Coarse Geometry and Operator Algebras

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Introduction to Coarse Geometry

- Studying the topology of metric spaces tells us about their 'small scale' structure, e.g. convergence properties...
- ... but doesn't see 'large scale' properties.
- For example, the usual metric on $\mathbb{Z}$ and the metric given by $d(z, z') = 1$ if $z \neq z'$ give the same topology on $\mathbb{Z}$, but only one metric is bounded.
- There are also natural situations where a metric space is only defined "coarsely" (e.g. finitely-generated groups)

Organization of this talk

- Develop an equivalence relation of metric spaces which only sees large scale structure (although coarse structures can be defined without a metric, we won't worry about that here)
- Discuss properties which are preserved by this equivalence relation and how they relate to each other
- See how coarse geometry relates to C^*-algebras
- Time permitting, discuss some of my work in coarse geometry, dynamics, and operator algebras

Coarse equivalence – definitions

- Suppose $\phi : X \to Y$ is a map of metric spaces.
- We say ϕ is *uniformly expansive* if there exists a non-decreasing function $\rho^+ : [0, \infty) \to [0, \infty)$ such that $d_Y(\phi(x), \phi(x')) \leqslant \rho^+(d_X(x, x'))$.
- We say ϕ is *effectively proper* if there exists a proper nondecreasing function $\rho_- : [0, \infty) \to [0, \infty)$ such that $d_Y(\phi(x), \phi(x')) \geqslant \rho_-(d_X(x, x'))$.
- If ϕ is both uniformly expansive and effectively proper, it's called a *coarse embedding*.
- Moreover, we say ϕ is *coarsely onto* if there is $R > 0$ such that for all $y \in Y$ there exists $x \in X$ such that $d_Y(\phi(x), y) \leqslant R$.
- A coarse embedding which is coarsely onto is called a *coarse equivalence*. This will be our notion of equivalence (this is an equivalence relation).

Examples

- Any bounded metric space is coarsely equivalent to a point
- $\mathbb{Z}^n$ (with its usual metric as a subspace of $\mathbb{R}^n$) is coarsely equivalent to $\mathbb{R}^n$.

Alternative definitions

- Other references may develop the definition of coarse equivalence in different ways.
- For instance, a map ϕ is called *coarse* if it is uniformly expansive, and a map $\psi : X \to Y$ is called a coarse equivalence if there is a coarse map $\phi : Y \to X$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are both a bounded distance away from the identity maps on X and Y respectively (so ψ has a 'coarse inverse').
- Note: These definitions of coarse equivalence are equivalent, but a coarse map is not in general a coarse embedding (Exercise: find an example showing this).
- Think the about analogy with topology: homeomorphism = topological embedding + surjective = continuous + continuous inverse

quasi-isometry – definition

- When the *control functions* ρ_+ and ρ_- have a specific form, we have different terminology for this stronger condition:

- A map of metric spaces $\phi : X \to Y$ is a *quasi-isometric embedding* if there are constants C and D such that $\frac{1}{C} d_X(x, x') - D \leqslant d_Y(\phi(x), \phi(x')) \leqslant C d_X(x, x') + D$. If it is also coarsely-surjective, we call ϕ a *quasi-isometry*.

More examples (graphs and Cayley graphs)

- If $G = (V, E)$ is a connected graph, we can give V a metric by defining $d(v, v')$ to be the least number of edges that must be traversed in a path from v to v'. In this case, any two n-regular trees with $n \geqslant 3$ are quasi-isometric.
- If Γ is a finitely-generated group, we can think of Γ as having the metric coming from a Cayley graph with respect to some finite generating set, and any choice of finite generating sets gives rise to quasi-isometric spaces, so we can talk about Γ as a coarse space without specifying a metric.
- Exercise: for graphs with metrics defined above, coarse equivalence and quasi-isometry are equivalent.

Coarse properties

- In topology, we first defined a notion of equivalence (homeomorphism) and then studied properties which were preserved by that equivalence (e.g. connectedness, compactness, simple-connectedness, etc.)

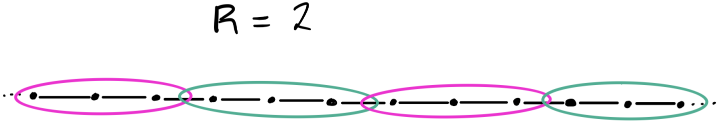
- We will now discuss some properties of metric spaces which are preserved by coarse equivalence. Some will be familiar, and many will be related to the study of operator algebras.

