Def: The <u>full</u> or <u>Universal</u> crossed product of a C^*-dynamical system (A,Γ,α) denoted $A \rtimes_\alpha \Gamma$ is the completion of $C_c(A,\Gamma)$ w.r.t. The norm

$$\|f\|_u := \sup\{ \|(\pi \rtimes u)(f)\| : (\pi, u) \text{ is a covarient-rep} \}$$

To Avoid Technicallities we assume Γ is discrete.

Prop: For every covarient rep (u, π, H) of (A, Γ, α) There exists a unique $*$-hom $\tilde{\pi} : A \rtimes_\alpha \Gamma \to \mathcal{B}(H)$ s.t.

$$\tilde{\pi}\left(\sum_{s\in\Gamma} f_s s\right) = \sum_{s\in\Gamma} \pi(f_s) u_s \quad \forall f \in C_c(\Gamma, A)$$

$\boxed{\text{Pf:}}$ first, one check that this is a $*$-hom when restricted to $C_c(\Gamma, A)$. $\tilde{\pi}(f) = \pi(f)$ $\forall f \in C_c(\Gamma, A)$

Next note that $\|\tilde{\pi}(f)\| = \|(\pi\rtimes u)(f)\| \leq \|f\|_u$ by def

Let $f_\gamma \to f \in A \rtimes_\alpha \Gamma$ where $\{f_\gamma\} \subseteq C_c(A,\Gamma)$. Note That this net is cauchy

So given $\varepsilon > 0$ $\exists$ γ_0 s.t. $\|f_\gamma - f_\beta\| < \varepsilon$ w/e $\gamma, \beta \geq \gamma_0$

Hence, $\|\tilde{\pi}(f_\gamma) - \tilde{\pi}(f_\beta)\| = \|\tilde{\pi}(f_\gamma - f_\beta)\| \leq \|f_\gamma - f_\beta\| < \varepsilon$ w/e $\gamma, \beta \geq \gamma_0$

So $\{\tilde{\pi}(f_\gamma)\}$ is cauchy in $\mathcal{B}(H)$ and has limit say y.

Defining $\tilde{\pi}(f) = y$ makes $\tilde{\pi}$ a $*$-hom. □

So This is like the universal prop for $C^*(\Gamma)$ or $A \otimes_{max} B$

fun fact:

Lemma 2.73 (williams)

If $\iota : \Gamma \to Aut(A)$ is trivial then $A \rtimes_\iota \Gamma \cong A \otimes_{max} C^*(\Gamma)$

(the question remains, do covarient representations exist?)

The reduced crossed product:

First we begin with a faithful representation $\rho : A \to \mathcal{B}(H)$. Since ρ is faithful $A \cong \rho(A)$ so we identify A with $\rho(A)$ and (Abusing the notation) veiw $A \subseteq \mathcal{B}(H)$. Next, define a new rep $\pi : A \to H \otimes \ell^2(\Gamma)$ by

$$\pi(a)(v \otimes \delta_g) = \alpha_{g^{-1}}(a)(v) \otimes \delta_g$$

Now the left regular rep spatialy implements the action of α

$$\begin{aligned}(1\otimes\lambda_s)\pi(a)(1\otimes\lambda_s^*)(v\otimes\delta_g) &= (1\otimes\lambda_s)\pi(a)(v\otimes\delta_{s^{-1}g}) \\ &= (1\otimes\lambda_s)(\alpha_{g^{-1}s}(a)(v)\otimes\delta_{s^{-1}g}) \\ &= \alpha_{g^{-1}}(\alpha_s(a)(v)) \otimes \delta_g \\ &= \pi(\alpha_s(a))(v\otimes\delta_g)\end{aligned}$$

So $(1\otimes\lambda_s)\pi(a)(1\otimes\lambda_s)^* = \pi(\alpha_s(a))$

Hence, we have an induced covarient representation $\pi \rtimes (1\otimes\lambda)$ called the regular-rep

Def: The reduced crossed product of The C^*-dynamical system (A,Γ,α) denoted $A \rtimes_{\alpha, r} \Gamma$ is the norm closure of the image of The regular rep.

prop:
The reduced crossed product does not depend on the choice of faithful rep $A \subseteq \mathcal{B}(\mathcal{H})$.

