Representations, Bimodules, and Approximations

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1 Representations of Groups

Definition 1.1. Let G be a discrete group. A *representation* of G, $(\pi, \mathcal{H})$, is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$. (All of my group representations will be unitary; $\mathcal{U}(\mathcal{H})$ denotes the set of unitary operators from $\mathcal{H}$ to $\mathcal{H}$ and is therefore a subset of $\mathcal{B}(\mathcal{H})$.)

Example 1.2.

1. The trivial representation

$$\begin{aligned} 1_G : G &\to \mathcal{U}(\mathbb{C}) = \mathbb{T} \\ g &\mapsto 1 \end{aligned}$$

2. The left regular representation

$$\begin{aligned} \lambda_G : G &\to \mathcal{U}(\ell^2 G) \\ g &\mapsto (\delta_h \mapsto \delta_{gh}) \end{aligned}$$

   The right regular representation ρ_G is defined analogously.

3. If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are representations of G then so is $(\pi \oplus \rho, \mathcal{H} \oplus \mathcal{K})$ defined by

$$\begin{aligned} \pi \oplus \rho : G &\to \mathcal{U}(\mathcal{H}) \oplus \mathcal{U}(\mathcal{K}) \subset \mathcal{U}(\mathcal{H} \oplus \mathcal{K}) \\ g &\mapsto \pi(g) \oplus \rho(g) \end{aligned}$$

Definition 1.3. If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are representations of G, we say that π is *contained* in ρ and write $\pi \subset \rho$ if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $V\pi(g)\xi = \rho(g)V\xi$ for all $\xi \in \mathcal{H}$ and all $g \in G$.

Remark 1.4. If $\pi \subset \rho$, then note that in particular, for all $\xi \in \mathcal{H}$ and all $g \in G$ there exists $\eta \in \mathcal{K}$ such that $\langle \xi, \pi(g)\xi \rangle_{\mathcal{H}} = \langle \eta, \rho(g)\eta \rangle_{\mathcal{K}}$. (Take $\eta = V\xi$.)

The quantities $\langle \xi, \pi(g)\xi' \rangle$ are called *matrix coefficients* of π. We would like an approximate version of inclusion where we might not have equality of matrix coefficients, but near equality up to an arbitrary $\varepsilon > 0$ and allowing convex combinations. This inspires our next definition.

Definition 1.5. Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be representations of G. We say π is *weakly contained* in ρ and write $\pi \prec \rho$ if for all $\varepsilon > 0$, all finite subsets $F \subset G$ and all $x \in \mathcal{H}$ there exist $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that for all $g \in F$,

$$\left| \langle \xi, \pi(g)\xi \rangle - \sum_{i=1}^{n} \langle \eta_i, \rho(g)\eta_i \rangle \right| < \varepsilon.$$

The notion of weak containment is actually intimately related to a (non-Hausdorff!!) topology on the space of representations of G, $\mathrm{Rep}(G)$. Let $(\pi, \mathcal{H})$ be a representation of G. For $\varepsilon > 0$, $F \subset G$ finite, and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, define

$$V(\pi; \varepsilon, F, \xi_1, \ldots, \xi_n)$$
$$:= \{(\rho, \mathcal{K}) \in \mathrm{Rep}(G) : \exists \eta_1, \ldots, \eta_n \in \mathcal{K}\ \forall 1 \leq i, j \leq n\ \forall g \in F \mid \langle \xi_i, \pi(g)\xi_j\rangle - \langle \eta_i, \rho(g)\eta_j\rangle \mid < \varepsilon\}$$

Definition 1.6. The *Fell topology* on $\mathrm{Rep}(G)$ is the topology generated by the $V(\pi; \varepsilon, F, \xi_1, \ldots, \xi_n)$ for $(\pi, \mathcal{H}) \in \mathrm{Rep(G)}$, $\varepsilon > 0$, $F \subset G$ finite, $n \in \mathbb{N}$, and $\xi_i \in \mathcal{H}$.

Remark 1.7. For a fixed representation $(\pi, \mathcal{H})$ of G, the collection of sets $V(\pi; \varepsilon, F, \xi_1, \ldots, \xi_n)$ (as ε, F, and the ξ_i vary) is a neighbourhood base for π in the Fell topology on $\mathrm{Rep}(G)$.

Exercise 1.8. Show that in a topological space X, we have that $x \in \overline{\{y\}}$ if and only if for all open sets U, $x \in U$ implies $y \in U$.

Fact 1.9. *$\pi \prec \rho$ if and only if $\pi \in \overline{\{\rho^{\oplus\infty}\}}$. Hint: First consider cyclic representations. ($\rho^{\oplus\infty}$ is the countably infinite direct sum of ρ with itself and the closure is taken in the Fell topology.)*

2 Bimodules

Definition 2.1. Let M, N be von Neumann algebras. An M-N *bimodule* (also *Hilbert bimodule* or *correspondence*) is a Hilbert space $\mathcal{H}$ with commuting normal *-representations $\pi_M : M \to \mathcal{B}(\mathcal{H})$ and $\pi_{N^{op}} : N^{op} \to \mathcal{B}(\mathcal{H})$. We often write ${}_M\mathcal{H}_N$ to denote a M-N bimodule.

