Problem Sets

Roy Araiza
Srivatsav Kunnawalkam Elayavalli
Matthew Lorentz
Stephanie Prahl

1. Show that $v \in B(\mathcal{H})$ is a partial isometry if and only if v^*v is a projection.
[**Hint:** expand $\|(v - vv^*v)\xi\|^2$ for $\xi \in \mathcal{H}$.]

2. For $x \in B(\mathcal{H})$ with $x = x^*$, show that

$$\sup_{\|\xi\|=1} |\langle x\xi, \xi\rangle| = \|x\|.$$

[**Hint:** show Re $\langle x\xi, \eta\rangle = \frac{1}{2}\langle x(\xi+\eta), \xi+\eta\rangle + \frac{1}{2}\langle x(\xi-\eta), \xi-\eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.]

3. For a Hilbert space $\mathcal{H}$, prove the inclusions

$$FR(\mathcal{H}) \subset L^1(B(\mathcal{H})) \subset HS(\mathcal{H}) \subset K(\mathcal{H}).$$

[**Hint:** approximate by finite-rank operators in the appropriate norm.]

4. Show that $v \in B(\mathcal{H})$ is a partial isometry if and only if there exists a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $v|_{\mathcal{K}}$ is an isometry and $v|_{\mathcal{K}^\perp} \equiv 0$.

5. Let $x \in B(\mathcal{H})$. We say x is *bounded below* if there exists $\epsilon > 0$ such that $\|x\xi\| \geq \epsilon\|\xi\|$ for all $\xi \in \mathcal{H}$. Determine the implications between the following properties for $x \in B(\mathcal{H})$:

(i) x is injective (i.e. $\ker(x) = \{0\}$)
(ii) x is left-invertible (i.e. $\exists\, y \in B(\mathcal{H})$ with $yx = 1$)
(iii) x is bounded below.

6. Let $\mathcal{H}$ be a Hilbert space and $1 \leq n < \infty$. We denote $\mathcal{H}^n = \bigoplus_{j=1}^n \mathcal{H}$. For $x_{i,j} \in B(\mathcal{H})$ for $1 \leq i, j \leq n$, define $[x_{i,j}] : \mathcal{H}^n \to \mathcal{H}^n$ by

$$[x_{i,j}] \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_{1,j}\xi_j \\ \vdots \\ \sum_{j=1}^n x_{n,j}\xi_j \end{pmatrix}.$$

Check that this gives an operator in $B(\mathcal{H}^n)$ (in fact $\|[x_{i,j}]\| \leq (\sum \|x_{i,j}\|^2)^{\frac{1}{2}}$). We denote by $M_n(B(\mathcal{H}))$ the operators in $B(\mathcal{H}^n)$ that can be written as $[x_{i,j}]$ for some $x_{i,j} \in B(\mathcal{H})$. Show $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$.
[**Hint:** How would you do this for $\mathcal{H} = \mathbb{C}^m$?]

7. Here's an intuition building exercise to think about for Wednesday:

(a) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) \ : \ f(t) = 0\}$ for some $t \in [0,1]$.
(b) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t \ : \ t \in [0,1]\}$.
(c) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi_0(f + \lambda 1) = \lambda$ for all $f \in A$.

C*.1 (Gelfand-Mazur) If A is a simple, unital, abelian Banach algebra, then $A = \mathbb{C}$.

[**Hint:** For each $a \in A$, consider $aA := \{ab : b \in A\}$]

C*.2 Here's an exercise to build intuition:

(a) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) : f(t) = 0\}$ for some $t \in [0,1]$.

(b) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t : t \in [0,1]\}$.

(c) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi(f + \lambda 1) = \lambda$ for all $f \in A$.

W*.1 Consider the shift operator S on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Show that $((S^*)^n)_{n\in\mathbb{N}}$ converges to zero in the SOT, but $(S^n)_{n\in\mathbb{N}}$ does not.