Asymptotic dimension

- There are many equivalent definitions of the asymptotic dimension first introduced by Gromov.
- Here is one: We say a metric space X has *asymptotic dimension* at most d if for all $R > 0$ there exists $M > 0$ and a cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_d$ where $d(U, V) > R$ for all $U, V \in \mathcal{U}_i$ with $U \neq V$ and all $0 \leqslant i \leqslant d$, and $\operatorname{diam}(U) < M$ for all $U \in \mathcal{U}_i$ and all $0 \leqslant i \leqslant d$.
- Other equivalent definitions make it clear that this is a large-scale version of covering dimension (see for example [Bell and Dranishnikov, 2005, Theorem 1])
- Exercise: show asdim$X$ is a coarse property

Examples

- $\text{asdim}\mathbb{R}^n = \text{asdim}\mathbb{Z}^n = n$. The picture below shows one of the covers showing that $\mathbb{Z}$ has asymptotic dimension $\leqslant 1$.



- Exercise: draw a picture for $n = 2$.

- (with the metric it gets as a subset of $\mathbb{R}$) $\text{asdim}\{n^2 : n \in \mathbb{N}\} = 0$

(some) Properties of asdim

- Asdim acts like a dimension theory in that it is subadditive on products $\mathrm{asdim}(X \times Y) \leqslant \mathrm{asdim} X + \mathrm{asdim} Y$ [Bell and Dranishnikov, 2008, Theorem 37]
- If $1 \to \Delta \to \Gamma \to \Lambda \to 1$ is an exact sequence of finitely generated groups, $\mathrm{asdim}\Gamma \leqslant \mathrm{asdim}\Delta + \mathrm{asdim}\Lambda$
- Both these theorems can be viewed as consequences of a Hurewicz-type theorem for asymptotic dimension [Brodskiy et al., 2006]

Asdim and algebra

Theorem

Suppose Γ *is finitely presented. If* $\mathrm{asdim}\Gamma = 1$ *and* Γ *is torsion free, then* Γ *is free.*

This follows from a result in [Gentimis, 2008] about asymptotic dimension and ends of groups, which implies Γ is virtually free.

Application: $BS(1,n)$

- Groups with presentation $\langle a, b | bab^{-1} = a^n \rangle$
- Fit into exact sequence $1 \to \mathbb{Z}[\frac{1}{n}] \to BS(1,n) \to \mathbb{Z} \to 1$
- So asdim$BS(1,n) \leqslant 2$ and it's torsion free, so can't have asdim $= 1$, therefore asdim$BS(1,n) = 2$

Amenability

- Amenability of groups is also a coarse property
- Exercise: Show this (I recommend using the Reiter condition)

Property A

Definition

A metric space X has *property A* if for any $R > 0$ and $\epsilon > 0$, there exists a Hilbert space $\mathcal{H}$, a map $\xi : X \to \mathcal{H}$ and a number S such that

(1) $\|\xi_x\| = 1$ for every $x \in X$,

(2) if $d(x, y) < R$, then $\|\xi_x - \xi_y\| < \epsilon$

(3) and if $d(x, y) \geqslant S$, then $\langle \xi_y, \xi_x \rangle = 0$.

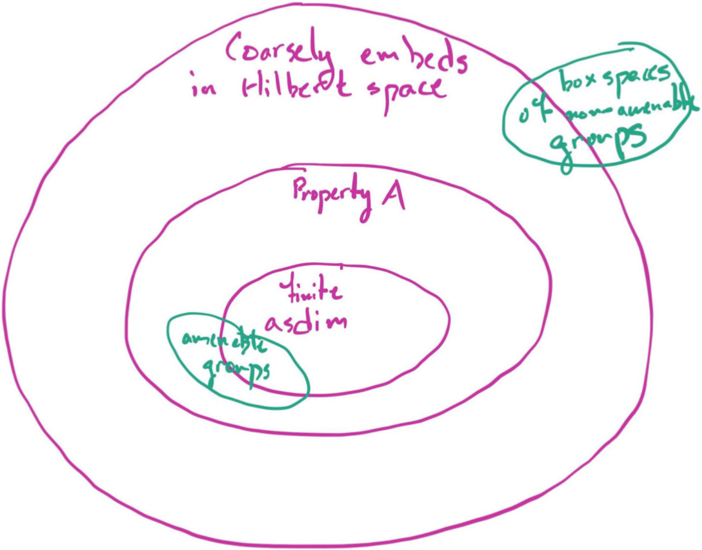
Property A (cont.)

