Show p is properly infinite in M if and only if $\pi(p)$ is properly infinite in N.

4.3.4. Let $\pi\colon M\to N$ be a normal $*$-isomorphism between von Neumann algebras. Show that M has type T for $T\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$ if and only if N has type T.

4.3.5. Let $\pi\colon M\to M_n(\mathbb{C})$ be the map defined at the end of the proof of Theorem 4.3.12. Show that π is a unital $*$-homomorphism.

4.3.6. Let $M\subset B(\mathcal{H})$ be properly infinite type I factor. In this exercise, you will show that $M\cong B(\mathcal{K})$ for some infinite dimensional Hilbert space $\mathcal{K}$.

(a) Show that M admits an infinite family $\{p_i\colon i\in I\}$ of pairwise orthogonal and equivalent minimal projections satisfying
$$\sum_{i\in I}p_i=1.$$

(b) Fix $i_0\in I$ and let $v_i\in M$ be a partial isometry satisfying $v_i^*v_i=p_i$ and $v_iv_i^*=p_{i_0}$. For each $x\in M$ and $i,j\in I$, show that there is a scalar $x_{i,j}\in\mathbb{C}$ so that $p_ixp_j=x_{i,j}v_i^*v_j$.

(c) Denote $\mathcal{K}_0:=\operatorname{span}\{p_i\colon i\in I\}$. Show
$$\left\langle\sum_{k=1}^m\alpha_kp_{i_k},\sum_{\ell=1}^n\beta_\ell p_{j_\ell}\right\rangle:=\sum_{k=1}^m\sum_{\ell=1}^n\alpha_k\bar{\beta}_\ell\delta_{i_k=j_\ell}$$
defines an inner product on $\mathcal{K}_0$.

(d) Let $\mathcal{K}$ be the completion of $\mathcal{K}_0$ with respect to this inner product. For each $x \in M$ show that

$$\pi(x) := \sum_{i,j\in I} x_{i,j} p_i \otimes \bar{p}_j \in B(\mathcal{K}),$$

where here we are viewing $p_i, p_j \in \mathcal{K}$ so that $p_i \otimes \bar{p}_j \in FR(\mathcal{K})$.

(e) Show that $\pi\colon M \to B(\mathcal{K})$ is normal $*$-isomorphism.

4.3.7. Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces, and for each $j = 1, \ldots, n$ and $x \in B(\mathcal{H}_j)$ define a a linear operator $\pi_j(x)$ on $\mathcal{H}_1 \odot \cdots \odot \mathcal{H}_n$ by

$$\pi_j(x)(\xi_1 \otimes \cdots \otimes \xi_j \otimes \cdots \otimes \xi_n) := \xi_1 \otimes \cdots \otimes (x\xi_j) \otimes \cdots \otimes \xi_n \qquad \xi_1 \in \mathcal{H}_1, \ldots, \xi_n \in \mathcal{H}_n.$$

(a) For $j = 1, \ldots, n$ and $x \in B(\mathcal{H}_j)$, show that $\pi_j(x)$ extends to a bounded operator on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with $\|\pi_j(x)\| = \|x\|$.

(b) Show that $\pi_j\colon B(\mathcal{H}_j) \to B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ is a unital $*$-homomorphism for each $j = 1, \ldots, n$.

(c) Show that $\pi_j(B(\mathcal{H}_j))$ and $\pi_k(B(\mathcal{H}_k))$ commute for $j \neq k$.

(d) Let $M_j \subset B(\mathcal{H}_j)$ be a von Neumann algebra for each $j = 1, \ldots, n$. Show that

$$M_1 \otimes \cdots \otimes M_n := \operatorname{span}\{\pi_1(x_1)\cdots\pi_n(x_n)\colon x_1 \in M_1, \ldots, x_n \in M_n\}.$$

is a unital $*$-algebra.

(e) The **tensor product** of $M_1, \ldots, M_n$ is the von Neumann algebra

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n := (M_1 \otimes \cdots M_n)''$$

Show that if $M_2 = \cdots = M_n = \mathbb{C}$, then $M_1 \bar{\otimes} \cdots \bar{\otimes} M_n \cong M_1$.

4.3.8. Using the notation from Example 4.3.15, show that $\mathcal{R}_0$ can be viewed as an inductive limit (see [Definition 6.1, GOALS Prerequisite Notes]).

4.3.9. Using the notation from Example 4.3.15, show that for verify that τ_0 is unital, positive, faithful, and tracial.

4.3.10. Using the notation from Example 4.3.15, show that τ is tracial. [**Hint:** first show $\tau(xy) = \tau(yx)$ for $x \in \mathcal{R}$ and $y \in \pi(\mathcal{R}_0)$ using the SOT density of $\pi(\mathcal{R}_0)$.]

4.3.11. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ for a countable discrete group Γ and a probability space (X, μ).

(a) For $f \in L^\infty(X, \mu)$, show that $\pi_\alpha(f)$ is a bounded operator on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ with $\|\pi_\alpha(f)\| = \|f\|_\infty$.

(b) Show that $\pi_\alpha\colon L^\infty(X, \mu) \to B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ is a unital $*$-homomorphism.

(c) Show that $\lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha_g(f))$ for all $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.

4.3.12. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ for a countable discrete group Γ and a probability space (X, μ). For $f \in L^\infty(X, \mu)$, define $\phi_\alpha(f) \in B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ by

$$\phi_\alpha(f)\left(\sum_{g\in\Gamma} \delta_g \otimes f_g\right) = \sum_{g\in\Gamma} \delta_g \otimes f f_g \qquad f_g \in L^2(X, \mu),$$

and define $\rho(g)$ for $g \in \Gamma$ by

$$\rho(g)\left(\sum_{h\in\Gamma} \delta_h \otimes f_h\right) = \sum_{h\in\Gamma} \delta_{hg^{-1}} \otimes \alpha_g(f_h) \qquad f_h \in L^2(X, \mu).$$

Show that $\phi_\alpha(L^\infty(X, \mu)) \cup \rho(\Gamma) \subset (L^\infty(X, \mu) \rtimes_\alpha \Gamma)'$.

4.3.13. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group Γ and a probability space (X,μ).

(a) Show that $\delta_e \otimes 1$ is a cyclic vector for $L^\infty(X,\mu) \rtimes_\alpha \Gamma$.

(b) Show that $\delta_e \otimes 1$ is a separating vector for $L^\infty(X,\mu) \rtimes_\alpha \Gamma$.

[**Hint:** use Exercise 4.3.12.]

4.3.14. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group Γ and a probability space (X,μ). For $x \in L^\infty(X,\mu) \rtimes_\alpha \Gamma$, define a linear operator x_g on $L^2(X,\mu)$ by

$$x_g(f) = [x(\delta_{g^{-1}} \otimes f)](e).$$

Show that $x_g \in L^\infty(X,\mu)$. [**Hint:** show that $x_g \in L^\infty(X,\mu)'$ by using ϕ_α as in Exercise 4.3.12.]

4.3.15. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group Γ and a probability space (X,μ).

(a) Show that $L^\infty(X,\mu)' \cap L^\infty(X,\mu) \rtimes_\alpha \Gamma = L^\infty(X,\mu)$ if and only if the action is free.

[**Hint:** using the notation from Exercise 4.3.14, compare $(xf)_g$ and $(fx)_g$ for $x \in L^\infty(X,\mu)' \cap L^\infty(X,\mu) \rtimes_\alpha \Gamma$ and $f \in L^\infty(X,\mu)$.]

(b) Assuming the action is free, show that $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is a factor if and only if the action is ergodic.

4.3.16. Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group Γ and a probability space (X,μ). Let $\tau\colon L^\infty(X,\mu) \rtimes_\alpha \Gamma \to \mathbb{C}$ be as in Example 4.3.16.

(a) Show that $\tau(\lambda(g)) = \delta_{g=e}$ for $g \in \Gamma$.

(b) Show that $\tau(\pi_\alpha(f)) = \int_X f\, d\mu$ for $f \in L^\infty(X,\mu)$.

(c) Assume that the action is probability measure preserving. Show that τ is a tracial.

4.3.17. In this exercise, you will show that $M_n(\mathbb{C})$ can be realized via a crossed-product construction. Consider $\Gamma := \mathbb{Z}_n$, the countable cyclic group of order n, and also set $X := \mathbb{Z}_n$ which we view as simply a space and equip with the counting (probability) measure.

(a) Show that $\alpha_g(f) := f(\,\cdot\, - g)$ for $g \in \Gamma$ defines an action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$.

(b) Show that $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ is free, ergodic, and probability measure preserving.

(c) Show that $1_{\{1\}}, \ldots, 1_{\{n\}} \in L^\infty(X,\mu)$ are pairwise orthogonal and equivalent minimal projections.

(d) Show that $L^\infty(X,\mu) \rtimes_\alpha \Gamma \cong M_n(\mathbb{C})$. What is the preimage of $E_{i,j}$ under this isomorphism?

(e) Explain why there does not exist a discrete group Γ such that $L(\Gamma) \cong M_n(\mathbb{C})$.

Chapter 5

The Trace

In the previous chapters we saw that the $M_n(\mathbb{C})$ group von Neumann algebras and the hyperfinite II_1 factor are all examples tracial von Neumann algebras. The main purpose of this chapter is to show admitting a trace that characterizes all finite von Neumann algebras.

The first section give some structural results for the projections on finite von Neumann algebras with an emphasis on the construction of the dyadic projections and the range of the trace on II_1 factors. The next section examines the outcome of the GNS contruction when applied to a trace on a factor, which is called the standard representation. There are many like it, but this one is ours. We also introduce ultrapower and ultraproduct constructions to help us define technical invariants for von Neumann algebras, namely the McDuff Property and Property Γ.

We leave the details of the construction of a center-valued trace to the very end of the chapter for those who want to punish themselves.

Lecture Preview: The content in this lecture will be covered over 2 days. The first of these lectures on the 10th of July will cover Pages 64–66 properties of the trace for finite von Neumann algebras. To prepare yourself for the lecture, it is highly encouraged that you know 5.4.1. Lemmas 5.1.1 through 5.1.3 will be briefly discussed, but proof will likely not be presented. Definition 5.1.4 onward will provide the bulk of the content. Please review the statements of Theorem 5.4.8, 5.4.9 , Theorem 5.4.10 as they will be referenced.

The lecture on Monday the 13th of July will describe the Standard Representation of a II_1 factor (Pages 67–70). If time allows, we will describe the ultraprocduct construction (see Definition 5.3.2).

5.1 Tracial von Neumann Algebras

Lemma 5.1.1. *Let $M \subset B(\mathcal{H})$ be a finite von Neumann algebra and $p \in \mathcal{P}(M)$ non-zero. If $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ is a family of pairwise orthogonal projections satisfying $p_i \sim p$ for all $i \in I$, then $|I| < \infty$.*

Proof. If I is infinite, then there exists a proper subset $J \subset I$ with $|J| = |I|$. But then

$$\sum_{i\in I} p_i \sim \sum_{j\in J} p_j < \sum_{i\in I} p_i,$$

contradicting M being finite. □

Lemma 5.1.2. *Let $M \subset B(\mathcal{H})$ be a type II_1 von Neumann algebra. Then there exists a projection $p_{1/2} \in \mathcal{P}(M)$ so that $p_{1/2} \sim 1-p_{1/2}$. Moreover, there exists a family of projections $\{p_r\}_r$ indexed by dyadic rationals $r \in [0,1]$ such that:*

(i) $p_r \leq p_s$ *if* $r \leq s$;

(ii) $p_s - p_r \sim p_{s'} - p_{r'}$ *whenever* $0 \leq r \leq s \leq 1$ *and* $0 \leq r' \leq s' \leq 1$ *satisfy* $s - r = s' - r'$;

(iii) $z(p_r) = 1$ *for every* r.

Proof. Let $\{p_i, q_i\}_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$. Define $p_{1/2} := \sum_i p_i$ and $q = \sum_i q_i$. Then $p_{1/2} \sim q$, and we further claim $q = 1 - p_{1/2}$. If not, then $1 - (p_{1/2} + q) \neq 0$. Since M is type II, $1 - (p_{1/2} + q)$ is not abelian and consequently there exists $p_0 \in \mathcal{P}([1 - (p_{1/2} + q)]M[1 - (p_{1/2} + q)])$ which is strictly less than its central support (in this corner), which we will denote by z. Therefore, if $q_0 = z - p_0$, then p_0 and q_0 are not centrally orthogonal, and consequently by Proposition 4.1.9 they have equivalent subprojections. However, this contradicts the maximality of $\{p_i, q_i\}_{i\in I}$. Thus $q = 1 - p_{1/2}$.