W*.2 Let $\mathcal{H}$ be a Hilbert space. Given $\xi, \eta \in \mathcal{H}$, recall that the rank one operator $\xi \otimes \bar{\eta} \in B(\mathcal{H})$ is defined by

$$(\xi \otimes \bar{\eta})(\zeta) := \langle \zeta, \eta \rangle \, \xi.$$

(a) Show that $x \in B(\mathcal{H})$ commutes with $\xi \otimes \bar{\eta}$ if and only if there exists $\lambda \in \mathbb{C}$ with $\xi \in \ker(x - \lambda)$ and $\eta \in \ker(x^* - \bar{\lambda})$.

(b) Show that $FR(\mathcal{H})' = \mathbb{C}$ and that $B(\mathcal{H})' = \mathbb{C}$.

C*.1 Let A be a C*-algebra. Show the following:

(a) If $a, b \in A$ are self-adjoint elements such that $a \leq b$ and $c \in A$, then $c^*ac \leq c^*bc$.
[**Hint:** Take a square root and use the fact that elements of the form x^*x are positive.]

(b) Assuming A is a unital C*-algebra and $a \in A$ positive, show that $a \leq \|a\|1$. Moreover, $\|a\| \leq 1$ iff $a \leq 1$. In this case we also have $1 - a \leq 1$ and $\|1 - a\| \leq 1$.

C*.2 Suppose A is a C*-algebra with closed two-sided ideal $J \triangleleft A$ and C*-subalgebra $I \subset A$ such that $I \triangleleft J$. Show that $I \triangleleft A$.

W*.1 Prove Lemma 2.1.2: Let $\mathcal{H}$ be a Hilbert space and suppose $q\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is linear in the first coordinate, conjugate linear in the second coordinate, and there exists $C > 0$ such that $|q(\xi, \eta)| \leq C\|\xi\|\|\eta\|$ for all $\xi, \eta \in \mathcal{H}$. Then there exists a unique $x \in B(\mathcal{H})$ satisfying

$$\langle x\xi, \eta\rangle = q(\xi, \eta) \qquad \forall \xi, \eta \in \mathcal{H},$$

and $\|x\| \leq C$.

[**Hint:** First fix $\xi \in \mathcal{H}$ and show for all $\eta \in \mathcal{H}$ that $q(\xi, \eta) = \langle \xi_1, \eta\rangle$ for some $\xi_1 \in \mathcal{H}$. Then show that $x(\xi) := \xi_1$ defines a bounded operator $x \in B(\mathcal{H})$.]

W*.2 Let $\mathcal{H}$ be a Hilbert space and let $p \in B(\mathcal{H})$ be a non-trivial projection: $p \neq 0$ and $p \neq 1$. Show that the algebra $A := pB(\mathcal{H})p$ has no cyclic vectors.

C*.1 Recall that the set $U(\mathcal{H})$ of unitaries in $B(\mathcal{H})$ is a group under multiplication. A *unitary representation* of a group G is a group homomorphism $\rho : G \to U(\mathcal{H})$. Show that representations of $\mathbb{C}G$ are in bijection with unitary representations of G.

C*.2 Complete the proof of Proposition 6.3 by showing that ψ is well-defined (independent of the choice of sequence $(a_n)_n$); $*$-preserving; and multiplicative.

W*.1 Suppose $(x_i)_{i\in I} \subset B(\mathcal{H})$ is a uniformly bounded net: $\sup_i \|x_i\| < \infty$.

(a) Show that $(x_i)_{i\in I}$ converges in the σ-SOT if and only if it converges in the SOT.

(b) Show that $(x_i)_{i\in I}$ converges in the σ-WOT if and only if it converges in the WOT.

(c) Show that the example $(x_{m,n})_{m\leq n}$ defined in lecture is **not** uniformly bounded.

W*.2 Recall that a $*$-isomorphism $\pi : M \to N$ between von Neumann algebras $M \subset B(\mathcal{H})$ and $N \subset B(\mathcal{K})$ is called a *spatial isomorphism* if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $\pi(x) = UxU^*$ for all $x \in M$. Show that a spatial isomorphism $\pi\colon M \to N$ is normal.