$\boxed{\text{Pf}}$ For a finite set $F \subseteq \Gamma$ let $P \in \mathcal{B}(\ell^2(\Gamma))$ be the Proj onto the Span$\{\delta_g : g \in F\}$. Let $\{e_{p,q}\}$ be the matrix units of $P\mathcal{B}(\mathcal{H})P \cong M_{|F|}(\mathbb{C})$ recall that $H \otimes \ell^2(\Gamma) \cong \bigoplus_{r \in \Gamma} H$. Thus, we may view $\pi(a)$ as a diagonal matrix in $\mathcal{B}(\bigoplus_{r\in\Gamma} H)$

$$\pi(a) \text{ "=" } \begin{bmatrix} \ddots & & & \\ & \alpha_{-1}^{-1}(a) & & \\ & & \alpha_0^{-1}(a) & \\ & & & \alpha_1^{-1}(a) \\ & & & & \ddots \end{bmatrix}$$

So in the tensor picture we have $\pi(a) = \sum_{r\in\Gamma} \alpha_r^{-1}(a) \otimes e_{r,r}$

So our representation looks like

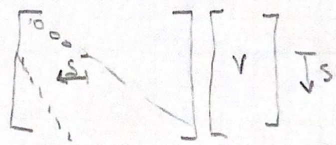
$$u : \Gamma \to \mathcal{U}(H \otimes \ell^2(\Gamma)) \qquad \pi : A \to \mathcal{B}(H \otimes \ell^2(\Gamma))$$
$$s \mapsto 1 \otimes \lambda_s \qquad\qquad a \mapsto \sum_{r\in\Gamma} \alpha_r^{-1}(a) \otimes e_{r,r}$$

$$(\pi \rtimes u)(\xi) = \sum_{s\in\Gamma} \pi(\xi_s)(1\otimes\lambda_s) = \sum_{s\in\Gamma}\Big(\sum_{r\in\Gamma}\alpha_r^{-1}(\xi_s)\otimes e_{r,r}\Big)(1\otimes\lambda_s)$$

Observe that

$$(1\otimes P)\pi(a)(1\otimes\lambda_s)(1\otimes P)$$
$$= (1\otimes P)\pi(a)(1\otimes P)(1\otimes\lambda_s)(1\otimes P)$$
$$= (1\otimes P)\pi(a)\underbrace{(1\otimes P\lambda_s P)}$$

$\lambda_s : \delta_g \mapsto \delta_{sg}$

$1\otimes\lambda_s$ "=" 

So $(1\otimes P)(1\otimes\lambda_s)(1\otimes P)\,[\oplus H]$ "=" $(1\otimes P)(1\otimes\lambda_s)\,[F]$ "=" $(1\otimes P)\,[sF]$ "=" $[F\cap sF]$

So $(1\otimes P\lambda_s P) = \sum_{r\in F\cap sF} 1\otimes e_{r,s^{-1}r}$ $**$

Thus, $(1\otimes P)\,[(\pi\rtimes(1\otimes\lambda))(\xi)]\,(1\otimes P)$

$$= \sum_{s\in\Gamma}(1\otimes P)\pi(\xi_s)(1\otimes\lambda_s)(1\otimes P)$$

$$= \sum_{s\in\Gamma}\Big(\sum_{q\in F}\alpha_q^{-1}(\xi_s)\otimes e_{q,q}\Big)\Big(\sum_{r\in F\cap sF} 1\otimes e_{r,s^{-1}r}\Big)$$

$$= \sum_{s\in\Gamma}\sum_{r\in F\cap sF}\alpha_r^{-1}(\xi_s)\otimes e_{r,s^{-1}r} \in M_{|F|}(A)$$

$M_n(A)$ has a unique C^*-norm

So

$(1\otimes P)[\pi\rtimes(1\otimes\lambda)(\xi)](1\otimes P)$ does not depend on the choice of embedding.

Taking limits over all finite $F\subseteq\Gamma$ gives the result,