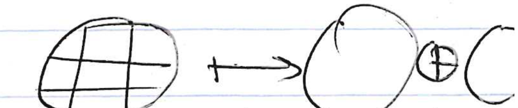
Amenability for groups and von Neumann algebras

Defn: G is amenable if $\exists\, m \in \ell^\infty(G)^*$ s.t. m is positive, $m(\mathbb{1}_G) = 1$. and $m(f) = m(gf)$. where $gf(h) = f(g^{-1}h)$.

Why?: Banach Tarski Paradox: 
works this way because $\exists$ a paradoxical decomposition.

i.e $\exists\, A, B$ st $A \cup B \cup \{1\} = G$
and $\exists\, g_1, g_2, g_3$ st $g_1 A \perp g_2 A \subset B$
and $g_3 B \subset A$

$$\Rightarrow m(\mathbb{1}_A) \le \tfrac{1}{2}\mu(B)$$
$$m(\mathbb{1}_B) \le m(\mathbb{1}_A). \Rightarrow m(\mathbb{1}_G) = 0.$$

Examples: Finite groups, $\mathbb{Z}$, $\mathbb{Z}^n$, abelian groups, Extensions, S_∞, unions, etc.

Non examples: $\mathbb{F}_2$, any group containing $\mathbb{F}_2$, Groups of piecewise linear transformations on $\mathbb{R}$.

Goal of this lecture:

Create a natural notion of amenability for von Neumann algebras. and show that G is amenable $\Leftrightarrow$ $L(G)$ is ~~amenable~~ amenable

Amenable groups are "Pseudo - compact"

<u>Defn:</u> Let (M, τ) be a tracial vNa. A state ψ on $B(L^2(M))$ is called a <u>hypertrace</u> if $\forall T \in B(L^2(M))$

$$\psi(xT) = \psi(Tx) \quad \forall x \in M \subset B(L^2(M)).$$

and $\psi|_M = \tau$.

<u>Observation:</u> If you are dealing with factors, the latter condition is automatic (why?).

<u>Theorem (conjecture):</u> M is amenable $\Leftrightarrow$ M has a hypertrace.

<u>Step 1:</u> Show that Defn (1) is equivalent to:

<u>Defn:</u> M is <u>amenable'</u> if $\exists$ a conditional expectation $E: B(L^2(M)) \longrightarrow M$.

<u>Proposition:</u> M is <u>amenable</u> $\Leftrightarrow$ M is <u>amenable'</u>

Pf: $\Leftarrow$: If $E: B(L^2(M)) \longrightarrow M$ is a conditional expn then $\tau \circ E: B(L^2(M)) \to \mathbb{C}$ is a <u>hypertrace</u>

why: $\tau \circ E(xT) = \tau(xE(T)) = \tau(E(T)x)$
$= \tau(E(Tx))$.

and $E(T) = T \ \forall T \in M \Rightarrow$
$\tau(E(T)) = \tau(T)$.

$\Rightarrow$ we will need some <u>tools</u>:

Little Radon Nikodym Theorem

For $a \in M$, denote $\tau_a : x \mapsto \tau(ax) \in M$.
$$= \langle ax \cdot 1, a^* 1 \rangle.$$

If $a \in M^+$,
$$\tau_a(x) = \tau(x^{1/2} a x^{1/2}) \leq \|a\| \tau(x)$$
for x positive.

~~Conv~~ Converse to this:

<u>Thm:</u> Let φ be a positive linear functional on M and ~~asst~~ assume that $\exists \lambda \in \mathbb{R}^+$ st $\varphi \leq \lambda \tau$.
Then, $\exists!$ $a \in M$ st $\varphi = \tau_a$, and
$$0 \leq a \leq \lambda \cdot 1.$$

<u>Back to proof:</u>

Given $T \in B(L^2(M))_+$, define $\varphi(x) = \psi(Tx)$.
For $x \in M_+$, we have
$$|\varphi(x)|^2 = |\psi(x^{1/2} T x^{1/2})|^2 \leq \|T\|^2 \psi(x)^2$$
$$\leq (\|T\| \tau(x))^2.$$
Applying previous theorem, $\exists$ $E(T) \in M_+$ st.
$$\forall x \in M, \quad \psi(Tx) = \tau(E(T) x).$$
We claim that this is a conditional expectation.

Indeed, $\psi(Tyx) = \psi(T(yx)) = \psi(E(T)yx)$

$= \psi([E(T)y]x)$.

more precisely,

$$\psi((aTb)x) = \psi(Tbxa) = \tau(E(T)bxa)$$
$$= \tau((aE(T)b)x).$$

as required.

In the remaining time, we shall prove

<u>Thm:</u> G is amenable $\Leftrightarrow$ $L(G)$ is <u>amenable</u>.

Pf. ~~Let u~~ Let u_g be the canonical unitaries in $L(G)$. Assume first the existence of $E: B(\ell^2) \to$ ~~M~~.
Let $f \in \ell^\infty(G)$. Set M_f to be the multiplication operator
$M_f \in B(\ell^2(G))$, $m_f(g) = f \cdot g$.
Let $m(f) = \tau(E(M_f))$.

Since $u_g M_f u_g^* = M_{gf}$, we have,

$$m(gf) = \tau(E(M_{gf})) =$$
$$\tau(E(u_g M_f u_g^*))$$
$$= \tau(E(M_f)) = m(f).$$

This concludes the first direction

Conversely, assume m is a left invariant mean for $\ell^\infty(G)$.

Given ξ & $\eta \in \ell^2(G)$, $T \in B(\ell^2(G))$

$$f^T_{\xi,\eta}(s) = \langle \xi, \rho(s) T \rho(s^{-1})\eta \rangle.$$

where ρ is the right regular rep.

By C-S, $\| f^T_{\xi,\eta}(s) | \leq \|T\| \, \|\eta\| \, \|\xi\|$

Define $(\xi, \eta) = m(f^T_{\xi,\eta})$

$\Rightarrow \exists \, E(T)$ s.t.

$$\langle \xi, E(T)\eta \rangle = m(f^T_{\xi,\eta})$$

observe that $\rho(s) E(T) \rho(s^{-1}) = E(T)$
so $E(T) \in L(G)$

$g(g^{-1}Eg)$

$\mu(g\cancel{E}Eg) = \mu(gEg^{-1}$

$=$

$\mu(gE) = \mu(E$