# Amenability and Property (T)

Gregory Patchell

June 2023

## 1 Amenability

**Definition 1.1.** Let $G$ be a discrete group and $M$ a tracial von Neumann algebra.

1. G is amenable if $1_G \prec \lambda_G$.
2. $M$ is amenable if $L^2(M) \prec L^2(M) \otimes L^2(M)$.

**Remark 1.2.** Another way to think of amenability is as "almost finite" in the sense that we can approximate matrix coefficients and almost take averages as we will see later. Amenability will also give us almost invariant/almost central vectors.

**Fact 1.3.** *$G$ is amenable if and only if $L(G)$ is amenable.*

**Example 1.4.**

1. All finite groups and finite dimensional von Neumann algebras are amenable.
2. $\mathbb{Z}$ and $L(\mathbb{Z}) \simeq L^\infty(X)$ are amenable. In fact, all abelian groups/tracial von Neumann algebras are amenable.
3. $S_\infty$, the group of finite permutations and $L(S_\infty) \simeq \mathcal{R}$, the hyperfinite factor, are amenable.
4. The classes of amenable discrete groups/tracial von Neumann algebras are closed under direct sums, tensor products, inductive limits, extensions, and substructures.
5. If $M$ is an amenable $\mathrm{II}_1$ factor, then $pMp$ is amenable for any projection $p \in M$.
6. The free group $\mathbb{F}_n$ is not amenable for any $n \geq 2$.

There are many equivalent definitions of amenability for groups. Here are just a handful:

**Theorem 1.5.** *A countable discrete group $G$ is amenable if and only if:*

1. *For all $F \subset G$ finite and $\varepsilon > 0$ there is $\xi \in \ell^2(G)$ such that $\|\lambda_G(g)\xi - \xi\| < \varepsilon$ for all $g \in F$.*
2. *There exists a state $\varphi : \ell^\infty(G) \to \mathbb{C}$ such that $\varphi(g \cdot f) = \varphi(f)$ for all $g \in G$ and $f \in \ell^\infty(G)$.*
3. *There exist finite subsets (Følner sets) $F_n \subset G$ such that $\bigcup_{n=1}^\infty F_n = G$ and for all $g \in G$ $\lim_{n\to\infty} \frac{|F_n \Delta g F_n|}{|F_n|} = 0$.*

**Remark 1.6.** All three of the above conditions have analogous statements in the language of von Neumann algebras.

1. Naively one would think the analogy of almost invariant vectors for a group representation is almost central vectors for a von Neumann algebra. But in fact we need vectors which are almost central *and* almost tracial. That is, $M$ is amenable if and only if $(L^2(M)\otimes L^2(M))^{\oplus\infty}$ contains, for each $F \subset M$ finite and $\varepsilon > 0$, a vector $\xi$ such that $\|x\xi - \xi x\| < \varepsilon$ and $|\tau(x) - \langle \xi, x\xi\rangle| < \varepsilon$ for all $x \in F$.
2. The invariant state is related to a conditional expectation from $\mathcal{B}(L^2(M))$ onto $M$.
3. The Følner sets $F_n$ are related to finite rank projections in $\mathcal{B}(L^2(M))$ that pointwise almost commute with $M$.

**Theorem 1.7** (Connes)**.** *If $(M, \tau)$ is an amenable $II_1$ factor, then it is the hyperfinite $II_1$ factor $\mathcal{R}$.*

# 2 Property (T)

**Definition 2.1.** Let $G$ be a discrete group and $(M, \tau)$ a tracial von Neumann algebra.

1. $G$ has Property (T) (or Kazhdan's Property (T)) if for all $\varepsilon > 0$ there is $F \subset G$ finite and $\delta > 0$ so that whenever $(\pi, \mathcal{H})$ is a representation of $G$ and $\xi$ is a unit vector $\xi \in \mathcal{H}$ such that $\|g\xi - \xi\| < \delta$ for all $g \in F$, then there is $\eta \in \mathcal{H}$ such that $g\eta = \eta$ for all $g \in G$ and $\|\eta - \xi\| < \varepsilon$.
2. $M$ has Property (T) if for all $\varepsilon > 0$ there there is $F \subset M$ finite and $\delta > 0$ so that whenever ${}_M\mathcal{H}_M$ is an $M$-$M$ bimodule and $\xi$ is a unit vector $\xi \in \mathcal{H}$ such that $\langle \xi, x\xi\rangle = \tau(x) = \langle \xi, \xi x\rangle$ for all $x \in M$ and $\|x\xi - \xi x\| < \delta$ for all $x \in F$, then there is $\eta \in \mathcal{H}$ such that $x\eta = \eta x$ for all $x \in M$ and $\|\eta - \xi\| < \varepsilon$.