Example 2.2. Let (M, τ) be a tracial von Neumann algebra.

1. Recall the standard representation $L^2(M)$ of M, namely the closure of M in the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$. For $x \in M$, denote by $\hat{x}$ the image of x in $L^2(M)$. Then $L^2(M)$ is an M-M bimodule where the left action on $\hat{M} \subset L^2(M)$ is given by $x\hat{y} = \widehat{xy}$ and the right action n $\hat{M}$ is given by $\hat{y}x = \widehat{yx}$, and both actions are extended to all of $L^2(M)$ by continuity. $L^2(M)$ is called the *trivial bimodule.*

2. $L^2(M) \otimes L^2(M)$ is also an M-M-bimodule with left action given by $x(\hat{1} \otimes \hat{1}) = \hat{x} \otimes \hat{1}$ and right action given by $(\hat{1} \otimes \hat{1})x = \hat{1} \otimes \hat{x}$. $L^2(M) \otimes L^2(M)$ is called the *coarse bimodule.*

Definition 2.3. If ${}_M\mathcal{H}_N$ and ${}_M\mathcal{K}_N$ are M-N bimodules, we say that ${}_M\mathcal{H}_N$ is *contained* in ${}_M\mathcal{K}_N$ and write ${}_M\mathcal{H}_N \subset {}_M\mathcal{K}_N$ if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $V(x\xi) = x(V\xi)$ and $V(\xi x) = (V\xi)x$ for all $\xi \in \mathcal{H}$ and all $x \in M$.

Remark 2.4. If $\pi \subset \rho$, then note that in particular, for all $\xi \in \mathcal{H}$ and all $g \in G$ there exists $\eta \in \mathcal{K}$ such that $\langle \xi, \pi(g)\xi\rangle_{\mathcal{H}} = \langle \eta, \rho(g)\eta\rangle_{\mathcal{K}}$. (Take $\eta = V\xi$.)

As in the group case, the space of M-N bimodules $\mathrm{Bimod}(M, N)$ can be topologized in such a way that allows us to analyze approximate inclusion of bimodules. Let ${}_M\mathcal{H}_N$ be an M-N bimodule. For $\varepsilon > 0$, $E \subset M$ finite, $F \subset N$ finite, and $\xi_1, \ldots, \xi_n \in {}_M\mathcal{H}_N$, define

$$V({}_M\mathcal{H}_N; \varepsilon, E, F, \xi_1, \ldots, \xi_n)$$
$$:= \{{}_M\mathcal{K}_N \in \mathrm{Bimod}(M, N) : \exists \eta_1, \ldots, \eta_n \in \mathcal{K}\ \forall i, j\ \forall x \in E, y \in F \mid \langle \xi_i, x\xi_j y\rangle - \langle \eta_i, x\eta_j y\rangle \mid < \varepsilon\}$$

Definition 2.5. The *Fell topology* on $\mathrm{Bimod}(M, N)$ is the topology generated by the $V({}_M\mathcal{H}_N; \varepsilon, E, F, \xi_1, \ldots, \xi_n)$ for ${}_M\mathcal{H}_N \in \mathrm{Bimod}(M, N)$, $\varepsilon > 0$, $E \subset M$ finite $F \subset N$ finite, $n \in \mathbb{N}$, and $\xi_i \in \mathcal{H}$.

Remark 2.6. For a fixed bimodule ${}_M\mathcal{H}_N$, the collection of sets $V(\pi; \varepsilon, E, F, \xi_1, \ldots, \xi_n)$ (as ε, E, F, and the ξ_i vary) is a neighbourhood base for ${}_M\mathcal{H}_N$ in the Fell topology on $\mathrm{Bimod}(M, N)$.

Definition 2.7. We say that ${}_M\mathcal{H}_N$ is *weakly contained* in ${}_M\mathcal{K}_N$ and write ${}_M\mathcal{H}_N \prec {}_M\mathcal{K}_N$ if $\mathcal{H} \in \overline{\{\mathcal{K}^{\oplus\infty}\}}$.

3 Amenability and Property (T)

Definition 3.1. Let G be a discrete group and M a tracial von Neumann algebra.

1. G is amenable if $1_G \prec \lambda_G$.
2. M is amenable if $L^2(M) \prec L^2(M) \otimes L^2(M)$.
3. G has Property (T) if for all representations π of G, $1_G \prec \pi \implies 1_G \subset \pi$.
4. If M is a II_1 factor, then M has Property (T) if for all M-M bimodules $\mathcal{H}$, $L^2(M) \prec \mathcal{H}$ implies $L^2(M) \subset \mathcal{H}$.

Theorem 3.2 (Dixmier Averaging)**.** *If M is a factor then for all $x \in M$ and $\varepsilon > 0$, there are unitaries $u_1, \ldots, u_n \in M$ and $\alpha \in \mathbb{C}$ such that $\left\| \frac{1}{n} \sum_{i=1}^{n} u_i x u_i^* - \alpha 1 \right\| < \varepsilon$. If M is tracial, then $\alpha = \tau(x)$.*