C*.1 Show that any positive linear functional $\phi : A \to \mathbb{C}$ is $*$-preserving, i.e. $\phi(a^*) = \overline{\phi(a)}$ for all $a \in A$.

C*.2 Show that for a unital C^*-algebra A, $\mathcal{S}(A)$ is a weak* closed convex subset of $A^*_{\leq 1}$. It follows from Alaoglu's theorem that it is weak*-compact. What does the Krein-Milman theorem say about $\mathcal{S}(A)$?

C*.3 Show that if the C*-algebra A is finite dimensional as a vector space, then we may take the Hilbert space $\mathcal{H}$ of the GNS Theorem to be finite dimensional.

[**Hint:** Show that you only need finitely many states $\phi \in F$, and that H_ϕ is finite dimensional for all ϕ.]

W*.1 Let $\mathcal{H}$ be a Hilbert space.

(a) For orthonormal sets $\{\xi_1, \ldots, \xi_n\}, \{\eta_1, \ldots, \eta_n\} \subset \mathcal{H}$, show that $\sum_{i=1}^n \xi_i \otimes \bar{\eta}_i$ is a partial isometry that implements the equivalence $(\sum_{i=1}^n \eta_i \otimes \bar{\eta}_i) \sim \left(\sum_{i=1}^n \xi_i \otimes \bar{\xi}\right)$.

(b) For finite-rank projections $p, q \in B(\mathcal{H})$, show that $p \sim q$ if and only if $\text{Tr}(p) = \text{Tr}(q)$.

(c) Let $\mathcal{E}, \mathcal{F} \subset \mathcal{H}$ be two orthonormal subsets with the same cardinality. Show that $[\mathcal{E}] \sim [\mathcal{F}]$.
[**Hint:** start with a bijection from $\mathcal{E}$ to $\mathcal{F}$ (as sets).]

W*.2 Let $M \subset B(\mathcal{H})$ be a factor. Show any two minimal projections are equivalent.

[**Hint:** use the Comparison Theorem.]

W*.3 Let (X, μ) be a positive σ-finite measure space. We call a measurable subset $A \subset X$ an **atom** of if $\mu(A) > 0$ and for all measurable subsets $E \subset A$ one has $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

(a) If $A_1, A_2 \subset X$ are atoms, show that either $1_{A_1 \cap A_2} = 0$ or $1_{A_1 \cap A_2} = 1_{A_1} = 1_{A_2}$.

(b) If $A \subset X$ is an atom, show that $f|_A$ is constant for all $f \in L^\infty(X, \mu)$.

(c) Show that 1_A is a minimal projection in $L^\infty(X, \mu)$ if and only if A is an atom.

Problem Session July 10th, 2020

C*.1 Show that the matrix amplification of any $*$-homomorphism between C*-algebras is again a $*$-homomorphism. Conclude that any $*$-homomorphism is completely positive.

C*.2 Let A and B be C*-algebras and $C \subset B$ a C*-subalgebra. Show that if $\theta : A \to C$ is a nuclear map, then so is θ when viewed as a map from A to B. Suppose we have a map $\rho : A \to C$ that is nuclear as a map from A to B. What could prevent ρ from being a nuclear map as a map from A to C?

C*.3 Partitions of unity are nicer when you have a concrete example. For each $n \geq 2$, cover $[0, 1]$ by $2^n - 1$ open intervals of equal length. (What are they? Also, we could start with $n = 1$, but it's too simple to pick up on a pattern.) Call this cover $\mathcal{U}_n$. Define (sketch) a partition of unity for $\mathcal{U}_n$. (Hint: think zig-zags.)

Now, construct a sequence of completely positive maps $C([0, 1]) \xrightarrow{\psi_n} \mathbb{C}^{k_n} \xrightarrow{\phi_n} C([0, 1])$, (what is k_n?) that give a completely positive approximation of $C([0, 1])$.