- Acts like amenability for metric spaces (but also passes to subspaces)
- See, for example [Brown and Ozawa, 2008, 5.5.6], for some other characterizations.
- Exercise: Show this is a coarse property (this takes some work, but there's no special trick to it)
- ([Brown and Ozawa, 2008, 5.5.7]) A countable group Γ has property A if and only if it is exact (i.e. its reduced group C^*-algebra has a faithful, nuclear representation)

Coarse embeddings in Hilbert space

- Every metric space coarsely embeds into a Banach space, but not necessarily a Hilbert space
- Every tree (hence every free group) coarsely embeds in Hilbert space
- Coarse embeddings in Hilbert space are used to verify the coarse Baum-Connes conjecture for certain groups, which in turn has applications to geometry and topology (Novikov conjecture)

Relation between coarse properties



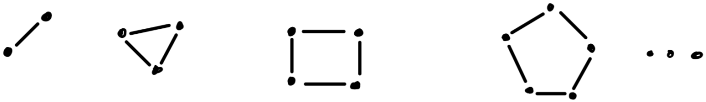
Box Spaces

Definition

Suppose $\Gamma = \langle F \rangle$ is finitely-generated. Let (N_n) be a decreasing sequence of finite-index normal subgroups of Γ. A *box space* of Γ is the disjoint union of Cayley graphs $C_F(\Gamma/N_n)$ with the metric given by $d(x, y) =$ usual distance between x and y if $x, y \in C_F(\Gamma/N_n)$ and $d(x, y) = \operatorname{diam} C_F(\Gamma/N_k) + \operatorname{diam} C_F(\Gamma/N_l)$ if $x \in C_F(\Gamma/N_k)$ and $y \in C_F(\Gamma/N_l)$.

Example

The full box space of $\mathbb{Z}$:



Box spaces and coarse properties

- A box space of Γ has property A iff Γ is amenable
- Some box spaces of non-abelian free groups are expanders, which do not coarsely embed Hilbert space. Some box spaces of such groups do admit such embeddings (and some which aren't expanders still don't) ([Arzhantseva and Guentner,] and [Delabie and Khukhro, 2018])
- If Γ is residually finite, then the asymptotic dimension of a box space of Γ is either infinite or equal to that of Γ. If Γ is virtually nilpotent, its box spaces are finite (asymptotic)-dimensional. [Delabie and Tointon, 2018]

Uniform Roe algebras

Definition

([Brown and Ozawa, 2008, 5.5.3]) Let X be a metric space with bounded geometry. An operator $A \in \mathbb{B}(l^2(X))$ is said to have *finite propagation* if $\langle A\delta_y, \delta_x \rangle \neq 0$ only if $d(x, y) < S$ (so thinking of A as an $X \times X$-matrix, the entries at distance greater than S from the diagonal are 0). The *translation algebra* $A(X)$ is the $*$-algebra of all operators with finite propagation. The closure of the translation algebra $A(X)$ in $\mathbb{B}(l^2(X))$ is called the *uniform Roe algebra* and is denoted by $C_u^*(X)$. If Γ is a countable group (e.g. a finitely-generated group) it has a natural metric and $C_u^*(\Gamma) \cong l^\infty(\Gamma) \rtimes_r \Gamma$.

Uniform Roe algebras and coarse geometry

- If d and d' are two metrics on X such that the identity map $(X, d) \to (X, d')$ is a coarse equivalence, then $C_u^*(X, d) = C_u^*(X, d')$ (this is because the two metrics lead to the same translation algebra $A(X)$)
- X and Y are coarsely equivalent iff $\mathcal{K}(l^2) \otimes C_u(X) \cong \mathcal{K}(l^2) \otimes C_u(Y)$
- Recall that the C^*-algebra of a group is nuclear iff the group is amenable. Similarly, a metric space X has property A if and only if $C_u^*(X)$ is nuclear. [Brown and Ozawa, 2008]
- Moreover, if asdim$X \leqslant d$, then the nuclear dimension of $C_u^*(X)$ is at most d (see [Winter and Zacharias, 2010]). It has been conjectured that the reverse inequality holds.

Coarse embeddings and the Baum-Connes conjecture

- The Baum-Connes conjecture states that a certain map called the assembly map $RK_*^\Gamma(\underline{E\Gamma}) \to K_*(C_r^*(\Gamma))$ is an isomorphism.
- When this holds, it is often helpful for computing K-theory and has applications to geometry and topology (verifying the Novikov conjecture for instance).
- There is also a coarse Baum-Connes conjecture regarding a different assembly map $K_*(X) \to K_*(C^*(X))$ (where $C^*(X)$ is the (not uniform) Roe algebra).

