Subalgebras of AF-algebras:

AF-alg's: A C^*-alg is ~~AF~~ approx fd (AF) if $\exists\, F_1 \subseteq F_2 \subseteq \cdots \subseteq A$, $\dim(F_n) < \infty$, $\overline{\bigcup F_n} = A$.

e.g.

$$M_2 \subseteq M_4 \subseteq M_8 \subseteq \cdots \quad \cdots \quad M_{2^\infty} = \overline{\bigcup M_{2^n}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right)$$

~~(CAR algebra)~~

Rmk Define $\mathrm{tr} : M_{2^n} \to \mathbb{C}$, $a \longmapsto \frac{1}{2^n} \sum_{i=1}^{2^n} a_{ii}$

tr induces $\mathrm{tr} : M_{2^\infty} \to \mathbb{C}$.

The GNS construction produces

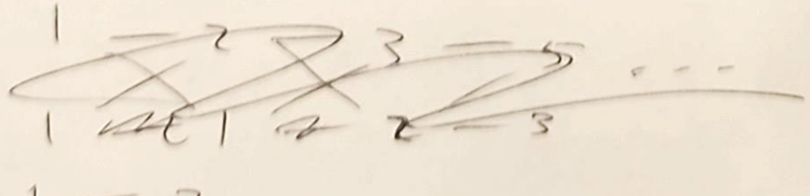
$$\pi_{\mathrm{tr}}(M_{2^\infty})'' =: R, \text{ hyperfinite } II_1\text{-factor}$$

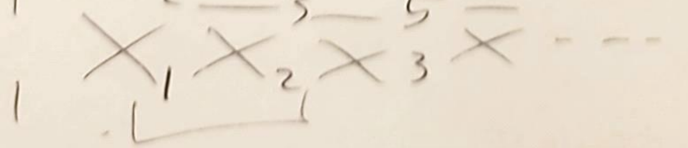
One also has, ~~for example~~ $\pi_{\mathrm{tr}}(M_{3^\infty}) \cong R$.

However $M_{2^\infty} \not\cong M_{3^\infty}$ ($\nexists$ proj $p, q, r \in M_{2^\infty}$, $p+q+r = 1$, $p \sim q \sim r$.

In M_{3^∞}, $p = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$, $r = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$)

e.g.





$$\begin{array}{ccc} F_2 & & F_3 \\ M_2 \oplus \mathbb{C} & \longrightarrow & M_3 \oplus M_2 \\ (a, b) & \longmapsto & \left(\begin{pmatrix} a & \\ & b \end{pmatrix}, a \right) \end{array}$$

$A = \overline{\bigcup F_n}$ AF-alg

~~$K_0(M_{2^\infty}) = \mathbb{Z}[\frac{1}{2}]$, $K_0 M$~~

A has ! trace tr, $\pi_{tr}(A)'' \cong R$.

~~$A \not\cong M$~~

$K_0 M_{2^\infty} \cong \mathbb{Z}[\frac{1}{2}]$, $K_0 M_{3^\infty} \cong \mathbb{Z}[\frac{1}{3}]$, $K_0 A \cong \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{5}}{2}\right)$

In all cases: $[p] \longmapsto tr(p)$.

In general: [Elliott '76]

For A, B AF, $K_0 A \cong K_0 B \iff (K_0 A, K_0^+ A, [1_A]) \cong (K_0 B, K_0^+ B, [1_B])$

~~ZN Center~~

Problem Which C^*-alg's embed into AF-alg's?

Thm [Connes '76] If M is a vNa and $M \hookrightarrow R$, then M is ~~AFD~~ injective ~~(…)~~

eg $C[0,1] \hookrightarrow M_{2^\infty}$.

Sketch: Set $h_n = \begin{pmatrix} 0/2^n & & & \\ & 1/2^n & & \\ & & \ddots & \\ & & & \frac{2^n-1}{2^n} \end{pmatrix} \in M_{2^n} \subseteq M_{2^\infty}$.

$(h_n)_{n=1}^\infty \subseteq M_{2^\infty}$ Cauchy. Set $h = \lim_{n\to\infty} h_n$. Then $0 \le h \le 1$, $\sigma(h) = [0,1]$.

Then $C[0,1] \hookrightarrow M_{2^\infty}: f \longmapsto f(h)$.

Note: $C[0,1] \cap M_{2^n} = \mathbb{C}$ $\forall n$.

More generally: $C(X) \hookrightarrow M_{2^\infty}$ $\forall X$ (compact metrizable)

Thm [Connes '76] If M is a vNa and $M \hookrightarrow R$, then M is hyperfinite (vNa-AF)

~~Harder examples:~~

Harder examples:

$\mathbb{Z} \overset{\alpha}{\curvearrowright} C(\mathbb{T})$; irrational rotation:

Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha : \mathbb{T} \xrightarrow{\cong} \mathbb{T} : \lambda \longmapsto e^{2\pi i \theta} \lambda$.

Then α induces $\hat{\alpha} : C(\mathbb{T}) \xrightarrow{\cong} C(\mathbb{T})$

$A_\theta := C(\mathbb{T}) \rtimes \mathbb{Z} \cong C^*\left(u, v \,\middle|\, \begin{array}{l} u, v \text{ unitary} \\ uv = e^{2\pi i \theta} vu \end{array}\right)$

Thm [Pimsner–Voiculescu] $A_\theta \hookrightarrow$ AF-alg.