**Remark 2.2.** There are a couple of strange qualities to the definition above of property (T).

1. The $F$ and $\delta$ are universal for all representations/bimodules. However, this is actually a non-issue; this is related to the proceeding exercise.
2. We require the invariant/central vector to be close to the almost invariant/central vector. This is not a problem in the group case nor in the case of $\text{II}_1$ factors but it is in the case of non-factors.
3. The requirement for the almost-central vectors to be tracial is a bit more stringent than we would get from assuming $L^2(M) \prec {}_M\mathcal{H}_M$. However, again this is not an issue in the setting of $\text{II}_1$ factors.

**Exercise 2.3.** Show that the following are equivalent:

1. For all $M$-$M$ bimodules $\mathcal{H}$, $L^2(M) \prec \mathcal{H}$ implies $L^2(M) \subset \mathcal{H}$.
2. There is a neighbourhood $V$ of $L^2(M)$ such that for all $M$-$M$ bimodules $\mathcal{H}$, $\mathcal{H} \in V$ implies $L^2(M) \subset \mathcal{H}$.

**Theorem 2.4.**

1. *$G$ has Property (T) if and only if for all representations $\pi$ of $G$, $1_G \prec \pi \implies 1_G \subset \pi$.*
2. *If $M$ is a $II_1$ factor, then $M$ has Property (T) if and only if for all $M$-$M$ bimodules $\mathcal{H}$, $L^2(M) \prec \mathcal{H}$ implies $L^2(M) \subset \mathcal{H}$.*

**Remark 2.5.** Another way to think about Property (T) is that it remembers the "discreteness" of finite groups. Indeed, Property (T) says that the trivial representation/bimodule is "isolated" in the Fell topology.

**Example 2.6.**

1. Finite groups and finite dimensional von Neumann algebras have property (T).
2. $\mathbb{Z}$ does not have property (T).
3. $\mathbb{F}_n$ does not have property (T).
4. $SL_n(\mathbb{Z})$ has property (T) for all $n \geq 3$.
5. $\mathrm{Aut}(\mathbb{F}_n)$ has property (T) for all $n \geq 4$.
6. $G$ has property (T) if and only if $L(G)$ does.
7. Property (T) is closed under direct sum, tensor product, extensions, and nontrivial central extensions by finitely generated abelian groups.
8. Property (T) of groups is closed under quotients.
9. Property (T) of $\mathrm{II}_1$ factors is closed under compressions ($pMp$).
10. Property (T) is *not* closed under substructures or inductive limits.

$\mathrm{II}_1$ factors with property (T) are often referred to as "rigid" since they have admit very few symmetries as seen in the below theorems.

**Theorem 2.7.** *If $M$ is a $II_1$ factor with property (T), then it is separable.*

**Theorem 2.8.** *If $M$ is a $II_1$ factor with property (T), then its outer automorphism group is countable.*

**Theorem 2.9** (Connes)**.** *If $M$ is a $II_1$ factor with property (T), then its fundamental group is countable.*

**Theorem 2.10** (Connes)**.** *For any icc property group $G$ with property (T), there are at most countably many countable groups $H$ such that $L(G) \simeq L(H)$.*

Conjecturally, there are no groups other than $G$ itself such that $L(G) \simeq L(H)$ when $G$ is icc and property (T). This would say $G$ is W$^*$-superrigid. Specific examples of W$^*$-superrigid groups exist, many of which have property (T).

Unlike in the amenable setting, there are uncountably many non-isomorphic property (T) $\mathrm{II}_1$ factors. In fact, something even stronger is true:

**Theorem 2.11** (Ioana, Chifan, Sun, Osin)**.** *Every separable $II_1$ factor $N$ is contained in a property (T) $II_1$ factor $M$.*

Property (T) can also be characterized in terms of having "spectral gap".

**Theorem 2.12** (Tan)**.** *Let $M$ be a $II_1$ factor. $M$ has property (T) if and only if whenever $M \subset N$ is an inclusion of tracial von Neumann algebras we have $(M' \cap N)^\omega = M' \cap N^\omega$.*