Coarse embeddings and the Baum-Connes conjecture continued

$$\begin{array}{ccc} K_*^{\Gamma}(\underline{E}\Gamma) & \longrightarrow & K_*(C_r^*\Gamma) \\ \downarrow & & \downarrow \\ K_*(\underline{E}\Gamma) & \longrightarrow & K_*(C^*(|\Gamma|)) \end{array}$$

- A commutative diagram relates these two maps (the vertical maps come from forgetting about equivariance), so the conjectures are related.
- A countable discrete group Γ which coarsely embeds in Hilbert space satisfies the coarse Baum-Connes conjecture
- This implies the (regular) Baum-Connes assembly map is injective, implying Γ satisfies the Novikov conjecture

Dynamic Asymptotic Dimension

Definition

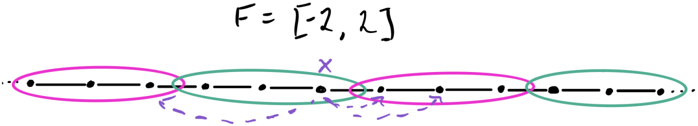
We say a free action $\Gamma \circlearrowright X$ has *dynamic asymptotic dimension* $\leqslant d$ if for all finite $F \subset \Gamma$ there is an open cover $\mathcal{U} = \{U_0, \ldots, U_d\}$ such that (for $f_i \in F$) the set $[x]_F = \{y : y = f_k \cdots f_1 \cdot x \text{ and } f_l \cdots f_1 \cdot x \in U_i \text{ for all } 1 \leqslant l \leqslant k\}$ is uniformly finite with respect to x.

DAD (cont.)

- Example: the action by an irrational rotation $\mathbb{Z} \circlearrowright S^1$ has dynamic asymptotic dimension 1.

- This dimension theory would appear to be sensitive both to the large scale structure of the Γ-orbits and to the topology of X

DAD and asdim



- For free actions $\Gamma \circlearrowright X$, $\mathrm{DAD}(\Gamma \circlearrowright X) \geqslant \mathrm{asdim}\Gamma$.
- When the dynamic asymptotic dimension of an isometric free action is finite, it is bounded above by $\mathrm{asdim}\Gamma + \dim X$ [Sawicki and Kielak, 2018]
- There are many cases where $\mathrm{DAD}\Gamma \circlearrowright X = \mathrm{asdim}\Gamma$, but nothing in general is known about this except when X is 0-dimensional.

DAD of odometers

- If Γ is a finitely-generated group and (N_n) is a decreasing sequence of finite-index normal subgroups, we can form an action $\Gamma \circlearrowright \prod_n \Gamma/N_n$ (or more generally, $\Gamma \circlearrowright \lim_{\leftarrow} \Gamma/N_n$). Such an action is called an odometer.

- If Γ is residually finite, then $\mathrm{DAD}(\Gamma \circlearrowright \lim_{\leftarrow} \Gamma/N_n)$ is equal to the asymptotic dimension of the box space $\sqcup_n C_F(\Gamma/N_n)$. [P.]

DAD and operator algebras

- The nuclear dimension of the crossed product product, $C(X) \rtimes_r \Gamma$ is bounded above by $(\text{dad}(\Gamma \circlearrowright X) + 1)(\dim(X) + 1) - 1$ where $\dim(X)$ is the covering dimension of X. [Guentner et al., 2015]

- Finite dynamic asymptotic dimension is also related to the Baum-Connes conjecture.

Dimension theory properties of DAD

- Product theorem:
  $\mathrm{DAD}(\Gamma \times \Lambda \circlearrowright X \times Y) \leqslant \mathrm{DAD}(\Gamma \circlearrowright X) + \mathrm{DAD}(\Lambda \circlearrowright Y)$.

- Extension theorem for odometers: If
  $1 \to \Delta \to \Gamma \to \Lambda \to 1$ is an exact sequence,
  $\mathrm{DAD}(\Gamma \circlearrowright \widehat{\Gamma}) \leqslant \mathrm{DAD}(\Delta \circlearrowright \widehat{\Delta}) + \mathrm{DAD}(\Lambda \circlearrowright \widehat{\Lambda})$.

Asymptotic dimension of $\Box BS(1, n)$

- Relation between box spaces and odometers described earlier now shows the dimension of box spaces is similarly subadditive over extensions

- Using the sequence $0 \to \mathbb{Z}[1/n] \to BS(1, n) \to \mathbb{Z} \to 0$ as before now shows $\operatorname{asdim} \Box BS(1, n) = 2$

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