Now, we construct the family of projections indexed by dyadic radicals $r \in [0, 1]$ inductively. We let $p_{1/2}$ be as above, and set $p_1 := 1$ and $p_0 := 0$. Let $v \in M$ be such that $v^*v = p_{1/2}$ and $vv^* = 1 - p_{1/2}$. Note that $p_{1/2}Mp_{1/2}$ is type II by Remark 4.3.9. Moreover, it is type II_1 since $p_{1/2}$ is a finite projection: if $q \sim p_{1/2}$ with $q < p_{1/2}$ then $q+(1-p_{1/2}) \sim p_{1/2}+(1-p_{1/2}) = 1$ by Lemma 4.1.10, but $q+(1-p1/2) < p_{1/2}+(1-p_{1/2}) = 1$ contradicts 1 being finite. Thus $p_{1/2}Mp_{1/2}$ is type II_1 and so the above argument yields $p_{1/4} \leq p_{1/2}$ such that $p_{1/4} \sim p_{1/2} - p_{1/4}$. Set $p_{3/4} := p_{1/2} + vp_{1/4}v^*$. It is easily observed that $p_0 \leq p_{1/4} \leq p_{1/2} \leq p_{3/4} \leq p_1$ and $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for each $k = 0, 1, 2, 3$. Induction then yields a family satisfying (i) and (ii).

To see (iii), fix a dyadic rational r and set $z := 1 - z(p_r)$. Let $n \in \mathbb{N}$ be large enough so that $s := \frac{1}{2^n} \leq r$. Then by (i) we have $zp_s \leq zp_r = 0$. Using (ii), we have $zp_s \sim z(p_{ks} - p_{(k-1)s})$ for every $k = 1, \ldots, 2^n$, and so it must be that $z(p_{ks} - p_{(k-1)s}) = 0$. We then have

$$z = z\sum_{k=1}^{n}(p_{ks} - p_{(k-1)s}) = 0,$$

so that $z(p_r) = 1$ as claimed. □

Lemma 5.1.3. *Let $M \subset B(\mathcal{H})$ be a type II_1 von Neumann algebra, and let $\{p_r\}_r \subset \mathcal{P}(M)$ be the family of projections indexed by dyadic rationals $r \in [0, 1]$ as in the previous lemma. If $p \in \mathcal{P}(M)$ is non-zero, then there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ and a dyadic rational $r \in (0, 1]$ so that $p_r z \preceq pz$ and $p_r z, pz \neq 0$.*

Proof. By considering the compression $Mz(p)$, we may assume $z(p) = 1$. By the Comparison Theorem, for each dyadic rational $r \in (0, 1]$ there exists a central projection z_r such that $p_r z_r \preceq pz_r$ and $p(1 - z_r) \preceq p_r(1 - z_r)$. Suppose, towards a contradiction, $pz_r = 0$ for every r. Since $z(p) = 1$, it must be that $z_r = 0$ and so $p \preceq p_r$ for all r. In particular, we have for each $k \in \mathbb{N}$

$$p \preceq p_{2^{-(k+1)}} \sim p_{2^{-k}} - p_{2^{-(k+1)}}.$$

For each $k \in \mathbb{N}$, let $q_k \leq p_{2^{-k}} - p_{2^{-(k+1)}}$ be such that $p \sim q_k$. But then $\{q_k\}_{k\in\mathbb{N}}$ is an infinite family or pairwise orthogonal projections that contradicts Lemma 5.1.1. Thus there must be some r such that $pz_r \neq 0$. Consequently, $z_r \neq 0$ and so $p_r z_r \neq 0$ since $z(p_r) = 1$. □

THe existence of the dyadic projections is one of the first steps in constructing a trace on a type II_1 von Neumann algebra. The general idea would be to create a map from the dyadic projections mapping $\phi(p_r) \to r$ and then attempting to extend this map from M to $\mathcal{Z}(M)$. In Section 5.4, take an alternate route applying the Ryll-Nardjewski Theorem and other Banach space techniques. Unfortunatley, both paths we described as long and highly technical which is why we are instead choosing to accept that finite von Neuamm algebras have traces.

Definition 5.1.4. Let M be a von Neumann algebra. If $\tau : M \to \mathbb{C}$ is if there exists a normal, faithful state which also satisfies the trace condition, $\tau(xy) = \tau(yx)$, the τ is called a **trace** on M. We say M is **tracial** if M admits a trace.

Assuming that a trace exists, we know that M is automatically finite. The converse, however, is much more difficult and relies upon the construction of a center-valued trace, (see definition 5.4.1). This can be done, and the approach we take relies on heavy-handed Banach space techniques.

The upshot is that once you know the center valued state $\phi : M \to \mathcal{Z}(M)$ exists, we identify $\pi : \mathcal{Z}(M) \to L^\infty(X, \mu)$ (assuming that M has a cyclic vector). An even better situation comes up when M a factor because the center-valued trace is automatically a trace and we can stop here.

Theorem 5.1.5. *A von Neumann algebra M is finite if and only if M has a trace M is a finite factor if and only if M admits a unique trace $\tau : M \to \mathbb{C}$.*

Theorem 5.1.6. *Let M be a finite factor equipped with its unique trace τ.*

- *If M is of type I, then M is of type I_n with n finite and $\tau(\mathcal{P}(M)) = \{0, \frac{1}{n}, \ldots, 1\}$.*
- *If M is of type II_1, then $\tau(\mathcal{P}(M)) = [0, 1]$.*

One interpretation of the values of the trace on projections of a finite type I_n is that it tells us the size of the space onto which p projects relative to the ambient space. The trace on a II_1 factor is similar, except now, the relative size of a projection can be associated to a number in the continuum $[0, 1]$ and moreover, every value is realized.

I like to remind myself that every projection in $M_n(\mathbb{C})$ can be unitarily conjugated to a diagonal projection with the only non-zero entries being 1 somewhere along the diagonal. Here, we can view the trace as something akin to the normalized counting measure on a set of n points.

The picture that I have for II_1 factors is remarkably similar, except first I start with a "matrix" indexed by the interval $[0, 1]$ and mentally identity the diagonal with the interval $[0, 1]$. We might imagine an projection of trace t in a II_1 factor with "1's along the interval $[0, t]$. This allows to view the trace as a non-commutative analog of the Lebesgue measure on a $[0, 1]$.

The fact that τ is normal implies that for a countable collection of orthogonal projections, $\tau(\sum p_i) = \sum \tau(p_i)$. Since projections are the analogs of characteristic functions and the trace is similar to a measure, we interpret this as a kind of countable additivity .

If $M \subseteq B(\mathcal{H})$ is a II_1 factor with trace τ, then for any non-zero projection p we have that $pMp \subseteq B(p\mathcal{H})$ is also a type II_1 factor with trace given by $\tau(pxp)/\tau(p)$ (remember, p is the identity element of pMp). Now suppose that q is another projection such that $\tau(q) = \tau(p)$. Since M is a factor, we have that $p \sim q$ and $1 - p \sim 1 - q$ and thus, there is a unitary $u \in M$ so that $u^*pMpu = qMq$ and thus the isomorphism class of pMp depends only on $t = \tau(p)$ and not the choice of projection. Then for any $0 < t \leq 1$, we define define $M^t := pMp$ where p is any trace t projection.

It's also possible to extend the definition of M^t for any $t \geq 1$ by first choosing $n \in \mathbb{N}$ with $n \geq t$, and considering $M_n(N)$. $M_n(\mathbb{C})$ is again a II_1 factor with trace $\tau_n([x_{i,j}]) = \sum_{i=1}^n \tau(x_{i,i})$. Choosing a projection $p \in M_n(M)$ with trace $\tau_n(p) = t/n$, $M^t = pM_n(M)p$. We can check that up to isomorphism, M^t does not depend on our p or n and thus is well defined.

Definition 5.1.7. Let M be a type II_1 factor. The **fundamental group** of M is the subgroup of $\mathbb{R}_+$

$$\mathcal{F}(M) := \{t \in (0, \infty) : M^t \cong M\}.$$

The terminology here is unfortunate since this has concept no relation to the better-know fundamental group from topology. Mentioning the fundamental group of a II_1 factor in a talk or in casual conversation will almost surely result in someone asking if this has any connection to topology. My advice, just say "no" and then change the subject.

It is in fact a multiplicative subgroup of $\mathbb{R}$, which can be checked by verifying that for any $s, t > 0$ we have $(M^t)^s \cong M^{st}$.

When M is a tracial factor, there is another norm that one frequently encounters called the *2-norm.* Letting τ be the unique trace on M, via the formula

$$\|x\|_2 = \sqrt{\tau(x^*x)}.$$

Since τ is faithful, we see that M this formula indeed defines a norm on M. The trace also induces a Hilbert space structure on M via the formlula $\langle x, y\rangle = \tau(y^*x)$. Unfortunately, M is not complete with respect to this norm but it's completion is of interest. We delay that discussion for now. Instead, let's compare the 2-norm and the oeprator norm of a finte von Neumann algebra.

Theorem 5.1.8. *Let M be a tracial von Neumann algebra with trace τ. Then for any $x, y \in M$ we have that*

$$\|xy\|_2 \leq \|x\|\|y\|_2.$$

In particular, $\|x\|_2 \leq \|x\|$

Proof. We first prove that for any self-adjoint $w \in M$, $w \leq \|w\|1$, were $1 \in M$ is the identity element. Define $f(t) = \|w\| - t$ on $[-\|w\|, \|w\|]$. Then by the continuous functional calculus, we have that $\sigma(f(a)) \subset f(\sigma(a)) \subseteq [0, \infty)$ and thus $\|w\| - w \geq 0$. In particular $x^*x \leq \|x\|^2$.

Now let us compute:

$$\|xy\|_2^2 = \tau(y^*x^*xy) \leq \tau(\|x\|^2 y^*y) = \|x\|^2\tau(y^2y) = \|x\|^2\|y\|_2^2.$$

□

Exercises

5.1.1. Let $\mathcal{R}$ be the hyperfinite II_1 factor.

(a) Show for every dyadic rational $r \in [0,1]$, there exists a projection $p_r \in \mathcal{R}$ with $\tau(t_r) = r$. Hint: think about the construction of $\mathcal{R}$ as an inductive limit.

(b) Now if $t \in [0,1]$, show that there exists a projection $p_t \in \mathcal{R}$ with $\tau(p_t) = t$. Hint: if $t \in [0,1]$, there exists an increasing sequence (r_n) of dyadic rationals such $r_n \to t$.

5.1.2. Show that a von Neumann algebra M is finite if and only if for every $x, y \in M$ such that $xy = 1$ we have $yx = 1$, i.e. if X is right invertible, it is invertible.

5.1.3. Let M be a type II_∞ factor and p a finite projection in M. Show that there exists an infinite family of orthogonal projections $\{p_{i_{i\in I}}\}$ with $p_i \sim p$ and $\sum_{i\in I} p_i = 1$. If $\tau : pMp \to \mathbb{C}$ is the trace on pMp and $v_i \in M$ with $v_i^*v_i = p, v_iv_i^* = p_i$, show that

$$\widetilde{\tau}(x) := \sum_{i\in I} \tau(v_i^*xv_i)$$

defines a normal tracial map. This is called a semi-finite trace on M.

5.1.4. Let M be a factor and $d : \mathcal{P}(M) \to [0, \infty]$ be a function such that

(i) $d(p + q) = d(p) + d(q)$ whenever $pq = 0$.

(ii) $d(p) = d(q)$ whenever $p \sim q$.

(iii) $d(p) = 0$ implies that $p = 0$.

Then any such d is called a dimension function.

(a) Show that M is finite if and only if there exists a dimension function d with $d(1) = 1$.

(b) When M is finite, show that $d = \tau|_{\mathcal{P}(M)}$ where τ is the trace on M.

(c) If M is type II_∞, show that any such function which is not identically 0 must take every value in $[0, \infty]$.

(d) If M is type III, show that $d(p) \in \{0, \infty\}$

5.1.5. Let M be a type II_1 factor.

(a) Show that $(M^t)^s \cong M^{ts}$.

(b) Conclude that the (poorly named IMO) fundamental group $\mathcal{F}(M)$ is in fact a subgroups of $\mathbb{R}_+$.

5.2 The Standard Representation

Let M be finite factor with unique faithful normal tracial state $\tau\colon M \to \mathbb{C}$. We denote by $L^2(M)$ the GNS Hilbert space associated to τ; that is,

$$\langle x, y\rangle_2 := \tau(y^*x) \qquad x, y \in M$$

defines an inner product on M and we take $L^2(M)$ to be its completion. For $x \in M$, we will sometimes add the decoration $\hat{x}$ when we want to emphasize that we are thinking of x as a vector in $L^2(M)$. We also obtain a faithful normal representation $\pi_\tau\colon M \to B(L^2(M))$ which is defined by $\pi_\tau(x)\hat{y} = \widehat{xy}$ for $x, y \in M$. Let us identify $M \cong \pi_\tau(M)$ so that we view M as a von Neumann algebra in $B(L^2(M))$, and for $x, y \in M$ we have $x\hat{y} = \widehat{xy}$.

Definition 5.2.1. For a finite factor M with unique trace τ, the representation $M \subset B(L^2(M))$ is called the **standard representation** of M.

Note that $x\hat{1} = \hat{x}$ implies $\hat{1}$ is a cyclic vector for M, and

$$\|x\hat{1}\|_2^2 = \langle \hat{x}, \hat{x}\rangle_2 = \tau(x^*x)$$

implies $\hat{1}$ is separating for M since τ is faithful.

Now, for $x \in M$ define $J\hat{x} := \widehat{x^*}$. We note that

$$\|J\hat{x}\|_2^2 = \|\widehat{x^*}\|_2^2 = \tau(xx^*) = \tau(x^*x) = \|\hat{x}\|_2^2.$$

Thus J extends to a conjugate linear isometry on $L^2(M)$.