W*.1 Let Γ be a countable discrete group. Show that all projections in $L(\Gamma)$ are finite.

[**Hint:** use the trace.]

W*.2 Let $\pi\colon M \to N$ be a $*$-isomorphism between von Neumann algebras and let $p \in \mathcal{P}(M)$.

(a) Show p is finite in M if and only if $\pi(p)$ is finite in N.

(b) Assuming π is normal, show p is semi-finite in M if and only if $\pi(p)$ is semi-finite in N.

(c) Show p is purely infinite in M if and only if $\pi(p)$ is purely infinite in N.

(d) Show p is properly infinite in M if and only if $\pi(p)$ is properly infinite in N.

W*.3 In this exercise, you will show that $M_n(\mathbb{C})$ can be realized via a crossed-product construction. Consider $\Gamma := \mathbb{Z}_n$, the countable cyclic group of order n, and also set $X := \mathbb{Z}_n$ which we view as simply a space and equip with the counting (probability) measure.

(a) Show that $\alpha_g(f) := f(\cdot - g)$ for $g \in \Gamma$ defines an action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$.

(b) Show that $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ is free, ergodic, and probability measure preserving.

(c) Show that $1_{\{1\}}, \ldots, 1_{\{n\}} \in L^\infty(X, \mu)$ are pairwise orthogonal and equivalent minimal projections.

(d) Show that $L^\infty(X, \mu) \rtimes_\alpha \Gamma \cong M_n(\mathbb{C})$. What is the preimage of $E_{i,j}$ under this isomorphism?

(e) Explain why there does not exist a discrete group Γ such that $L(\Gamma) \cong M_n(\mathbb{C})$.

Michael Brannan: *Quantum Groups: what are they and what are they good for?*

Recall that the *Brown Algebra* B_n is the unital C*-algebra satisfying the following universal property: B_n is generated by the elements $u_{ij}, 1 \leq i, j \leq n$ satisfying the property that $[u_{ij}]_{ij}$ is a unitary in $M_n(B_n)$, and if A is another unital C*-algebra generated by elements v_{ij} satisfying the same relations then there exists a unique unital $*$-homomorphism $\pi : B_n \to A$ such that $u_{ij} \mapsto v_{ij}$ for all i, j.

Exercise: Prove that Brown's universal unitary algebras B_n, equipped with their canonical co-products, do **not** define compact quantum groups.

If you have extra time, consider the following exercise:
Let $G = (A, \Delta)$ be a compact quantum group with comultiplication $\Delta : A \to A \otimes_{\min} A$. Define $\Delta^{\text{opp}} := {}^t(\Delta) := t \circ \Delta$, where $t : A \otimes_{\min} A \to A \otimes_{\min} A$ denotes the flip map, i.e., $a \otimes b \mapsto b \otimes a$. Show that $G^{\text{opp}} = (A, \Delta^{\text{opp}})$ is a compact quantum group.

Dawn Archey: *A Crash Course in Crossed Product C*-Algebras*

We say that a C^*-algebra A has **real rank zero** if the invertible elements in $A_{s.a.}$ are dense in $A_{s.a.}$.

Exercise: Show that $C([0,1])$ does not have real rank zero, by finding a function $f \in C([0,1])_{s.a.}$ which cannot be approximated within $\epsilon = 1/4$ by an invertible self-adjoint element.