Aside: $A_\theta \cong A_{\theta'} \iff \theta \equiv \pm\theta' \pmod{\mathbb{Z}}$, $\tau_*(K_0(A_\theta)) = \mathbb{Z} + \theta\mathbb{Z} \subseteq \mathbb{R}$

Idea ~~Write $\theta = \lim \frac{p_n}{q_n}$~~

$\theta \approx \frac{p}{q}$, $p, q \in \mathbb{N}$ (WLOG $\theta > 0$)

Consider $u \longmapsto$

$A_\theta \longrightarrow$ $u_0 = \begin{pmatrix} \omega_p & & & \\ & \omega_p^2 & & \\ & & \ddots & \\ & & & \omega_p^n \end{pmatrix}$, $v_0 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix} \in M_q$

Then M_q: $u_0 v_0 = e^{2\pi i p/q} v_0 u_0 \approx e^{2\pi i \theta} v_0 u_0$

with some care, write $\theta = \lim_{n\to\infty} \frac{p_n}{q_n}$.

~~$\exists\ \varphi_n : A_\theta \xrightarrow{cpc} M_q$~~

with some care,

$$\exists\ \varphi_n : A_\theta \xrightarrow{ucp} M_{q_n}, \quad \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \forall a,b \in A_\theta$$

$$\|\varphi_n(a)\| \to \|a\| \quad \forall a \in A_\theta$$

Hard part: Arrange embeddings

~~$M_{q_n} \hookrightarrow M_{q_{n+1}}$ s.t. $(\varphi_n)_n$ Cauchy~~ ~~(cont pos)~~

$$M_{q_n} \oplus M_{q_{n+1}} \hookrightarrow M_{q_{n+1}} \oplus M_{q_{n+2}} \quad (\varphi_n \oplus \varphi_{n+1})$$

[Pimsner '82] $\forall\ \mathbb{Z} \curvearrowright X$ minimal, $C(X) \rtimes \mathbb{Z} \hookrightarrow AF$.

Necessary Condition:

If $A \subseteq AF$-alg, then A is quasidiagonal:

$$\exists\ \varphi_n : A \to M_{k_n}, \quad \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$$

$$\|\varphi_n(a)\| \to \|a\|$$

and

exact i.e. $\exists$

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "ucp"'] & \approx & \mathcal{B}(H) \\ & F_n \arrow[ur, "ucp"'] & \end{tikzcd}$$

F_n f.d.

Problem: Are these criteria sufficient?

Elliott question:

~~When does $A \hookrightarrow$~~
~~For a C*-alg A,~~

~~$\exists ? \; A \subset$~~

When is there an embedding $A \hookrightarrow$ simple AF-alg?

Thm [S '18]

Assume A separable ~~...~~ the UCT ($KK(A,-) \cong KK(C_0(X),-)$, $\exists X$)

then $A \hookrightarrow$ simple AF-alg $\iff$ A is exact and

$\exists\, \tau \in A^*$ s.t.

- τ is a trace: $\tau(ab) = \tau(ba)$
- τ is faithful: $\tau(a^*a) = 0 \Rightarrow a = 0$
- τ is amenable: $\pi_\tau(A)''$ ~~...~~ R injective ($\hookrightarrow R$)

~~...~~

Corollary: For G amenable group, $G \curvearrowright X$ minimal,

$C(X) \rtimes G \hookrightarrow$ AF-alg

Rmk The proof uses vNa techniques:

Define $\|\cdot\|_2 : R \longrightarrow [0,\infty)$, $\|a\|_2 = tr(a^*a)^{1/2}$

~~Then~~

Viewing $R \subseteq B(L^2(R, tr))$, $\|\cdot\|_2$ induces the SOT on ball(R).

Step 1 ("easy") $\exists$ AF ~~embeds~~ simple B with ! trace tr. $K_0 A \xrightarrow{\alpha} K_0 B$, $\tau \searrow \mathbb{R} \swarrow tr$, $\cong$

Assume $\pi_\tau(A)''$ injective:

(∃) $\exists\ \theta : A \hookrightarrow R = \pi_{tr}(B)''$, $tr \circ \theta = \tau$,

(!) If $\theta, \theta' : A \hookrightarrow R$, $tr \circ \theta = tr \circ \theta' = \tau$, then $\exists (u_n)_{n=1}^\infty \subseteq \mathcal{U}(R)$, $\|u_n \theta(a) u_n^* - \theta'(a)\|_2 \longrightarrow 0$

Use Kaplansky density then,

$\exists\ \theta_n : A \longrightarrow B$,

($\approx_2 \exists$) $\|\theta_n(ab) - \theta_n(a)\theta_n(b)\|_2 \longrightarrow 0$ $\quad \forall a,b \in A$

$|tr\, \theta_n(a) - \tau(a)| \longrightarrow 0$

($\approx_2$!) For another seq ρ_n, $\exists (u_n)_n \subseteq \mathcal{U}(B)$

$\|u_n \theta_n(a) u_n^* - \rho_n(a)\|_2 \longrightarrow 0$ $\quad \forall a \in A$

__If__ these hold in $\|\cdot\|$, can find unitaries $(v_n) \subseteq \mathcal{U}(B)$ s.t. $(v_n \theta_n(\cdot) v_n^* : A \longrightarrow B)_{n=1}^\infty$ $\|\cdot\|$-Cauchy

This implies $A \hookrightarrow B$

~~let~~ ~~J~~

~~V~~ ~~J_B~~

$$\text{let } J_B = \frac{\{(b_n)_n \subseteq B \mid \|b_n\|_2 \to 0,\ \sup_n \|b_n\| < \infty\}}{\{(b_n)_n \subseteq B \mid \|b_n\| \to 0\}}.$$

"trace-kernel ideal"

~~Can find such that~~

on other $\|\cdot\|$-~~extension~~ $\Longleftrightarrow$ absolute on $KK(A, J_B)$ under

hold under UCT,

~~easily~~ ~~evaluate~~ at α