Definition 5.2.2. For a finite factor M, the conjugate linear isometry J on $L^2(M)$ is called the **canonical conjugation operator**.

Note that since J is conjugate linear, we have $\langle J\xi, J\eta\rangle_2 = \langle \eta, \xi\rangle_2$ for $\xi, \eta \in L^2(M)$. You should also convince yourself that $(JxJ)^* = Jx^*J$ for $x \in B(L^2(M))$ (Exercise 5.2.1). Also observe that for $x, y, z \in M$ we have

$$\begin{aligned} x(JyJ)\hat{z} &= xJy\widehat{z^*} = xJ\widehat{yz^*} = x\widehat{zy^*} = \widehat{xzy^*} \\ &= J\widehat{yz^*x^*} = Jy\widehat{z^*x^*} = JyJ\widehat{xz} = (JyJ)x\hat{z}. \end{aligned}$$

Thus $x(JyJ) = (JyJ)x$ since $\widehat{M}$ is dense in $L^2(M)$. This implies $JMJ \subset M' \cap \mathcal{B}(L^2(M))$. We will show the reverse inclusion holds, but we first need to develop a few concepts. The following definition should remind you of left and right convolvers in $L(\Gamma)$ for a discrete group Γ (see Definition 1.3.4).

Definition 5.2.3. For $\xi \in L^2(M)$ define (potentially unbounded) linear operators $\lambda(\xi)\colon \widehat{M} \to L^2(M)$ and $\rho(\xi)\colon \widehat{M} \to L^2(M)$ by

$$\begin{aligned} \lambda(\xi)\hat{x} &:= (Jx^*J)\xi \qquad x \in M \\ \rho(\xi)\hat{x} &:= x\xi. \end{aligned}$$

We will call $\xi \in L^2(M)$ a **left bounded** (resp. **right bounded**) vector if $\lambda(\xi)$ (resp. $\rho(\xi)$) extends to a bounded operator on $L^2(M)$, and in this case we also denote this extension by $\lambda(\xi)$ (resp. $\rho(\xi)$). We denote by $LB(M)$ (resp. $RB(M)$) the collection of $\lambda(\xi)$ (resp. $\rho(\xi)$) for left-bounded (resp. right-bounded) vectors $\xi \in L^2(M)$.

We make a few observations about left and right bounded vectors. For $\lambda(\xi) \in LB(M)$ and $x \in M$

$$J\lambda(\xi)J\hat{x} = J\lambda(\xi)\widehat{x^*} = J(JxJ)\xi = xJ\xi = \rho(J\xi)\hat{x}.$$

This shows that $\rho(J\xi) \in RB(M)$ and $J\lambda(\xi)J = \rho(J\xi)$. Similarly, we have $J\rho(\xi)J = \lambda(J\xi)$ and hence $J(LB(M))J = RB(M)$. Additionally, for $\rho(\xi) \in RB(M)$ and $x, y \in M$

$$\langle \rho(J\xi)\hat{x}, \hat{y}\rangle_2 = \langle xJ\xi, \hat{y}\rangle_2 = \langle J\xi, x^*\hat{y}\rangle_2 = \left\langle J\widehat{x^*y}, \xi\right\rangle_2 = \left\langle \widehat{y^*x}, \xi\right\rangle_2 = \langle \hat{x}, y\xi\rangle_2 = \langle \hat{x}, \rho(\xi)\hat{y}\rangle_2 = \langle \rho(\xi)^*\hat{x}, \hat{y}\rangle_2 .$$

Thus $\rho(\xi)^* = \rho(J\xi)$, and using our previous identities we see that

$$\lambda(J\xi) = J\rho(\xi)J = J\rho(J\xi)^*J = (J\rho(J\xi)J)^* = \lambda(\xi)^*,$$

so $\lambda(xi)^* = \lambda(J\xi)$. Lastly, we observe that for $\lambda(\xi) \in LB(M)$, $\rho(\eta) \in RB(M)$, and $x, y \in M$

$$\begin{aligned}\langle \lambda(\xi)\rho(\eta)\hat{x}, \hat{y}\rangle_2 &= \langle x\eta, \lambda(J\xi)\hat{y}\rangle_2 = \langle x\eta, Jy^*J(J\xi)\rangle_2 = \langle JyJx\eta, J\xi\rangle_2 = \langle xJyJ\eta, J\xi\rangle_2 \\ &= \langle \xi, JxJyJ\eta\rangle_2 = \langle Jx^*J\xi, yJ\eta\rangle_2 = \langle \lambda(\xi)\hat{x}, \rho(J\eta)\hat{y}\rangle_2 = \langle \rho(\eta)\lambda(\xi)\hat{x}, \hat{y}\rangle_2 .\end{aligned}$$

Thus $\lambda(\xi)\rho(\eta) = \rho(\eta)\lambda(\xi)$, and so $LB(M) \subset RB(M)'$. We collect these observations in the following proposition.

Proposition 5.2.4. *Let M be a finite factor. For $\lambda(\xi) \in LB(M)$ and $\rho(\eta) \in RB(M)$ we have*

$$\begin{aligned}\lambda(\xi)^* &= \lambda(J\xi) = J\rho(\xi)J \\ \rho(\eta)^* &= \rho(J\eta) = J\lambda(\eta)J.\end{aligned}$$

Moreover, $J(LB(M)J = RB(M)$ and $LB(M) \subset RB(M)'$.

Just as left and right bounded vectors should remind of left and right convolvers, the proof of the following theorem should remind you of how we showed $R(\Gamma) = L(\Gamma)'$ (see Theorem 1.3.7).

Theorem 5.2.5. *Let M be a finite factor with trace τ. Under the standard representation $M \subset B(L^2(M))$, we have $M' = JMJ$ where J is the canonical conjugation operator on $L^2(M)$.*

Proof. For $x, y \in M$ we have

$$\lambda(\hat{x})\hat{y} = (Jy^*J)\hat{x} = Jy^*\widehat{x^*} = J\widehat{y^*x^*} = \widehat{xy} = x\hat{y},$$

so that $\lambda(\hat{x}) = x$. Hence $M \subset LB(M)$. Also, for $x \in M'$ and $y \in M$ we have

$$\rho(x\hat{1})\hat{y} = yx\hat{1} = xy\hat{1} = x\hat{y},$$

so that $\rho(x\hat{1}) = x$. Hence $M' \subset RB(M)$. Thus

$$M \subset LB(M) \subset RB(M)' \subset (M')' = M,$$

where the second inclusion follows from Proposition 5.2.4. Thus $M = LB(M) = RB(M)'$. Similarly,

$$M' \subset RB(M) \subset LB(M)' \subset M',$$

and so $M' = RB(M) = LB(M)'$. Thus $M' = RB(M) = J(LB(M))J = M$. $\square$

One consequence of the above theorem is that $M' \cap B(L^2(M))$ is also a finite factor. Indeed, $\tau'(Jx^*J) := \tau(x)$ for $x \in M$ defines a trace on M'. However, this need not be true for an arbitrary representation $M \subset B(\mathcal{H})$ of a finite factor.

We change topics slightly here and derive another important concept from the standard representation.

Definition 5.2.6. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $1_M \in N \subset M$ a von Neumann subalgebra. A **conditional expectation** from M to N is a linear map $E\colon M \to M$ satisfying

(i) $E(a) = a$ for all $a \in N$;

(ii) $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in M$;

(iii) $E(x) \geq 0$ whenever $x \geq 0$.

Observe for $x \in M$ that one has

$$\begin{aligned} 0 \leq E\left((x-E(x))^*(x-E(x))\right) &= E\left(x^*x - x^*E(x) - E(x)^*x + E(x)^*E(x)\right) \\ &= E(x^*x) - E(x^*)E(x) - E(x)^*E(x) + E(x)^*E(x) = E(x^*x) - E(x^*)E(x). \end{aligned}$$

So $E(x^*)E(x) \leq E(x^*x)$. Since E preserves positive elements, decomposing x as a linear combination of four positive elements yields $E(x^*) = E(x)^*$. Thus we have $E(x)^*E(x) \leq E(x^*x) \leq E(\|x\|^2 1_M) = \|x\|^2 1_M$, which implies $\|E(x)\| \leq \|x\|$. That is, E is automatically a contraction.

In general, a conditional expectation from M to a subalgebra N need not exist. However, when M is a finite factor the situation is quite nice:

Theorem 5.2.7. *Let M be a finite factor with trace τ. If $1_M \in N \subset M$ is a von Neumann subalgebra, then there exists a unique conditional expectation $E_N\colon M \to N$ satisfying $\tau \circ E_N = \tau$. Moreover, E_N is normal and faithful.*

The proof of this theorem is beyond the scope of these notes, but can be found in *An introduction to* II_1 *factors* by Claire Anantharaman-Delaroche and Sorin Popa. We mention that if $e_N := [N\hat{1}]$ then one can show $e_N\hat{x}$ is left bounded for all $x \in M$ and $E_N(x) = \lambda(e_N\hat{x})$ for $x \in M$. Observe that

$$\widehat{E_N(x)} = E_N(x)\hat{1} = \lambda(e_N\hat{x})\hat{1} = J1^*Je_N\hat{x} = e_N\hat{x}.$$

Since $\hat{1}$ is separating for M (and hence N), $E_N(x)$ is the unique $a \in N$ satisfying $\hat{a} = e_N\hat{x}$. Moreover, $E_N(x)$ is the unique $a \in N$ satisfying

$$\langle \hat{x}, \hat{b}\rangle_2 = \langle \hat{a}, \hat{b}\rangle_2 \qquad \forall b \in N.$$

Indeed,

$$\langle \hat{x}, \hat{b}\rangle_2 = \langle \hat{x}, e_N\hat{b}\rangle_2 = \langle e_N\hat{x}, \hat{b}\rangle_2 = \langle \widehat{E_N(x)}, \hat{b}\rangle_2,$$

implies that $\langle \hat{a} - \widehat{E_N(x)}, \hat{b}\rangle_2 = 0$ for all $b \in N$. Choosing $\hat{b} = \hat{a} - \widehat{E_N(x)}$ shows that $\hat{a} = \hat{E_N}(x)$ and so $a = E_N(x)$ since $\hat{1}$ is separating for M.

Exercises

5.2.1. For $x \in B(L^2(M))$, show that $(JxJ)^* = Jx^*J$.

5.2.2. For $n \in \mathbb{N}$, show that $L^2(M_n(\mathbb{C})) = M_n(\mathbb{C})$ with inner product

$$\langle A, B\rangle_2 = \frac{1}{n}\sum_{i,j=1}^{n} A_{i,j}\overline{B_{i,j}}.$$

5.2.3. For a discrete i.c.c. group Γ, let $M := L(\Gamma)$.

(a) Show that $L^2(M) = \ell^2(\Gamma)$.

(b) Show that $LB(M) = LC(\Gamma)$ and $RB(M) = RC(\Gamma)$.

5.2.4. Let M be a finite factor with trace τ. For $N := \mathbb{C} \subset M$, show that the conditional expectation $E_N\colon M \to N$ is given by $E_N(x) = \tau(x)1_M$.

5.2.5. For $n \in \mathbb{N}$, let $D \subset M_n(\mathbb{C})$ be the subalgebra of diagonal matrices. Show that the conditional expectation $E_D\colon M_n(\mathbb{C}) \to D$ is given by

$$E_D\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & & 0 \\ & \ddots & \\ 0 & & a_{n,n} \end{pmatrix}$$

5.2.6. Let Γ be a discrete i.c.c. group. Let $\Lambda < \Gamma$ be a subgroup, and view $L(\Gamma)$ as a von Neumann subalgebra of $L(\Gamma)$. Show that the conditional expectation $E_{L(\Lambda)}\colon L(\Gamma) \to L(\Lambda)$ satisfies $E_{L(\Lambda)}(\lambda(g)) = 1_\Lambda(g)\lambda(g)$.

5.3 The Tracial Ultraproduct and Ultrapowers

For this portion, we fix a family of von Neumann algebras $(M_n)_{n\in\mathbb{N}}$ such that every M_n is finite with trace τ_n. We fix an ultrafilter $\omega \in \beta(\mathbb{N})$, where $\beta(\mathbb{N})$. We let

$$\ell^\infty(\mathbb{N}, (E_n)) = \left\{ (x_n) \in \prod_{n=1}^{\infty} M_n : \sup_{n\in\mathbb{N}} \|x_n\| < \infty \right\},$$

denote the $*$-algebra of bounded sequences. We define the *trace ideal* to be

$$\mathcal{I} = \left\{ (x_n) \in \ell^\infty(\mathbb{N}, (M_n) : \lim_{n\to\omega} \tau_n(x_n^* x_n) = 0 \right\}$$

Lemma 5.3.1. *$\mathcal{I}$ as defined above is an operator closed 2-sided ideal of*

$$\ell^\infty(\mathbb{N}, (M_n))$$

.