If you have extra time, consider the following exercises:

1. Let G be a finite group. Let A be a unital $C*$-algebra. Let $\alpha\colon G \to \text{Aut}(A)$ be a homomorphism. As short hand, write α_t instead of $\alpha(t)$. Consider the algebra AG of all sums $\sum_{t \in G} a_t t$.
   (a) We will define multiplication on AG by the formal rule $tat^{-1} = \alpha_t(a)$.
   Work out an explcit formula for the product fg where $f = \sum_{t \in G} a_t t$ and $g = \sum_{s \in G} b_s s$. Your final answer should be in the same format (a sum of things of the form: algebra element times group element).
   (b) Later we will complete this to create a C^*-algebra. So we will need an adjoint. The adjoint is determined by $s^* = s^{-1}$. Use this to determine a formula for the adjoint of f as defined in the previous part of the problem.
2. Let $h : X \to X$ be a homeomorphism. We say (H, h) is a **minimal dynamical system** if X has no proper closed h invariant subsets. Let $X = S_1$. Let $h(z) = e^{-2\pi\theta z}$. If $\theta \in \mathbb{Q} \setminus \{0\}$ then show h is not minimal.

Lecture Exercises

C*.1 Finish the proof of the following proposition from the lecture notes:

Proposition 0.1. *For* C*-*algebras* A_1 *and* A_2, *and* $x = \sum_{j=1}^{n} a_j \odot b_j \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup\{\|\sum_{j=1}^{n} \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \text{ (nondegenerate) representations}\}.$$

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}'_i)$ be faithful representations. Then by Exercise 4.16, $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2$... □

(This is an example of a technique where one can *dilate* a map to one with a desired property (e.g. faithfulness) and then *cut down* to the original map to draw the desired conclusion.)

W*.1 For each $N \subset M$ below, compute the conditional expectation $E_N \colon M \to N$. Recall that the conditional expectation is determined by the formula

$$\langle E_N(x), y\rangle_2 = \langle x, y\rangle_2 \qquad x \in M,\ y \in N$$

where $\langle a, b\rangle_2 = \tau(b^* a)$ for $a, b \in M$.

(a) For $d \in \mathbb{N}$, let $M := M_d(\mathbb{C})$ and let N be the subalgebra of diagonal matrices.

(b) Let M be an arbitrary finite factor and let $N := \mathbb{C}$.

(c) Let Γ be a discrete i.c.c. group. Let $\Lambda < \Gamma$ be a subgroup. Take $M := L(\Gamma)$ and $N := L(\Lambda)$.

Mark Tomforde *K-theory: An Elementary Introduction*

Let A be a C^*-algebra, and let I be an ideal of A. Prove that if $K_0(I) \cong K_1(I) \cong \{0\}$, then $K_0(A) \cong K_0(A/I)$ and $K_1(A) \cong K_1(A/I)$.

If you have extra time, consider the following exercises:

1. Suppose A is a unital C^*-algebra that is Morita equivalent to a crossed product of an AF-algebra by $\mathbb{Z}$; that is, there exists an AF-algebra B and an automorphism $\alpha\colon B \to B$ such that A is Morita equivalent to the crossed product $B \rtimes_\alpha \mathbb{Z}$. Prove that

   $$K_0(A) \cong \operatorname{coker}(id - \alpha_0) \qquad \text{and} \qquad K_1(A) \cong \ker(id - \alpha_0)$$

   where $(id - \alpha_0)\colon K_0(B) \to K_0(B)$. Also show that $K_1(A)$ is torsion-free abelian group.

   (Recall: if $h\colon G \to H$ is a homomorphism between abelian groups, then the *cokernel* of h is defined $\operatorname{coker}(h) := H/\operatorname{im}(h)$.)

   [**Hint:** use the Pimsner–Voiculescu (PV) sequence.]

2. Prove that K_0 and K_1 distribute over a direct sum; that is, for any C^*-algebras A and B prove that

   $$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B) \qquad \text{and} \qquad K_1(A \oplus B) \cong K_1(A) \oplus K_1(B).$$

   There are several ways to do this problem. The hints below outline one possible approach.

   [**Hint 1:** use the fact that K_0 and K_1 each take split exact sequences to split exact sequences.]

   [**Hint 2:** obtain the following commutative diagram

   $$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus K_0(B) & \longrightarrow & K_0(B) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A \oplus B) & \longrightarrow & K_0(B) & \longrightarrow & 0 \end{array}$$

   and apply the three-lemma (i.e. a special case of the five-lemma). Similarly for K_1.]

