Exercises with Solutions

1. (Finishing proof of Free CLT; doing hands-on moment calculation).
Suppose $(a_i)_{i\in\mathbb{N}}$ is a family of self-adjoint, freely independent, identically distributed nc random variables with $\phi(a_i) = 0$ and $\phi(a_i^2) = \sigma^2$. Compute the following:

   (a) $\phi(a_1 a_2 a_3)$
   Solution: $a_1 a_2 a_3$ is an alternating word of centered elements, so by the definition of freeness, $\phi(a_1 a_2 a_3) = 0$.

   (b) $\phi(a_1 a_2 a_1)$
   Solution: Again by the definition of freeness, $\phi(a_1 a_2 a_1) = 0$.

   (c) $\phi(a_1 a_1 a_2 a_2)$
   Solution: We do a standard trick, which is to add zero in the form of $\pm\phi(\text{each element})$, and isolate the part that is zero by freeness.

$$\begin{aligned}\phi(a_1 a_1 a_2 a_2) &= \phi(a_1^2 a_2^2)\\ &= \phi[(a_1^2 - \phi(a_1^2) + \phi(a_1^2))(a_2^2 - \phi(a_2^2) + \phi(a_2^2))]\\ &= \phi[(a_1^2 - \phi(a_1^2))(a_2^2 - \phi(a_2^2))] + \phi((a_1^2 - \phi(a_1^2))\phi(a_2^2))\\ &\quad + \phi(a_1^2)\phi(a_2^2 - \phi(a_2^2)) + \phi(a_1^2)\phi(a_2^2);\end{aligned}$$

   Now note the first term above is zero by freeness, while the second and third terms are zero since $\phi(a_i^2 - \phi(a_i^2)) = \phi(a_i^2) - \phi(a_i^2) = 0$. We conclude that

$$\phi(a_1 a_1 a_2 a_2) = \phi(a_1^2)\phi(a_2^2) = \sigma^4.$$

   (d) $\phi(a_1 a_2 a_1 a_2)$
   Solution: Again by the definition of freeness, $\phi(a_1 a_2 a_1 a_2) = 0$.

   (e) $\phi(a_1 a_2 a_2 a_1)$
   Repeat the same kind of trick as in part (c). If you work out the algebra correctly, you should again get

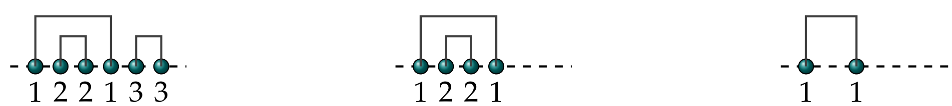
$$\phi(a_1 a_2 a_2 a_1) = \phi(a_1^2)\phi(a_2^2) = \sigma^4.$$

(f) Generalize the process in (3) and (5) above to arbitrary even-length products with 2 of each index, such as $\phi(a_1a_2a_3a_3a_2a_1)$.
For $\pi \in \mathcal{P}_2(2n)$, what conditions are needed to get $\phi(\pi) = \sigma^{2n}$? What about to get $\phi(\pi) = 0$? Are there any other possible values for $\phi(\pi)$?
Note: I expect this problem to take quite some time! Students will probably need to work out a few examples before they start to see the pattern emerge. As long as they are talking/working through examples, there's no need to try to hint at the solution!
If students are stuck, give them some starting configurations which correspond to either crossing or non-crossing partitions, draw these partitions, and ask what $\phi()$ of the corresponding word is.
Some example words you can give students:

Crossing:
$a_1a_2a_1a_3a_3a_2$
$a_1a_2a_3a_1a_3a_2$
$a_1a_2a_2a_3a_1a_3$

Non-crossing:
$a_1a_2a_2a_3a_3a_1$
$a_1a_2a_2a_1a_3a_3$
$a_1a_1a_2a_3a_3a_2$

I will cover the following solution/fact at the beginning of Lecture 2: We see that for $\pi \in \mathcal{P}_2(2n)$, we have $\phi(\pi) = \sigma^{2n}$ if and only if we can successively remove pairs of matching random variables until we end with a single pair, for example:



Actually, this occurs if and only if π is non-crossing.

Otherwise, $\phi(\pi) = 0$.

2. (Applying relationship between moment & cumulant functionals).
Show that if ϕ is a trace, then the cumulants κ_n are invariant under cyclic permutations, i.e.

$$\kappa_n(a_1, a_2, \ldots, a_n) = \kappa_n(a_2, a_3, \ldots, a_n, a_1).$$

Hints to give students if they are stuck: Go by induction on n, and use the moment-cumulant formula. Is there a way to rearrange the moment-cumulant formula so it is of the following form?

$$\kappa_n(\cdots) = \phi(\cdots) - \sum \textit{(some terms)}.$$

Solution: There is nothing to show in the case $n = 1$, and for the $n = 2$ case, we saw in the example on page 6 that

$$\kappa_2(a_1, a_2) = \phi(a_1 a_2) - \phi(a_1)\phi(a_2),$$

but since ϕ is a trace we have that this is equivalent to

$$\phi(a_2 a_1) - \phi(a_2)\phi(a_1) = \kappa_2(a_2, a_1).$$

Now suppose we know that $\kappa_\ell(\cdots)$ is invariant under cyclic permutation for all $\ell < n$, for some $n > 2$. By the moment-cumulant formula, we have

$$\phi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n) \implies \kappa_n(a_1, \ldots, a_n) = \phi(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \kappa_\pi(a_1, \ldots, a_n).$$

I use the notation 1_n to mean the maximal partition, i.e. the one with a single block containing all elements $\{1, 2, \ldots, n\}$. Now simply note that if $\pi \in NC(n)$ is not the maximally connected partition 1_n, then $\kappa_\pi(\cdots)$ is a product of cumulants of the form $\kappa_\ell(\cdots)$, with $\ell < n$. By the induction hypothesis, these are invariant under cyclic permutation. Combining this with the fact that ϕ is a trace, we have

$$\begin{aligned}
\kappa_n(a_1, a_2, \ldots, a_n) &= \phi(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \kappa_\pi(a_1, a_2 \ldots, a_n) \\
&= \phi(a_2 \cdots a_n a_1) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \kappa_\pi(a_2, \ldots, a_n, a_1) \\
&= \kappa_n(a_2, \ldots, a_n, a_1).
\end{aligned}$$