Proof. Letting $(x_n) \in \mathcal{I}$, notice that for any $a_n, b_n \in M_n$, we have that

$$\|a_n x_n b_n\|_2 \leq \|a_n\| \|b_n\| \|x_n\|_2,$$

and hence

$$\lim_{n\to\omega} \tau_n(b_n^* x_n a_n^* a_n x_n b_n) \leq \lim_{n\to\omega} (\|a_n\| \|b_n\| \|x_n\|_2)^2 = 0$$

□

Definition 5.3.2. Consider a family of finite von Neumann algebras $(M_n)_{n\in\mathbb{N}}$ such that every M_n is finite with fixed trace τ_n, and fix an ultrafilter $\omega \in \beta(\mathbb{N})$ where $\beta(\mathbb{N})$ is the Stone-Cech compactification of $\mathbb{N}$. The algebra $\ell^\infty(\mathbb{N}, (M_n))/\mathcal{I}$, called the *ultraproduct of the family* (M_n). $\|(x_n)\| = \lim_{n\to\omega} \|x_n\|$ is a norm on the ultraproduct. When $M_n = M$ is a fixed finite von Neumann algebra, then this is called the ultrapower of M, and is denoted by M^ω.

There is a natural embedding of $M \subseteq M^\omega$ which is defined by mapping x to the equivalence class of the constant sequence $(x) \in M^\omega$.

Observe that since every element of $\ell^\infty(\mathbb{N}, (M_n))/\mathcal{I}$ with $\lim_{n\to\omega} \|x_n\| = 0$ is contained in the trace ideal $\mathcal{I}$, and thus the $\|(x_n)\| = \lim_{n\to\omega} \|x_n\|$.

Theorem 5.3.3. *The ultrapoduct of a family of finite von Neumann algebras is again a finite von Neumann algebra with trace $\tau_\omega := \lim_{n\to\omega} \tau_n$. Additionally, the ultraproduct is a factor whenever each of the M_n's are factors.*

Notice that the definition above is in some sense uninteresting when ω is a principle ultrafilter. Hence, we often make the standing assumption that an utlrafilter is non-principle.

Definition 5.3.4. Let M be a tracial von Neumann algebra. M has Property Gamma if and only if $M' \cap M^\omega \neq \mathbb{C}$ where ω is a non-principle ultrafilter on $\mathbb{N}$. M has the McDuff property if and only if $M' \cap M^\omega$ is non-abelian.

The advantage of working with an ultrapower von Neumann algebar is that it converts asymptotic behavior within a von Neumann algebra into something exact. To say the same thing more concretely, the key property of ulttrapowers is countable saturation, which essentially enables us to pass from approximately satisfying a certain property to exactly satisfying that property. On the flip side, if an ultrapower of a von Neumann M^ω algebra satisfies a certain property, then there should be some kind of sequential version of that same statement for M.

This is not the definition of Property Γ or the McDuff property that one usually encounters. However, the ultrapower version of these concepts simplifies things quite a bit. For example, here are the version of Property Γ and the McDuff property that is frequently found in the literature.

Definition 5.3.5. Let M be a tracial von Neumann algebra. M is said to have **property** Γ if there exist a sequence of unitaries $(u_n)_{n\in\mathbb{N}}$ with $\tau(u_n) = 0$ and

$$\lim_{n\to\infty} \|ux_n - xu_n\|_2 = 0$$

for every $x \in M$. This sequence $(u_n)_{n\in\mathbb{N}}$ is said to be an asymptotically central sequence of M. M is McDuff (or has the McDuff property) if $M \cong M \otimes \mathcal{R}$.

$\mathcal{R}$, the hyperfinite II_1 factor, has both Property Γ and the McDuff property, an hence any McDuff von Neumann algebra has property Γ, though my proof does depend on the model I create for $\mathcal{R}$ $\mathcal{R}$ is isomorphic to infinite tensor product of $M_2(\mathbb{C})$. Other examples McDuff von Neumann algebras include infinite tensor products of II_1 factors. Murray and von Neumann were able to show that $L(\mathbb{F}_2)$ does not have Property Γ and hence $L(\mathbb{F}_2) \ncong \mathcal{R}$. The issue here is that we have not talked about tensor products of von Neumann algebras,

We now have the terminology to state the infamous Connes Embedding Problem (sometimes called the Connes Embedding Conjecture). Does every II_1 factor M admit an embedding into $\mathcal{R}^\omega$ wehere R^ω is some ultrapower of $\mathcal{R}$? There are a myriad of equivalences that one can formulate here. In the language of free probability theory, the existence of an embedding of M into $\mathcal{R}^\omega$ is equivalent to M admitting *micorstates.* In C^* algebras, this question about embeddings of every possible II_1 factor is logically equivalent verifying that the tensor square of $C^*(\mathbb{F}_n)$ admits exactly one C^* norm. The language of operator spaces and quantum information theory allow for equivalent rephrasings of the Connes Embbedding Problem that, while notable, will not be discussed here. Talk to Roy...

Property Gamma is an invariant of the algebra and allowed Murray and von Neumann to show that there are at least 2 type II_1 factors. Here, we show that while the group of finitely supported permutations of $\mathbb{N}$, S_∞, has Gamma, produces a von Neumann algebra $L(S_\infty)$ with Gamma while $L(\mathbb{F}_2)$ does not. We will proceed by getting our hands dirty by looking at sequences of unitaries in the von Neumann algebras.

Embed $S_n \subseteq S_{n_1}$ my mapping $g \in S_n$ a permutation of the set $\{1, \ldots, n\}$ into $\{1, \ldots, n, n+1\}$ by considering it as a permutation that leave $n+1$ fixed. We now have trace-preserving, unital, embeddings of the von Neumann algebras

$$L(S_2) \subseteq L(S_3) \subseteq \cdots \subseteq L(S_\infty).$$

Notice that $\bigcup_{n\in\mathbb{N}} L(S_n)$ is a weak-operator topology dense subset of $L(S_\infty)$. Let g_k be the transposition that swaps $2k$ and $2k+1$. Notice that if $n < k$ then $g_k g = g g_k$ for every $g \in S_n$ precisely because these group elements permute disjoint subsets of $\mathbb{N}$. Let $u_k = \lambda_{g_k}$ (left-regular representation). If $x \in L(S_n)$ then we may write $x = \sum_{g\in S_n} c_g \lambda_g$. So now, we have that whenever $k > n$

$$u_k x = u_k \sum_{g\in S_n} c_g \lambda_g = \sum_{g\in S_n} c_g u_k \lambda_g = \sum_{g\in S_n} c_g \lambda_{g_k g} = \sum_{g\in S_n} c_g \lambda_{g g_k} = \sum_{g\in S_n} c_g \lambda_g \lambda_{g_k} = x u_k$$

Thus, we have that $\|xu_k - u_k x\| \to 0$ for every $x \in \bigcup_{n\in\mathbb{N}} L(S_n)$. Since the trace is WOT continuous and the aformentioned algebra is WOT dense in $L(S_\infty)$ this holds for every $x \in L(S_\infty)$ show that this von Neumann algebra has property Gamma.

Moving to a new algebra $L(\mathbb{F}_2)$, we first study the representation $\pi : \mathbb{F}_2 \to \mathcal{U}(\ell^2(\Gamma))$ given by mapping $\xi \in \ell^2(\mathbb{F}_{\not{2}})$ to $\pi_g(\xi)(x) = \xi(g^{-1}xg)$. This is given by applying the left and right regular representation at the same time. Notice that this leaves $C\delta_e$ fixed, so instead we will consider the restriction to

$$\pi : \mathbb{F}_2 \to \mathcal{U}(\ell^2(\mathbb{F}_2 \setminus \{e\})).$$

For notation's sake, I'm gonna write $\mathcal{H} = \ell^2(\mathbb{F}_2 \setminus \{e\}) = \ell^2(\mathbb{F}_2) \ominus \mathbb{C}\delta_e$, and $\mathbb{F}_2 = \langle a, b\rangle$. Let's decompose $\mathbb{F}_2 \setminus \{e\}$ into S_a and S_b, the words beginning with a power of a and b, respectively. This decomposes $\mathcal{H} = \ell^2(S_a) \oplus \ell^2(S_b)$, and we'll call each summmand into H_a and H_b. $a, a^2 \in S_a$ and $b \in S_b$ by definition. Moreover, $\pi_a(H_b) \subset H_a, \pi_{a^2}(H_b) \subseteq H_a$, $\pi_a(H_b) \perp \pi_{a^2}(H_b)$, and $\pi_b(H_a) \subseteq H_b$. More notation: P_H is the orthogonal projection onto a closed subspace $H \subseteq \mathcal{H}$.

$$\|P_{H_b}(\pi_a(\xi))\|_2^2 + \|P_{H_b}(\pi_{a^2}(\xi))\|_2^2 = \|P_{\pi_a(H_b)}(\xi)\|_2^2 + \|P_{\pi_{a^2}(H_b)}(\xi)\|_2^2 \le \|P_{H_a}(\xi)\|_2^2,$$

and

$$\|p_{H_a}(\pi_b(\xi))\|_2 = \|P_{\pi_b(H_a)}(\xi)\| \leq \|P_{H_b}(\xi)\|.$$

Thus, if ξ_n is any bounded sequence which is almost invariant ($\|\pi_g(\xi_n) - \xi\|_2 \to 0$ for every $g \in \Gamma$), we would have that $\sqrt{2}\limsup\|P_{H_b}(\xi_n)\| \leq \limsup\|P_{H_a}(\xi_n)\|$ and $\limsup\|P_{H_a}(\xi_n)\| \leq \limsup\|P_{H_b}(\xi_n)\|$. and thus $\|\xi_n\|_2 \to 0$, a contradiction. Hence, we cannot have almost invariant vectors, and instead this representation has spectral gap. That is, there exists a constant $k \geq 0$ and $g_1, \ldots, g_k \in \mathbb{F}_2$ so that for every $\xi \in \mathcal{H}$ we have

$$\|\xi\|_2^2 \leq k \sum_{i=1}^{j} \|\pi_{g_k}(\xi) - \xi\|_2^2.$$

Let's use these group elements to show that $L(\mathbb{F}_2)$ does not have property Gamma. For any $x \in L(\mathbb{F}_2)$ we have that $(x - \tau(x))\delta_e \in \mathcal{H}$. We define $v_i = \lambda_{g_i}$ so show that

$$\|x - \tau(x)1\|_2^2 \leq k \sum_{i=1}^{k} \|v_i^* x v_i - v_i x\|_2^2 = k \sum_{i=1}^{k} \|x v_i - v_i x\|_2^2$$

In particular, any trace zero unitary must have that

$$\|u\|_2^2 \leq k \sum_{i=1}^{k} \|u v_i - v_i u\|_2^2$$

preventing us from forming a sequence of unitaries which asymptotically commute with v_i's, showing that $L(\mathbb{F}_2)$ does not have Gamma.

So at this point, we have shown that $L(S_\infty) \neq L(\mathbb{F}_2)$ but not much else...

5.4 Center-Valued traces

Fixing for a moment a finite dimensional factor, $M_n(\mathbb{C})$, there is a distinguished state $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ which we call the trace and it is characterized by the so-called *tracial property* which means that $\tau(xy) = \tau(yx)$, c.f. Exercise 1.3.2. Now, suppose that $M = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C})$ is a direct sum of 2 finite dimensional factors. M admits multiple traces, for example $\tau = \frac{1}{2}(\tau_{n_1} \oplus \tau_{n_2})$ and $\tau' = \frac{1}{4}\tau_{n_1} \oplus \frac{3}{4}\tau_{n_2}$. We will soon see that the existence and uniqueness of a trace finite factors. Even when M is not a factorm, there is however a surrogate for the trace, a unique map $\phi : M \to \mathcal{Z}(M)$ that reduces to the trace when M is a factor. Moreover, the existence of such a map completely characterizes finite von Neumann algebras.

Definition 5.4.1. Let M be a von Neumann algebra and $\mathcal{Z}(M)$ its center. A map $\phi : M \to \mathcal{Z}(M)$ is a **center-valued state** if

(i) ϕ is linear and bounded,

(ii) and $\phi(zm) = z\phi(m)$ for any $z \in \mathcal{Z}(M)$,

If in addition we have that

(iii) $\phi(xy) = \phi(yx)$ for every $x, y \in M$,

then ϕ is called a **center-valued trace**.

Lemma 5.4.2. *Let M be a von Neumann algebra and $\phi : M \to \mathbb{C}$ be any linear functional. The following are equivalent:*

(i) $\phi(xy) = \phi(yx)$ for all $x, y \in M$.

(ii) $\phi(x) = \phi(u^* x u)$ for all $x \in M$ and all unitaries $u \in M$.

A linear functional as in Lemma 5.4.2 is called a *tracial* linear functional. A warning: some authors use the word *central* to describe linear functions which satisfy this property.

Proof. This is left as Exercise 5.4.2. □

We now discuss the structure theory of states, in particular the polar decomposition. If $\phi : M \to \mathbb{C}$ is a normal positive linear functional, then $\{x \in M : \phi(x^*x) = 0\}$ is a left ideal which is closed in the WOT, thus by Exercise 4.2.8 there exists a projection $p \in \mathcal{P}(M)$ such that $\phi(x^*x) = 0$ if and only if $x \in Mp$. We denote by $s(\phi) = 1 - p$ the support projection of ϕ. Note that if $q = s(\phi)$ then

$$\phi(xq) = \phi(qx) = \phi(x)$$

for all $x \in M$, and moreover, ϕ will be faithful when restricted to qMq.