Ian Charlesworth *Free Probability*

Let G and H be countable discrete groups and let $G * H$ denote their free product. View $L(G)$ and $L(H)$ as subalgebras of $L(G * H)$, whose trace we denote by τ.

(a) For $g_1, \ldots, g_n \in G \setminus \{e\}$ and $h_1, h_2, \ldots, h_n \in H \setminus \{e\}$, show that

$$\tau(\lambda(g_1)\lambda(h_1) \cdots \lambda(g_n)\lambda(h_n)) = 0.$$

(b) For $x \in \mathbb{C}[\lambda(G)]$, characterize when $\tau(x) = 0$. Similarly for $y \in \mathbb{C}[\lambda(H)]$.

(c) For $x_1, \ldots, x_n \in \mathbb{C}[\lambda(G)]$ and $y_1, \ldots, y_n \in \mathbb{C}[\lambda(H)]$ assume $\tau(x_i) = \tau(y_i) = 0$ for $i = 1, \ldots, n$. Show that

$$\tau(x_1 y_1 \cdots x_n y_n) = 0.$$

(d) Show that the previous part holds for $x_i \in L(G)$ and $y_i \in L(H)$.

Lauren Ruth *Operator Algebras and Equivalences between Groups*

Exercise: Show that measure equivalence of groups is an equivalence relation.

If you have extra time, consider the following exercises:

1. Give an example of a space measure (X, μ) and a measure-preserving action of $\mathbb{Z}$ on X along with a fundamental domain.
2. Let $\Gamma = SL_2(\mathbb{Z})$ act on the upper-half plane $H \subseteq \mathbb{C}$ by fractional linear transformations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$
   (a) Show that if $\mu(E) := \int_E \frac{dx\,dy}{y^2}$ then
$$\mu(E) = \mu(g \cdot E)$$
   for every $g \in \Gamma$ and $E \subset H$ measurable.
   (b) Show that the set $\mathcal{F} = \{z \in H : -\frac{1}{2} \leq \text{Re }(z) < \frac{1}{2}, |z| \geq 1\} \cup \{z \in H : |z| = 1, \text{Re }(z) \leq 0\}$ is a fundamental domain for Γ.
   [**Hint:** Use the fact that Γ is generated by the elements
$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
   How do the elements a and b act on a point $z \in H$?]

Nate Brown *Duality as the bridge between C*- and W*-algebras*

We say a von Neumann algebra $M \subset B(\mathcal{H})$ is **injective** if there is a contractive linear map $\Phi \colon B(\mathcal{H}) \to M$ such that $\Phi(x) = x$ for all $x \in M$.

Exercise: For a direct sum of von Neumann algebras $M := M_1 \oplus M_2$, show that M is injective if and only if M_1 and M_2 are injective.

Exercise: Suppose M is injective and $I \subset M$ is a σ-WOT closed ideal. Show that I and M/I are injective.

Lecture Exercises

W*.1 Let Γ be an i.c.c. group, let $\Lambda < \Gamma$ be a finite index subgroup, and set $M := L(\Gamma)$ and $N := L(\Lambda)$.

(a) Show that Λ is i.c.c.
(b) Suppose $\Gamma = \Lambda \sqcup \Lambda g_2 \sqcup \cdots \sqcup \Lambda g_n$ for $g_2, \ldots, g_n \in \Lambda \setminus \Lambda$. For each $i = 2, \ldots, n$, show that $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J \in N'$ and that this is the projection onto $\ell^2(\Lambda g_i)$.
(c) For each $i = 2, \ldots, n$, show that e_N is equivalent to $J\lambda(g_i^{-1})Je_N J\lambda(g_i)J$ in N'.
(d) Compute $\tau_{N'}(e_N)$ and $[M : N]$.
(e) Show that $\langle M, e_N \rangle$ is isomorphic to $M_n(N)$. What is the image of M under this isomorphism?