Theorem 5.4.3 (Polar Decomposition for States)**.** *Suppose M is a von Neumann algebra and $\phi \in M_*$, then there exists a unique partial isometry $v \in M$ and positive linear functional $\psi \in M_*$ such that $\phi(x) = \psi(xv)$ for every $x \in M$ and $v^*v = s(\psi)$*

Proof. Assume for now that $\|\phi\| = 1$. There exists some $a \in (M)_1$ so that $\phi(a) = \|\phi\|$. Let $a^* = v|a^*|$ denote the polar decomposition of a^*. Letting $\psi(x) = \phi(xv)$, we have that $\psi(|a^*|) = \phi(a) = \|\phi\| = 1$. Since $0 \le |a^*| \le 1$, it follows that for every $t \in \mathbb{R}$

$$\||a^*| + e^{it}(1 - |a^*|)\| \le 1.$$

Fix t so that $e^{it}\psi((1 - |a^*|) \ge 0$. Then we have

$$\psi(|a^*|) \le \psi(|a^*|) + e^{it}\psi((1 - |a^*|) \le \|\phi\| = \phi(|a^*|),$$

and thus $\psi(1) = \psi(|a^*|) = \|\phi\|$ implying that ψ is a positive linear functional.

Let $p = v^*v$. Since we may replace a with $avs(\phi)s$, we may assume that $p \le s(\phi)$. For every $x \in M$ such that $\|x\| \le 1$, we have that

$$\psi(|a^*| + (1 - p)x^*x(1 - p)) \le \|\psi\|$$

which shows that $\psi((1 - p)x^*x(1 - p)) = 0$ and thus $p \ge s(\phi)$.

We leave out the proof of the uniqueness for now.

To see that$\phi(x) = \psi(xv)$ it suffices to show that $\phi(x(1 - p)) = 0$ for all $x \in M$. Suppose that $\|x\| = 1$ and $\phi(x(1 - p)) = \beta \ge 0$. Then for $n \in \mathbb{N}$ we have

$$\begin{aligned} n + \beta =& \phi(na + x(1 - p)) \\ \le& \|na + x(1 - p)\| \\ =& \|(na + x(1 - p))(na + x(1 - p)^*\|^{1/2} \\ \le& \|n^2|a^*|^2 + x(1 - p)x^*\|^{1/2} \\ \le& \sqrt{n^2 + 1} \end{aligned}$$

implying that $\beta = 0$. □

Our goal with the next few lemmas is to characterize finite von Neumann algebras in terms of the existence of a center-valued state. The presentation contained here is an existence result that relies on the Ryll-Nardzewski fixed point theorem. We exclude the proof for now; instead, accept it as fact and acknowledge that it bestows upon us the existence of a fixed point in an appropriate setting.

Theorem 5.4.4 (Ryll-Nardzewski)**.** *Let X be a Hausdorff locally convex vector space, $K \subseteq X$ a non-empty, weakly compact, convex subset and E a non-contracting semigroup of weakly continuous affine mappings of K into K . Then there exists an $x_0 \in K$, such that $T(x_0) = x_0$ for every $T \in E$.*

Lemma 5.4.5. *Let M be a von Neumann algebra, $\mathcal{Z}(M)$ its center, and $\phi \in M_*$ a normal tracial linear functional. Then $\|\phi\| = \|\phi|_{\mathcal{Z}(M)}\|$. In particular, ϕ is positive if and only if $\phi|_{\mathcal{Z}(M)}$ is positive.*

Proof. Let $\phi = R_v|\phi|$ be the polar decomposition of ϕ. The for any unitary $u \in M$, we have that

$$\phi = R_{u^*vu} T_u |\phi|.$$

From the uniqueness of the polar decomposition for linear functionals and the centrality of ϕ, it follows that that $u^*vu = v$ and $T_u|\phi| = |\phi|$ for every unitary $u \in M$. Thus, $v \in \mathcal{Z}(M)$ and $|\phi|$ is also tracial. Thus, we have that

$$\|\phi\| = \| \, |\phi| \, \| = |\phi|(1) = \phi(v^*) \leq \|\phi|_{\mathcal{Z}(M)}\| \|v^*\| \leq \|\phi\|.$$

□

Lemma 5.4.6. *Let M be a finite von Neumann algebras with $\mathcal{Z}(M)$ its center. Then any normal linear functional $\omega : \mathcal{Z}(M) \to \mathbb{C}$ extends uniquely to a bounded normal tracial linear function ϕ_ω on M. Moreover, $\|\phi_\omega\| = \|\omega\|$, ϕ_ω is positive whenever ω is positive, and the map $\psi : \mathcal{Z}(M)_* \to M_*$ defined by $\omega \mapsto \phi_\omega$ is linear.*

Proof. The uniqueness will be left as an exercise (see 5.4.4). If we can indeed show that such an extension exists, then the norm preserving property, and positivity follow from the previous lemma. To show the existence, let $\phi \in M_*$ be any normal linear functional extending ω to M. For notational convenience, whenever u is a unitary in M we let $T_u : M_* \to M_*$ denote the transformation mapping $\psi \mapsto \psi \circ \mathrm{Ad}(u)$ where $\mathrm{Ad}(u)(x) = u^*xu$ for every $x \in M$. In the statement of the Ryll-Nardzewski theorem, let $X = M_*$, K be the norm closed convex hull of $= \{T_u\phi : u \in \mathcal{U}(M)\} \subseteq M_*$, and $E = \{T_u|_K\}$. We claim without proof that K is a weakly compact, convex, non-empty subset of $X = M_*$. Further, observe that $T_u|_K : K \to K$ and that T_u is an isometry, making E a collection of non-contracting semi-group of weakly continuous affine mappings of K to itself.

Then Ryll-Nardzewski Theorem provides the existence of a fixed point $\phi_\omega \in K$, i.e. $T_u\phi_\omega = \phi_\omega$ for every $u \in M$ implying that ϕ_ω is a normal tracial linear functional on M.

Finally, we show that $\phi_\omega|_{\mathcal{Z}(M)} = \omega$. Notice that by construction, $\phi|_{\mathcal{Z}(M)} = \omega$, and hence, $T_u\phi|_{\mathcal{Z}(M)} = \omega$ for every $u \in \mathcal{U}(M)$. Thus, any convex combination and therefore any element of K will also equal ω when restricted to the center of M.

Now to show linearity, assume that $\omega_1, \omega_2 \in \mathcal{Z}(M)_*$ and $c \in \mathbb{C}$. Then, $\psi_{\omega_1 + c\omega_2}$ and $\psi_{\omega_1} + c\psi_{\omega_2}$ are extensions of $\omega_1 + c\omega_2$, and by uniqueness they are equal. □

Theorem 5.4.7. *If M if a finite von Neumann algebra, then admits a center-valued trace, namely the adjoint of the map $\psi : \mathcal{Z}(M)_* \to M_*$ defines a center-valued state on a finite von Neumann algebra.*

Proof. Now consider a finite von Neumann algebra M. By Lemma 5.4.6, there is a linear and isometric map $\psi : \mathcal{Z}(M)_* \to M_*$ taking normal linear functionals on $\mathcal{Z}(M)$ to tracial linear functional on M. Since we may identify $\mathcal{Z}(M)_*)^*$ with $(\mathcal{Z}(M)$ and $(M_*)^*$ with M, we let $\phi : M \to \mathcal{Z}(M)$ be the map determined by the relation

$$\psi_\omega(x) = \omega(\phi(x))$$

for every $\omega \in \mathcal{Z}(M)_*$ and $x \in M$. In other words, $\phi : M \to \mathcal{Z}(M)$ is the (Banach space) adjoint of the map ψ. □

Theorem 5.4.8. *Let M be a von Neumann algebra, $\mathcal{Z}(M)$. If $\phi : M \to \mathcal{Z}(M)$ is a center-valued trace, then ϕ has the following additional properties:*

(i) *ϕ is unique.*

(ii) $\|\phi\| = 1$,

(iii) *ϕ is σ-WOT continuous (normal),*

(iv) *$\phi(zx) = z\phi(x)$ for every $x \in M$ and $z \in \mathcal{Z}(M)$ (bimodular),*

(v) *$\phi(x^*x) \geq 0$ (positive),*

(vi) *$\phi(x^*x) = 0 \implies x = 0$ (faithful),*

Proof. If we suppose that there was another center-value trace $\tilde{\phi}$ on M distinct from ϕ, there would exist $x \in M$ so that $\phi(x) \neq \tilde{\phi}(x)$. But this would imply that we can find a normal linear functional $\omega \in \mathcal{Z}(M)$ so that $\omega(\phi(x)) \neq \omega(\tilde{\phi}(x))$. However, since $\omega \circ \phi$ and $\omega \circ \tilde{\phi}$ are distinct bounded normal extensions of ω which are tracial, this contradicts Lemma 5.4.6. Hence, the center-valued trace from Theorem 5.4.7 is the unique such map on M.

The normality and the the fact that the ϕ has norm 1 arises from the fact that ϕ is the (Banach space) adjoint of the map from Lemma 5.4.6.

Now to prove the bimodularity, we start by fixing a unitary $u \in \mathcal{Z}(M)$ and defining $\psi : M \to \mathcal{Z}(M)$ by $\psi(x) = u^*\phi(ux)$. Notice that ψ is a center-valued trace and thus must equal ϕ, i.e. $\phi(x) = u^*\phi(ux)$ for every $x \in M$. Replacing x with u^*x shows that $\phi(u^*x) = u^*\phi(x)$ for every $x \in M$ and for every unitary $u \in \mathcal{Z}(M)$. Since every element $z \in \mathcal{Z}(M)$ is a linear combination of 4 unitaries and ϕ is a linear map, we now have that $\phi(zx) = z\phi(x)$ for every $z \in \mathbb{Z}(M)$ and every $x \in M$.

In order to verify postivity, we will show that $\omega(\phi(x^*x)) \geq 0$ for every positive linear functional $\omega \in \mathcal{Z}(M)_*$. Notice that ϕ must satisfy

$$\omega(\phi(x^*x)) = \phi_\omega(x^*x),$$

where ϕ_ω is the normal tracial linear functional extending ω. Since Lemma 5.4.6 shows that $\phi_\omega(x^*x) \geq 0$, which is what we wanted to show.

We will verify the defniteness of ϕ by proving the contrapositive. In particular, if $y \in M$ and $y > 0$ we will show that there exists $\omega \in \mathcal{Z}(M)_*$ so that $\omega(\phi(y)) \neq 0$. To this end, fix $y \in M$ be a positive element and $z = \mathbf{z}(y)$ its central support projection, choose ω a positive normal linear functional such that $p = s(\omega) \leq z$. If ψ_ω is the tracial extension of ω to M, it is invariant under conjugation by any unitary in M. Hence, its support projection is also invariant under conjugation by all unitaries of M implying that $s(\phi_\omega)$ is in the center of M and in particular $s(\phi_\omega) = s(\omega) = p$. If $\phi_\omega(y) = 0$, then $xp = 0$; however this is not possible since $0 \neq p \leq z$. Thus, $\phi_\omega(y) = \omega(\phi(x)) \neq 0$, finishing the final claim. ☐

In light of the first item in the previous lemma, we are justified calling $\phi : M \to \mathcal{Z}(M)$ *the canonical center valued trace* on a finite von Neumann algebra M, whenever such a map exists. We should observe that the canonical center valued trace ϕ is an example of a *conditional expectation*. That is, ϕ is a positive, bimodular, norm 1, linear functional from M to the subalgebra $\mathcal{Z}(M)$. We will explore general conditional expectations in a later section.

Corollary 5.4.9. *M is a finite von Neumann algebra if and only if M has a unique center-valued trace.*

Proof. Assume that ϕ is a center-valued trace on M. If p is a projection on in M such that $p \leq 1$ and $p \sim 1$. In this case, $0 \leq 1 - p$ and $1 = \phi(1) = \phi(p)$. It follows that $0 \leq \phi(1-p) = \phi(1) - \phi(p) = 0$. Thus, $1 = p$ and hence M is finite. ☐

One of the main uses of a center-valued trace is that it detects equivalence of projections.

Theorem 5.4.10. *Let M be a von Neumann algebra with center-valued trace ϕ. If p and q are projections in M , $p \preceq q$ if and only if $\phi(p) \leq \phi(q)$. Specifically, $p \sim q$ if and only if $\phi(p) = \phi(q)$.*

Proof. If $p \preceq q$ and v is a partial isometry such that $v^*v = p$ and $vv^* \leq q$, then $\phi(p) = \phi(v^*v) = \phi(vv^*) \leq \phi(q)$, where the last line follows from the fact that $q - vv^* \geq 0$ and the linearity of ϕ.

Conversely, assume that $\phi(p) \leq \phi(q)$. Linearity of ϕ shows that $\phi(q - p) \leq 0$. By the Comparison Theorem, there exists a central projection z so that $zp \preceq zq$ and $(1-z)q \preceq (1-z)p$. Using the tracial property of ϕ in conjunction with the , we have that $\phi((1-z)(q-p)) \geq 0$. The bimodularity of ϕ now implies $0 \leq (1-z)\phi(q-p)$ From here, we use the initial assumption to conclude that $(1-z)\phi(q-p) \leq 0$, which when combined with the positive definiteness implies that $(1-z)q \sim (1-z)p$. Thus, $p \preceq q$.

The fact that $p \sim q$ is logically equivalent to $\phi(p) = \phi(q)$ follows from an application of Proposition 4.1.5 (Cantor-Schoder-Bernstein for projections). ☐

For this portion, we fix a finite von Neumann algebra M with center valued trace $\phi : M \to \mathcal{Z}(M)$.

Definition 5.4.11. M is homogeneous of type I_n if there exists a family of n equivalent abelian mutually orthogonal projections,$e_1, \ldots, e_n$, such that $\sum_{i=1}^n p_i = 1$.

An elementary example of a homogenous von Neumann algebra of type I_n is the $n \times n$ matrices, $M_n(\mathbb{C})$.

Theorem 5.4.12. *Let M be a finite homogeneous type I_n von Neumann algebra and $\phi : M \to \mathcal{Z}(M)$ its center valued trace. Then the range of ϕ restricted to the projections of M coincides with*

$$\sum_{k=1}^{n} \frac{k}{n} z_k$$

where $z_1, \dots, z_n$ are mutually orthogonal central projections

Proof. First, we show that there exists a projection p_0 so that $z(p_0) = 1$. Letting $\{p_1, \dots, p_k\}$ be a maximal family of mutually centrally orthogonal abelian projections ($\mathbf{z}(p_i)\mathbf{z}(p_j) = 0$ whenever $i \neq j$). Then $p_0 = \bigvee p_i$ is also abelian, and by maximality we must have that $\mathbf{z}(p_0)) = 1$.

Since M is homogeneous, of type I_n, there exists a family of n abelian, mutually orthogonal projections, equivalent to p_0 whose sum equals 1. Hence

$$1 = \phi(1) = \phi\left(\sum_{i=1}^{n} q_i\right) = \sum_{i=1}^{n} \phi(q_i) = n\phi(p_0).$$

Now, for any central projection z we see that $\phi(p_0 z) = \frac{1}{n} z$ and thus the range of $\phi|_{P(M)}$ contains elements of the form indicated above.

□

A center valued trace on a II_1 von Neumann algebra satisfies an analog of the intermediate value property.

Theorem 5.4.13. *Let M be a type II_1 von Neumann algebra with center valued trace ϕ. If p, q are projections in M and $z \in \mathcal{Z}(M)$ is a central projection with $\phi(q) \leq z \leq \phi(q)$, then there exists some projection r with $p \leq r \leq f$ and $\phi(r) = q$.*

Proof. First an observation about II_1 von Neumann algebras: if s is any projection and $\varepsilon > 0$ then there exists a non-zero projection $s_\varepsilon \leq s$ such that $\phi(s_\varepsilon) \leq \varepsilon \mathbf{z}(s_\varepsilon)$. To this end, choose n so that $\frac{1}{2^n} \leq \varepsilon$. Since M is type II_1, Lemma 5.1.2 shows that there is a family of 2^n equivalent, mutually orthogonal, non-zero subprojections of s whose sum is s. Letting s_ε be any one of these, this now gives that $\phi(s_\varepsilon) = \frac{1}{2^n}\phi(s) \leq \varepsilon\phi(s) \leq \varepsilon\mathbf{z}(e_\varepsilon)$.

Now, let P a maximal family of totally ordered projections in M such that if $s \in P$ then $p \leq s \leq q$ and $\phi(s) \leq z$. Such a collection exists and is non-empty since $p \in P$. Letting $r = \bigvee_{s \in P} s$, we have that $p \leq r \leq q$ and $\phi(p) \leq z$.

Let's suppose that $z - \phi(r) > 0$. Then in this case, there is some $\varepsilon > 0$ and a no-zero central projection w so that

$$z - \phi(r)w \geq \varepsilon w.$$

Notice that this would imply that $(q - r)w \neq 0$; otherwise we would have that $\phi(r)w = \phi(p) \geq zp$, a contradiction . So, we can find some non-zero projection s_ε with $s_\varepsilon \leq (p - r)w$ and $\phi(s_\varepsilon) \leq \varepsilon w$. But this would imply that $r + S_\varepsilon \in P$, contradicting the maximality of P. So we must have that $z = \phi(r)$.

□

Exercises

5.4.1. Show that if M is finite and separable, then M is tracial. When M is not a factor show that a trace is not unique.

5.4.2. Prove Lemma 5.4.2. [**Hint:** use Exercise 3.1.7.]

5.4.3. Let $n_1, \dots, n_k$ be a collection of natural numbers and consider $M = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$.

(a) Compute the center of M. Show that M has a continuum of faithful states $\phi : M \to \mathbb{C}$ with the tracial property.

(b) $M = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ has two non-zero orthogonal central projections which sum to the identity of M, which we call z_1, z_2. For each $a \in \{0, \frac{1}{2,} 1\}$ and $b \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, find a projection $p \in \mathcal{P}(M)$ such that $\phi(p) = az_1 + bz_2$ where ϕ is the center-valued trace.

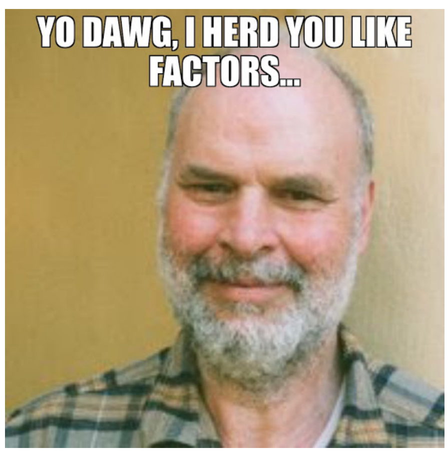
5.4.4. Show that under the conditions in 5.4.6, the extension of ω to a tracial state defined all of M is unique.

Chapter 6

Subfactors

In this chapter we will study *subfactors*: an inclusion of factors $N \leq M$ satisfying $1_M \in N$. We will restrict ourselves to the case when N and M are both II_1 factors, though more general inclusions have been studied extensively in the literature. Note that if τ_M is the unique trace on M, then $\tau_M|_N$ is necessarily the unique trace on N. Despite the starting point sounding like a an Xzibit meme, subfactors result in an incredibly rich theory with deep connections to knot polynomials and tensor categories.

This chapter will not be as thorough as the other chapters, and part of the reason is because entire books can be written about this subject alone. We present here only a starting point for learning about subfactors, though we will strive to present complete details whenever possible.



6.1 Index for Subfactors

Let $1_M \in N \subset M$ be an inclusion of II_1 factors, and let τ_M and τ_N be the unique traces on M and N, respectively. We will identify M (and consequently N) with its representation on $L^2(M)$. In this context we will denote $N' \cap B(L^2(M))$ simply by N', which satisfies $N' \supset M'$. Note that N' is a factor, and by Remark 4.3.9 we know that N' is type II. Consequently, N' is either a II_1 factor or a II_∞ factor. In the former case, we will denote its unique trace by $\tau_{N'}$.

Noting that $\tau_M|_N = \tau_N$, we see that the closure of $N\hat{1}$ in $L^2(M)$ is a copy of $L^2(N)$. Thus we can view $L^2(N)$ as a closed subspace of $L^2(M)$ and we let $e_N \in B(L^2(M))$ be the projection onto $L^2(N)$. Since $L^2(N)$ is reducing for N, we have $e_N \in N'$ by Lemma 1.2.5.

Definition 6.1.1. Let $1_M \in N \subset M$ be an inclusion of II_1 factors. We define the **index** of N inside M as the quantity

$$[M : N] := \frac{1}{\tau_{N'}(e_N)}$$

when N' is a II_1 factor, and otherwise set $[M : N] := \infty$.

Assuming N' is a II_1 (i.e. finite) factor, we have $\tau_{N'}(e_N) \leq 1$ and consequently $[M : N] \geq 1$. In particular, we have $[M : N] = 1$ if and only if $\tau_{N'}(e_N) = 1$. Since $\tau_{N'}$ is a faithful state this is further equivalent to $e_N = 1$, which means $L^2(N) = L^2(M)$ and $N = M$. Thus $[M : N] = 1$ if and only if $N = M$. Roughly speaking, $[M : N]$ measures how much larger M is than N. The notation should remind of you the notation for group indices, and the following example makes this explicit.

Example 6.1.2. Let $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ be a free ergodic p.m.p action of a countably infinite discrete group on a probability space (X, μ). Then $M := L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is a II_1 factor by Example 4.3.16. Let $\Lambda \leq \Gamma$ be a subgroup such that $\alpha|_\Lambda$ is still ergodic (it is automatically free and p.m.p.). Then $N := L^\infty(X, \mu) \rtimes_{\alpha|_\Lambda} \Lambda$ is a II_1 subfactor of M. In this case, we have

$$[M : N] = [\Gamma : \Lambda].$$

We provide only a sketch of the proof. Assume $[\Gamma : \Lambda] = n < \infty$ so that

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$$

for some $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. By Exercise 4.3.13, we have $L^2(M) = \ell^2(\Gamma) \otimes L^2(X, \mu)$ and $L^2(N) = \ell^2(\Lambda) \otimes L^2(X, \mu)$. Consequently

$$\begin{aligned} L^2(M) &= \left[\ell^2(\Lambda) \otimes L^2(X, \mu)\right] \oplus \left[\ell^2(\Lambda g_2) \otimes L^2(X, \mu)\right] \oplus \cdots \oplus \left[\ell^2(\Lambda g_n) \otimes L^2(X, \mu)\right] \\ &= L^2(N) \oplus \left[\ell^2(g_2 \Lambda) \otimes L^2(X, \mu)\right] \oplus \cdots \oplus \left[\ell^2(g_n \Lambda) \otimes L^2(X, \mu)\right]. \end{aligned}$$

It can be shown that the projections onto each of the remaining direct summands is equivalent to e_N in N' (see Exercise 6.1.3). Consequently, $\tau_{N'}(e_N) = \frac{1}{n}$ and so $[M : N] = n = [\Gamma : \Lambda]$. ■

Remark 6.1.3. There is an alternate formula for the index. Suppose $M \subset B(\mathcal{H})$ is a finite factor such that $M' \cap B(\mathcal{H})$ is also finite. Denote their respective traces by τ_M and $\tau_{M'}$. For any non-zero $\xi \in \mathcal{H}$, $M\xi$ and $M'\xi$ are reducing for M' and M, respectively, and so $[M\xi] \in M'$ and $[M'\xi] \in M$ by Lemma 1.2.5. Murray and von Neumann defined the *coupling constant* of M over $\mathcal{H}$ to be the ratio

$$\frac{\tau_M([M'\xi])}{\tau_{M'}([M\xi])},$$

and they showed that it is independent of the choice of ξ. When $1_M \in N \subset M \subset B(\mathcal{H})$ is a subfactor, it can be shown that the ratio of the coupling constants for N and M

$$\frac{\tau_N([N'\xi])}{\tau_{N'}([N\xi])} \frac{\tau_{M'}([M\xi])}{\tau_M([M'\xi])} \tag{6.1}$$

is further independent of the representation $M \subset B(\mathcal{H})$. This expression is in fact Jones' original definition for $[M : N]$, and since it does not depend on either $\mathcal{H}$ or ξ we can check that it matches with Definition 6.1.1. Indeed, take $\mathcal{H} = L^2(M)$ and $\xi = \hat{1}$, then $\hat{1}$ being cyclic and separating for M implies $[M\hat{1}] = [M'\hat{1}] = [N'\hat{1}] = 1$. Consequently

$$\frac{\tau_N([N'\hat{1}])}{\tau_{N'}([N\hat{1}])} \frac{\tau_{M'}([M\hat{1}])}{\tau_M([M'\hat{1}])} = \frac{1}{\tau_{N'}([N\hat{1}])} = [M : N].$$

Thus (6.1) gives us a more flexible definition for the $[M : N]$.

Given a projection $p \in N' \cap M$, we can consider the compressed inclusion $p \in Np \subset pMp$. Note that Np and pMp are both type II factors by Corollary 4.2.3 and Remark 4.3.9, and since $\frac{1}{\tau_M(p)} \tau_M$ defines a trace on pMp we see that they are in fact II_1 factors. Thus we can consider the index $[pMp \colon Np]$. Using Remark 6.1.3 and a few facts about the coupling constant, one can show

$$[pMp : Np] = [M : N] \tau_M(p) \tau_{N'}(p). \tag{6.2}$$

We can use this fact to derive some nice consequences for certain values of the index.