C*.1 Prove the following fact used in the proof of Theorem 12.7: if $f \in \ell^\infty(G)$, $f = \sum_{g \in G} a_g u_g$, then $\lambda_s(f) = u_s f u_s^*$ as operators on $\ell^2(G)$. In other words, left translation is spatially implemented.

Problem Session July 16th, 2020

Robin Deeley *Groupoid C*-algebras*

Exercise: Suppose X is a nonempty finite set and R is an equivalence relation on X.

(a) Prove that $C^*(R)$ is the direct sum of finitely many matrix algebras.

(b) Compute the K-theory of $C^*(R)$.

If you have extra time:

Let G be a group. Examine the structure of the following two groupoids constructed from G.

1. Let $\mathcal{G} = G$ and $\mathcal{G}^2 = G \times G$; the multiplication map $\mathcal{G}^2 \to \mathcal{G}$ is given by group multiplication, and the map $\mathcal{G} \to \mathcal{G}$ is given by taking the inverse.
   (a) Prove that $\mathcal{G}$ is a groupoid.
   (b) What is $\mathcal{G}^0$ in this case?
   (c) Prove that $\mathcal{G}$ is étale iff G is discrete.
2. Let G be a finite group, and set $\tilde{\mathcal{G}} = G \times G$, with $\tilde{\mathcal{G}}^2 = \{((g,h),(gh,k)) : g,h,k \in G\}$. Then the map $\tilde{\mathcal{G}}^2 \to \tilde{\mathcal{G}}$ is given by $((g,h),(gh,k)) \mapsto (g,hk)$ and the map $\tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ is defined via $(g,h) \mapsto (gh,h^{-1})$.
   (a) Prove that $\tilde{\mathcal{G}}$ is a groupoid.
   (b) What is $C_c(\tilde{\mathcal{G}})$? What changes if G is a countable discrete group?

Corey Jones *Subfactors and quantum symmetries*

Exercise: Let $N \subset B(\mathcal{H})$ be a II_1 factor. For $d \in \mathbb{N}$, embed $N \hookrightarrow M_d(N)$ by

$$x \mapsto \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \qquad x \in N.$$

In this exercise, you will compute $[M_d(N) : N]$.

(a) Show that $B(L^2(M_d(N))) = M_{d^2}(B(L^2(N)))$, where the entries in the latter space are indexed by pairs of pairs: $((i,j),(k,\ell))$ for $i,j,k,\ell = 1,\dots,d$.

[**Hint:** first show that $L^2(M_d(N)) \cong L^2(N)^{\oplus d^2}$.]

(b) Show that $N' \cap B(L^2(M_d(N))) = M_{d^2}(N' \cap L^2(N))$.

(c) For $X = (x_{i,j})_{i,j=1}^d \in M_d(N)$, show that

$$e_N X = \begin{pmatrix} \frac{1}{d}\sum_{i=1}^d x_{i,i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d}\sum_{i=1}^d x_{i,i} \end{pmatrix}.$$

as vectors in $L^2(M_d(N))$.

(d) Viewing $e_N \in M_{d^2}(N' \cap L^2(N))$, show that the $((i,j),(k,\ell))$-entry of e_N is $\frac{1}{d}\delta_{i=j}\delta_{k=\ell}$.

(e) Compute $\tau_{M_d(N)}(e_N)$ and $[M_d(N) : N]$.

(f) Show that $\langle M_d(N), e_N \rangle \cong M_{d^2}(N)$.

Isaac Goldbring *Model theory and von Neumann algebras*

Exercise: Let N be a tracial von Neumann algebra, and construct N^ω the tracial ultrapower, where ω is a non principal ultrafilter on the natural numbers. Note that N embeds diagonally in N^ω. Suppose $N' \cap N^\omega$ is not contained in the diagonal embedding of N, we will show in this exercise that $N' \cap N^\omega$ is infinite dimensional.