Proposition 6.1.4. *If $[M : N] < \infty$, then $N' \cap M$ is finite dimensional.*

Proof. Let $p_1, \ldots, p_n \in \mathcal{P}(N' \cap M)$ be non-zero pairwise orthogonal projections. Then since the index is always greater than or equal to one, (6.2) implies

$$[M : N] \geq [M : N] \sum_{i=1}^{n} \tau_M(p_i) = \sum_{i=1}^{n} \frac{1}{\tau_{N'}(p_i)} [p_i M p_i : N p_i] \geq \sum_{i=1}^{n} \frac{1}{\tau_{N'}(p_i)}.$$

Note that the condition $\sum_{i=1}^{n} \tau_{N'}(p_i) \leq 1$ implies $\tau_{N'}(p_i) \leq \frac{1}{n}$ for some $i = 1, \ldots, n$. Consequently, $[M : N] \geq n$, and so for any family of non-zero pairwise orthogonal projections $\mathcal{P} \subset \mathcal{P}(N' \cap M)$ we must have $|\mathcal{P}| \leq [M : N] < \infty$. Suppose $\mathcal{P}$ is a maximal family of pairwise orthogonal projections. We must have

$$\sum_{p \in \mathcal{P}} p = 1,$$

since otherwise $\{1 - \sum_p p\} \cup \mathcal{P}$ contradicts the maximality of $\mathcal{P}$. Also, each $p \in \mathcal{P}$ must be minimal in $N' \cap M$ because otherwise for $0 < q < p$ the maximality of $\mathcal{P}$ is contradicted by $\{q, p - q\} \cup \mathcal{P} \setminus \{p\}$. Now, as minimal projections, $p, q \in \mathcal{P}$ are either centrally orthogonal or equivalent in $N' \cap M$ by Proposition 4.1.9. If they are centrally orthogonal, then the same proposition implies $pxq = 0$ for all $x \in N' \cap M$. If they are equivalent, say by $vv^* = p$ and $v^*v = q$, then for $x \in N' \cap M$ we have

$$pxq = pxqq = px(v^*v)(v^*v) = pxv^*(vv^*)v = pxv^*pv = cpv = cv$$

for some $c \in \mathbb{C}$. Denote $v := v_{p,q}$, and if p and q are centrally orthogonal set $v_{p,q} := 0$. Thus for any $x \in N' \cap M$, we have

$$x = (\sum_{p \in \mathcal{P}} p) x (\sum_{q \in \mathcal{P}} q) = \sum_{p,q \in \mathcal{P}} pxq = \sum_{p,q \in \mathcal{P}} c_{p,q} v_{p,q},$$

for $c_{p,q} \in \mathbb{C}$. Hence $N' \cap M = \operatorname{span}\{v_{p,q} \colon p, q \in \mathcal{P}\}$, and since $\mathcal{P}$ is a finite set we see that $N' \cap M$ is finite dimensional. ☐

Proposition 6.1.5. *If $[M \colon N] < 4$, then $N' \cap M = \mathbb{C}$.*

Proof. Suppose, towards a contradiction that $p, q \in \mathcal{P}(N' \cap M)$ are orthogonal and non-zero. Then (6.2) implies (by the same argument as in the proof of the previous proposition)

$$[M : N] \geq \frac{1}{\tau_{N'}(p)} + \frac{1}{\tau_{N'}(q)} \geq \frac{1}{\tau_{N'}(p)} + \frac{1}{1 - \tau_{N'}(p)}.$$

This last expression is minimized at $\tau_{N'}(p) = \frac{1}{2}$, and hence we obtain the contradiction $[M : N] \geq 4$. ☐

We present the next result without proof, but we direct the interested reader to Jones' original paper.

Theorem 6.1.6 (Jones, 1983)**.** *Let $1_M \in N \subset M$ be an inclusion of II_1 factors. Then*

$$[M : N] \in \{4\cos^2(\pi/n) \colon n \geq 3\} \cup [4, \infty].$$

Moreover, every value in the set above occurs as the index of some unital inclusion of II_1 factors.

This result is part of the work that would ultimately earn Vaughan Jones the Fields Medal. That the index has a discrete component to its range was a remarkable revelation at the time[1].

[1] Masamichi Takesaki says he first heard about the result when picking Vaughan Jones up from the airport for a visit to UCLA, and was so startled by it that he nearly crashed the car.

Exercises

6.1.1. Let $N \subset P \subset M$ be inclusions of II_1 factors. Show that $[M : N] = [M : P][P : N]$. [**Hint:** use (6.1).]

6.1.2. Let $N \subset B(\mathcal{H})$ be a II_1 factor. For $d \in \mathbb{N}$, embed $N \hookrightarrow M_d(N)$ by

$$x \mapsto \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \qquad x \in N.$$

In this exercise, you will compute $[M_d(N) : N]$.

(a) Show that $B(L^2(M_d(N))) = M_{d^2}(B(L^2(N)))$, where the entries in the latter space are indexed by pairs of pairs: $((i, j), (k, \ell))$ for $i, j, k, \ell = 1, \ldots, d$.
[**Hint:** first show that $L^2(M_d(N)) \cong L^2(N)^{\oplus d^2}$.]

(b) Show that $N' \cap B(L^2(M_d(N))) = M_{d^2}(N' \cap L^2(N))$.

(c) For $X = (x_{i,j})_{i,j=1}^d \in M_d(N)$, show that

$$e_N X = \begin{pmatrix} \frac{1}{d}\sum_{i=1}^d x_{i,i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d}\sum_{i=1}^d x_{i,i} \end{pmatrix}.$$

as vectors in $L^2(M_d(N))$.

(d) Viewing $e_N \in M_{d^2}(N' \cap L^2(N))$, show that the $((i, j), (k, \ell))$-entry of e_N is $\frac{1}{d}\delta_{i=j}\delta_{k=\ell}$.

(e) Compute $\tau_{M_d(N)}(e_N)$ and $[M_d(N) : N]$.

6.1.3. Let $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ be a free ergodic p.m.p action of a countably infinite discrete group on a probability space (X, μ). Let $\Lambda < \Gamma$ be a finite index subgroup with

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n.$$

for some $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. Assume $\alpha|_\Lambda$ is ergodic and set

$$M := L^\infty(X, \mu) \rtimes_\alpha \Gamma$$
$$N := L^\infty(X, \mu) \rtimes_{\alpha|_\Lambda} \Lambda.$$

Recall that $L^2(M) = \ell^2(\Gamma) \otimes L^2(X, \mu)$ and $L^2(N) = \ell^2(\Lambda) \otimes L^2(X, \mu)$.

(a) For each $i = 2, \ldots, n$, show that $\ell^2(\Lambda g_i) \otimes L^2(X, \mu)$ is reducing for N.

(b) Let J be the canonical conjugation operator on $L^2(M)$: $J\hat{x} = \widehat{x^*}$. Show that

$$J(\delta_g \otimes f) = \delta_{g^{-1}} \otimes \alpha_{g^{-1}}(\bar{f})$$

for $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.

(c) For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto the subspace $\ell^2(\Lambda g_i) \otimes L^2(X, \mu)$.

(d) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.

(e) Compute $\tau_{N'}(e_N)$ and $[M : N]$.

6.1.4. Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$.

(a) Show that Λ is i.c.c.

(b) Suppose $\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$ for $g_2, \ldots, g_n \in \Lambda \setminus \Lambda$. For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto $\ell^2(\Lambda g_i)$.

(c) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.

(d) Compute $\tau_{N'}(e_N)$ and $[M : N]$.

6.2 The Basic Construction

Once more we let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of II_1 factors with unique traces τ_N and τ_M, respectively. Let $e_N \in B(L^2(M))$ be the projection onto the subspace $L^2(N) \subset L^2(M)$, so that $e_N \in N'$. Recall from Theorem 5.2.5, that if J is the canonical commutation on $L^2(M)$ then $JMJ = M'$. Consequently, $JNJ \subset M'$. However, $N \subset M$ implies $N' \subset M'$. Thus we cannot have $JNJ = N'$ unless $N' = M'$, in which case $N = M$. Since this case corresponds to $[M : N] = 1$, we see that whenever $[M : N] > 1$ we have M' is a strict subset of N', and $JN'J \supset JM'J = M$. We summarize these various relations in the diagram below.

$$\begin{tikzcd}
 & B(L^2(M)) & \\
JN'J \arrow[ur, phantom, "\subset" sloped] & & N' \arrow[ul, phantom, "\supset" sloped] \\
\cup & J \;\vdots\; J & \cup \\
M & & M' \\
\cup & & \cup \\
N & & JNJ \\
 & \mathbb{C}1_M \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] &
\end{tikzcd}$$

Horizontal reflection in the above diagram corresponds to conjugating by J. There is another important symmetry: reflecting through the center of the diagram corresponds to taking the commutant. This is clear for the pairs $(\mathbb{C}1_M, B(L^2(M)))$, (M, M'), and (N, N'), but it also holds for $(JN'J, JNJ)$. That is, $(JN'J)' = JNJ$. Indeed, $x \in (JN'J)'$ if and only if $x(JyJ) = (JyJ)x$ for all $y \in N'$, and conjugating the equation by J shows this is equivalent to $(JxJ)y = y(JxJ)$ for all $y \in N'$. Consequently, $x \in (JN'J)'$ if and only if $JxJ \in N'' = N$, and thus the claimed equality holds. In particular, this implies $JN'J$ is a factor:

$$\mathcal{Z}(JN'J) = (JN'J) \cap (JN'J)' = (JN'J) \cap JNJ = J(N' \cap N)J = \mathbb{C},$$

since N is a factor. Thus using only conjugation by J and taking commutants, we have produced a new factor extending our original inclusion: $N \subset M \subset JN'J$. We will study this new factor further, but first we require a lemma.

Recall from Theorem 5.2.7 that there is a faithful normal trace-preserving conditional expectation $E_N \colon M \to N$. This map is positive, restricts to the identity on N, and satisfies $E_N(axb) = aE_N(x)b$ for all $a, b \in N$ and $x \in M$. Also recall that for each $x \in M$, $E_N(x)$ is uniquely determined by $E_N(x)\hat{1} = e_N\hat{x}$.

Lemma 6.2.1.

(i) *For $x \in M$, $e_N x e_N = E_N(x)e_N$.*

(ii) *$N = \{e_N\}' \cap M$.*

(iii) *$N' = \{M' \cup \{e_N\}\}''$.*

(iv) *$Je_N = e_N J$.*

Proof.

(i): For $y \in M$, we have

$$e_N x e_N \hat{y} = e_N x E_N(y)\hat{1} = e_N \widehat{xE_N(y)} = E_N(xE_N(y))\hat{1} = E_N(x)E_N(y)\hat{1} = E_N(x)e_N\hat{y}.$$

Since $\widehat{M}$ is dense in $L^2(M)$, we have $e_N x e_N = E_N(x)e_N$.

(ii): Since $e_N \in N'$, we have $N \subset \{e_N\}' \cap M$. On the other hand, for $x \in \{e_N\}' \cap M$ we have

$$E_N(x)\hat{1} = e_N\hat{x} = e_N x\hat{1} = x e_N\hat{1} = x\hat{1}.$$

Since $\hat{1}$ is separating for M, we must have $x = E_N(x) \in N$.

(iii): The Bicommutant Theorem implies it suffices to show $\{M' \cup \{e_N\}\}' = N$. Note that $\{M' \cup \{e_N\}\}' \subset M'' \cap \{e_N\}' = M \cap \{e_N\}' = N$ by the previous part. The reverse inclusion follows from $N \subset M$ and the previous part.

(iv): For $x \in M$ we have

$$Je_N\hat{x} = JE_N(x)\hat{1} = E_N(x)^*\hat{1} = E_N(x^*)\hat{1} = e_N\widehat{x^*} = e_N J\hat{x}.$$

Thus the density of $\widehat{M}$ in $L^2(M)$ yields $Je_N = e_N J$. □

Proposition 6.2.2. *Let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of* II_1 *factors. If $e_N \in B(L^2(M))$ is the projection onto $L^2(N)$, then the factor $JN'J$ is generated by $M \cup \{e_N\}$. In fact, $JN'J$ is generated by the $*$-algebra span$(M \cup Me_NM)$.*

Proof. Recall that we have already seen that $JN'J$ is a factor in the discussion at the beginning of the section. From Lemma 6.2.1.(iii), we see that N' is the von Neumann algebra generated by $M' \cup \{e_N\}$. Note the unital *-algebra generated by $M' \cup \{e_N\}$ is spanned by elements of the form $y_1e_Ny_2e_N \cdots e_Ny_d$ for $d \geq 1$ and $y_1, \ldots, y_d \in M'$. Using Lemma 6.2.1.(iv) to assert $e_N = Je_NJ$ we have

$$J(y_1e_Ny_2e_N \cdots e_Ny_d)J = (Jy_1J)e_N(Jy_2J)e_N \cdots e_N(Jy_dJ).$$

Since $JM'J = M$ by Theorem 5.2.5, the above element is in the *-algebra generated by $M \cup \{e_N\}$. Consequently, $JN'J$ is the von Neumann algebra generated by $M \cup \{e_N\}$.

The $*$-algebra generated by $M \cup \{e_N\}$ is $\operatorname{span}\{x_1e_Nx_2e_N \cdots e_Nx_d \colon d \geq 1,\ x_1, \ldots, x_d \in M\}$. But Lemma 6.2.1.(i),(iii) imply for $d \geq 3$

$$x_1e_Nx_2e_Nx_3e_N \cdots e_Nx_d = x_1E_N(x_2)e_NE_N(x_3)e_N \cdots e_Nx_d = x_1E_N(x_2)E_N(x_3) \cdots E_N(x_{n-1})e_Nx_d.$$

So span$(M \cup Me_NM)$ is a $*$-algebra generating $JN'J$.

□

In light of the above proposition, we make the following definition.

Definition 6.2.3. The **basic construction** for $N \subset M$ is $\langle M, e_N \rangle := \{M \cup \{e_N\}\}'' \subset B(L^2(M))$.

By the discussion of at the beginning of the section, we know the commutant of $\langle M, e_N \rangle = JN'J$ is JNJ, which is a II_1 factor since N is a II_1 factor. So by Remark 4.3.9 we know $\langle M, e_N \rangle$ is a type II factor, but it could be either type II_1 or type II_∞. As we will see in the next theorem, the former case happens precisely when the index $[M : N]$ is finite.

Theorem 6.2.4. *Let $1_M \in N \subset M \subset B(L^2(M))$ be an inclusion of* II_1 *factors, and let $\langle M, e_N \rangle$ be its basic construction. Then $\langle M, e_N \rangle$ is a* II_1 *factor if and only if $[M : N] < \infty$. In this case, we have*

$$[\langle M, e_N \rangle : M] = [M : N].$$

If $\tau_{\langle M, e_N \rangle}$ is the unique trace on $\langle M, e_N \rangle$, then

$$\tau_{\langle M, e_N \rangle}(xe_N) = \frac{1}{[M : N]}\tau_M(x) \qquad \forall x \in M,$$

and in particular $\tau_{\langle M, e_N \rangle}(e_N) = [M : N]^{-1}$.

Proof. By the discussion preceding the theorem we know that $\langle M, e_N \rangle$ is a type II factor, and so it suffices to show $\langle M, e_N \rangle$ is finite if and only if $[M : N] < \infty$. Recall that $[M : N] < \infty$ if and only if N' is a finite by definition of the index. Thus it further suffices to show $\langle M, e_N \rangle$ is finite if and only if N' is finite, and by Theorem 5.1.5 it yet further suffices to show $\langle M, e_N \rangle$ has a trace if and only if N' has a trace. But this follows from $\langle M, e_N \rangle = JN'J$ because a trace on one algebra can be used to define a trace on the other:

$$\begin{aligned} \tau_{\langle M, e_N \rangle}(x) &:= \tau_{N'}(JxJ) & \qquad & x \in \langle M, e_N \rangle. \\ \tau_{N'}(y) &:= \tau_{\langle M, e_N \rangle}(JyJ) & & y \in JN'J. \end{aligned}$$

Thus $\langle M, e_N\rangle$ is II_1 factor if and only if $[M : N] < \infty$.

Let $\tau_{\langle M,e_N\rangle}$ be the unique trace on $\langle M, e_N\rangle$. By the above, we have $\tau_{\langle M,e_N\rangle}(x) = \tau_{N'}(JxJ)$ for all $x \in \langle M, e_N\rangle$, and in particular

$$\tau_{\langle M,e_N\rangle}(e_N) = \tau_{N'}(Je_N J) = \tau_{N'}(e_N) = \frac{1}{[M : N]},$$

where in the second equality we have used $e_N = Je_N J$ from Lemma 6.2.1.(iv). Now, by Lemma 6.2.1.(ii) we see that $N \ni x \mapsto \tau_{\langle M,e_N\rangle}(xe_N)$ defines a tracial positive linear functional on N, and so must equal $c\tau_N$ for some $c \in \mathbb{C}$ by the uniquness of τ_N. Setting $x = 1$ reveals

$$c = c\tau_N(1) = \tau_{\langle M,e_N\rangle}(1e_N) = \tau_{\langle M,e_N\rangle}(e_N) = \frac{1}{[M : N]}.$$

Thus $\tau_{\langle M,e_N\rangle}(xe_N) = \frac{1}{[M:N]}\tau_N(x)$ for $x \in N$. Using Lemma 6.2.1.(i), we can show this also holds for $x \in M$:

$$\begin{aligned}\tau_{\langle M,e_N\rangle}(xe_N) &= \tau_{\langle M,e_N\rangle}(e_N xe_N) = \tau_{\langle M,e_N\rangle}(E_N(x)e_N)\\ &= \frac{1}{[M : N]}\tau_N(E_N(x)) = \frac{1}{[M : N]}\tau_M(E_N(x)) = \frac{1}{[M : N]}\tau_M(x),\end{aligned}$$

where the last equality uses the fact that E_N is trace-preserving.

Finally, we compute the index $[\langle M, e_N\rangle : M]$ using (6.1). We take $\mathcal{H} = L^2(M)$ and $\xi = \hat{1}$. Note that $\hat{1}$ is cyclic for $\langle M, e_N\rangle$ since it is cyclic for M, and it is cyclic for M' since it is separating for M. Consequenlty, $[\langle M, e_N\rangle\, \hat{1}] = [M'\hat{1}] = [M\hat{1}] = 1$. Thus

$$[\langle M, e_N\rangle : M] = \frac{\tau_M([M'\hat{1}])}{\tau_{M'}([M\hat{1}])}\frac{\tau_{\langle M,e_N\rangle'}([\langle M, e_N\rangle]\hat{1})}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])} = \frac{\tau_M(1)}{\tau_{M'}(1)}\frac{\tau_{\langle M,e_N\rangle'}(1)}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])} = \frac{1}{\tau_{\langle M,e_N\rangle}([\langle M, e_N\rangle'\,\hat{1}])}$$

Now, as we saw above $\langle M, e_N\rangle' = JNJ$ and so $[JNJ\hat{1}] = [JN\hat{1}] = [N\hat{1}] = e_N$. Since $\tau_{\langle M,e_N\rangle}(e_N) = [M : N]^{-1}$, the above computation yields $[\langle M, e_N\rangle : M] = [M : N]$. $\square$

We now see that a finite index inclusion of II_1 factors $N \subset M$ begets another finite index inclusion of II_1 factors: $M \subset \langle M, e_N\rangle$. Moreover, the index of this new inclusion equals the original index and is therefore finite. Consequently, we can iterate this process and generate a tower of II_1 factors:

$$N \subset M \subset \langle M, e_N\rangle \subset \langle\langle M, e_N\rangle, e_M\rangle \subset \cdots$$

If we relabel these von Neumann algebras by $M_0 := N$, $M_1 := M$, $M_2 := \langle M, e_N\rangle$, etc. then we have

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots,$$

and $[M_i : M_{i-1}] = [M : N]$ for all $i \geq 1$. Moreover, by Exercise 6.1.1 for any $i > j \geq 0$ we have

$$[M_i : M_j] = \prod_{k=j+1}^{i}[M_k : M_{k-1}] = [M : N]^{i-j}.$$

In particular, $[M_i : M_0] = [M : N]^i < \infty$ and $[M_i : M_1] = [M : N]^{i-1} < \infty$, and so $M_0' \cap M_i$ and $M_1' \cap M_i$ are finite dimensional by Proposition 6.1.4.

Definition 6.2.5. The **Jones tower** for a finite index inclusion of II_1 factors $N \subset M$ is series of inclusions constructed above:

$$\begin{array}{ccccccccc} M_0 & \subset & M_1 & \subset & M_2 & \subset & M_3 & \subset & \cdots \\ \shortparallel & & \shortparallel & & \shortparallel & & & & \\ N & & M & & \langle M, e_N\rangle & & & & \end{array}$$

The **standard invariant** of $N \subset M$ is the collection of finite dimensional relative commutants $\{M_0' \cap M_i\}_{i \geq 0} \cup \{M_1' \cap M_i\}_{i \geq 1}$:

$$\begin{array}{ccccccccc} \mathbb{C} = M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & M_0' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} = M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & M_1' \cap M_2 & \subset & \cdots \end{array}$$

While the standard invariant may seem to be a dizzying array of von Neumann algebras, remember that each $M_j' \cap M_i$ is finite-dimensional and consequently is isomorphic to a direct sum of matrix algebras. Moreover, one can diagrammatically encode the data of these relative commutants and their various inclusions using planar algebras. These objects also provide a bridge between subfactors and category theory, and although they are worthy of their own entire course we will not go into further detail on them here.

We conclude with an example where the basic construction can be explicitly described. The resulting von Neumann algebra is the generalization of the crossed product construction from Example 4.3.16, where $L^\infty(X, \mu)$ (i.e. a commutative von Neumann algebra) has been replaced with II_1 factor.

Example 6.2.6. Consider a II_1 factor $M \subset B(L^2(M))$. Let $\mathcal{U}(L^2(M))$ denote the group of unitary operators on $L^2(M)$, and suppose $U < \mathcal{U}(L^2(M))$ is a finite subgroup satisfying $U \cap M = \{1\}$, $uMu^* = M$ for all $u \in U$, and $u\hat{1} = \hat{1}$ for all $u \in U$. Denote

$$M^U := \{x \in M \colon uxu^* = x \ \forall u \in U\}.$$

The hypotheses on U imply that this is a factor. This is not obvious but we will assume it as a fact. Then

$$p := \frac{1}{|U|} \sum_{u \in U} u \in (M^U)',$$

and p is a projection (Exercise 6.2.3). Observe for $x \in M$ that

$$p\hat{x} = \frac{1}{|U|} \sum_{u \in U} ux\hat{1} = \frac{1}{|U|} \sum_{u \in U} uxu^*u\hat{1} = \frac{1}{|U|} \sum_{u \in U} uxu^*\hat{1},$$

and $\frac{1}{|U|} \sum_u uxu^* \in M^U$. Thus $p = e_{M^U}$. We claim that

$$\langle M, e_{M^U} \rangle = \left\{ \sum_{u \in U} x_u u \colon x_u \in M \right\}''.$$

Denote the set on the right by B. For $x, y \in M$ we have

$$xe_{M^U}y = \frac{1}{|U|} \sum_{u \in U} xuy = \frac{1}{|U|} \sum_{u \in U} x(uyu^*)u \in B.$$

Since the identity of the group U is 1, we have $x = x1 \in B$ for $x \in M$. Thus $\mathrm{span}(M \cup Me_N M) \subset B$, and the former is a $*$-algebra generating $\langle M, e_{M^U} \rangle$ by Proposition 6.2.2. Thus to prove the claim it suffices to show $B \subset \langle M, e_{M^U} \rangle$, and this will follow if $U \subset \langle M, e_{M^U} \rangle$. For $x \in M$ we have

$$JuJ\hat{x} = Jux^*\hat{1} = Jux^*u^*u\hat{1} = Jux^*u^*\hat{1} = \widehat{uxu^*} = uxu^*\hat{1} = ux\hat{1} = u\hat{x}.$$

So $JuJ = u$ by the density of $\widehat{M} \subset L^2(M)$. Since $U \subset (M^U)'$, this shows $U = JUJ \in J(M^U)'J = \langle M, e_{M^U} \rangle$, and so the claim holds. The trace on $\langle M, e_{M^U} \rangle$ is given by

$$\tau_{\langle M, e_{M^U} \rangle} \left(\sum_{u \in U} x_u u \right) = \tau_M(x_1)$$

(see Exercise 6.2.4). In particular,

$$\tau_{\langle M, e_{M^U} \rangle}(e_{M^U}) = \tau_{\langle M, e_{M^U} \rangle} \left(\frac{1}{|U|} \sum_{u \in U} u \right) = \frac{1}{|U|} \tau_M(1) = \frac{1}{|U|}.$$

So by Theorem 6.2.3 we have $[M : M^U] = |U|$. ■

Exercises

6.2.1. Show that the basic construction for $N \subset M_d(N)$ is $M_{d^2}(N)$. [**Hint:** use Proposition 6.2.2 and the computation of e_N in Exercise 6.1.2.]

6.2.2. Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$. Suppose

$$\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$$

for $g_2, \ldots, g_n \in \Gamma \setminus \Lambda$. Set $p_1 := e_N$ and $p_i = \lambda(g_i) e_N \lambda(g_i^{-1})$ for $i = 2, \ldots, n$.

(a) Show that $p_i M p_i$ is spatially isomorphic to $N e_N$ for each $i = 2, \ldots, n$.

(b) Show that $\langle M, e_N \rangle$ is isomorphic to $M_n(N)$. What is the image of M under this isomorphism?

6.2.3. Let $\mathcal{U}(\mathcal{H})$ be the group of unitaries on a Hilbert space $\mathcal{H}$. For a finite subgroup $U < \mathcal{U}(\mathcal{H})$, show that

$$\frac{1}{|U|} \sum_{u \in U} u$$

is a projection.

6.2.4. Let $M \subset B(L^2(M))$ be a II_1 factor and let $U < \mathcal{U}(L^2(M))$ be a finite subgroup satisfying $U \cap M = \{1\}$ and $uMu^* = M$ for all $u \in M$.

(a) Show that $\tau_M(uxu^*) = \tau_M(x)$ for all $u \in U$ and $x \in M$. [**Warning:** since $u \notin M$ when u is non-trivial, this is **not** simply a consequence of the tracial property of τ_M.]

(b) Show that

$$\tau\left(\sum_{u \in U} x_u u\right) = \tau_M(x_1)$$

defines a faithful trace on the $*$-algebra $\{\sum_u x_u u \colon x_u \in M\}$.