(a) Suppose $x_0 = (a_i)_\omega$ is an element of $N' \cap N^\omega$ that is not in the diagonal embedding of N. Show that there is an $\epsilon > 0$ such that for all $y \in N$, $\lim_{i\to\omega} ||a_i - y||_2 > \epsilon$.

(b) Show that there exists $x_1 \in N' \cap N^\omega$ such that $||x_0|| = ||x_1||$ and $||x_0 - x_1||_2 > \epsilon/2$.

[**Hint:** try working with the same x_0, but speeding up the sequence.]

(c) Show that one can find a sequence x_i as above, such that they all have the same norm, and they are pairwise 2-norm distance $\epsilon/2$ apart. Conclude that the commutant is infinite dimensional.

Sam Kim *An introduction to Operator Systems*

For a C*-algebra, let $M_{m,n}(A)$ denote the vector space of $m \times n$-matrices with entries in A. For all $n \geq 1$, let 1_n denote the identity element in $M_n(A)$. It follows from [Theorem 3.10, C*-Algebra Notes] that for a C*-algebra A, an element $a \in A$ is positive if and only if there is some $b \in A$ such that $a = b^*b$. The proof of the following lemma is almost exactly the same as the proof of [Lemma 9.16, C*-Algebra Notes] and you can feel free to fill in the details if you are interested.

Lemma. Let $m, n \geq 1$. Let $A \in M_n(B(H))$ be a positive operator, let $X \in M_{m,n}(B(H))$, and let λ be a non-negative number. We have the inequality

$$\begin{bmatrix} \lambda \cdot 1_m & X \\ X^* & A \end{bmatrix} \geq 0$$

if and only if $X^*X \leq \lambda A$.

The next two exercises use the above result to give us alternative ways to describe multiplication in a C*-algebra.

Exercise (Walter's Lemma): Let $U, V \in B(H)$ be unitary operators and let $X \in B(H)$. Show that $X = UV$ if and only if

$$\begin{bmatrix} 1 & U & X \\ U^* & 1 & V \\ X^* & V^* & 1 \end{bmatrix} \geq 0\,.$$

Exercise[1,2]**:** Let $X, Y, Z \in B(H)$ be contractions. Show that $Z = XY$ if and only if for all $B \in B(H)$, we have the norm identity

$$\left\| \begin{bmatrix} 2\cdot 1 & X & -Z & B \\ 0 & Y^* & 1 & 0 \end{bmatrix} \right\|^2 = \| \begin{bmatrix} 2\cdot 1 & X & -Z & B \end{bmatrix} \|^2\,.$$

[**Hint:** you may require the following two facts.

(1) For any $m \times n$-block matrix X, $\|X^*X\| = \|X\|^2$.

[1] This is a result of David Blecher and Matthew Neal

[2] This was also used by Isaac Goldbring and Thomas Sinclair to show that the class of unital C*-algebras is *first order axiomatizable* in the language of operator systems (see Isaac Goldbring's talk).

(2) For a positive operator T, we have the inequality $T \leq \|T\| 1$.]

Lecture Exercises

C*.1 Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a self-adjoint element. Show that b lifts to a self-adjoint element $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.

W*.1 Suppose that Γ is a discrete group. We say that Γ is **inner amenable** if the representation $\pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma \setminus \{e\}))$ given by

$$\pi_(g)(\xi(x)) = \xi(g^{-1}xg) \quad \forall\, \xi \in \ell^2(\Gamma \setminus \{e\}), x \in \Gamma \setminus \{e\},\, g \in \Gamma$$

has a sequence of unit vectors $(\xi_n)_{n=1}^{\infty}$ such that

$$\|\pi_g(\xi_n) - \xi_n\|_2 \to 0 \qquad \forall\, g \in \Gamma$$

(a) Show that any non-inner amenable group is an i.c.c. group.

(b) Let $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$, $|\Gamma_2| \geq 3$. Show that Γ is an i.c.c. group that is not inner amenable by showing that π, as described above, has spectral gap. Conclude that $L(\Gamma) \ncong L(S_\infty)$.