Operator Systems I

Sam Kim

University of Waterloo

July 18, 2020

What is an operator system?

An operator system is any subspace $S \subseteq B(H)$ such that

1. the identity map 1 belongs to S and
2. for any $x \in S$, $x^* \in S$.

Some people assume S is norm closed, but this largely doesn't affect the theory

Operator systems are a strict generalization of unital C*-algebras, since operator systems are just unital C*-algebras without the multiplication.

We shall see that many new and strange behaviour occur for operator systems even when they have dimension 3 or 4.

Talk outline

For the first part of our talk, we will describe some basic properties of operator systems and its connections to C*-algebras.

In the second part of our talk, we shall describe some examples of operator systems that admit properties that one might not expect, considering all we have learned about C*-algebras.

Although operator systems admit an abstract characterization like C*-algebras, we shall not consider this today.

Recall an operator $T \in B(H)$ is positive if it is of the form $T = X^*X$ for some operator X.

Proposition

Let S be an operator system. Every element is spanned by positive elements in S.

Proof.

If $x \in S$, we know that the self-adjoint operators $\mathrm{Re}(x) = \frac{1}{2}(x + x^*)$ and $\mathrm{Im}(x) = \frac{1}{2i}(x - x^*)$ both belong to S. If x is self-adjoint, then $x = \frac{1}{2}(\|x\|1 + x) - \frac{1}{2}(\|x\|1 - x)$. □

We say that a map $\phi : S \to T$ is *positive* if for all $x \geq 0$ in S, $\phi(x) \geq 0$. The map ϕ is unital if $\phi(1) = 1$.

Proposition

Let $\phi : S \to T$ be a positive map. For all $x \in S$, $\phi(x^*) = \phi(x)^*$.

Proof.

Let $x = (a - b) + i(c - d)$, where $a, b, c, d \geq 0$. Since $\phi(a), \phi(b), \phi(c), \phi(d) \geq 0$,

$$\begin{aligned} \phi(x^*) &= \phi((a - b) - i(c - d)) = (\phi(a) - \phi(b)) - i(\phi(c) - \phi(d)) \\ &= \phi(x)^* . \end{aligned}$$

□

Example

The transpose map $[\cdot]^T : M_n \to M_n$ is a unital positive map. If $X \in M_n$ is a positive operator, then $X = Y^*Y$ for some operator $Y \in M_n$. Thus, $X^T = (Y^T)(Y^T)^*$ is positive.

Example

Any unital *-homomorphism between unital C*-algebras are unital positive maps.

Given an operator system $S \subseteq B(H)$, denote by $M_n(S)$ the $n \times n$-matrices with coefficients in S as operator subsystem of $B(H^n)$.

Given a map $\phi : S \to T$, define the nth amplification of ϕ by

$$\phi^{(n)} : M_n(S) \to M_n(T) : [x_{i,j}]_{i,j} \mapsto [\phi(x_{i,j})]_{i,j} \ .$$

Definition

A map $\phi : S \to T$ is said to be a unital, completely positive (ucp) map if for all $n \geq 1$, $\phi^{(n)}$ is a unital, positive map.

Example

Suppose $\pi : A \to B$ is a unital *-homomorphism between unital C*-algebras A and B. Since $\pi^{(n)}$ is also a unital *-homomorphism, π is a ucp map.

Example

The transpose map $[\cdot]^T : M_2 \to M_2$ is not a completely positive map. The matrix

$$M := \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

is positive, while

$$([\cdot]^T)^{(2)}(M) = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

is not positive.

Just as *-homomorphisms are automatically contractive, so are ucp maps.

Proposition (Lemma 9.16)

Let X be an operator on a Hilbert space H. The following are equivalent:

1. *The matrix*
$$\begin{bmatrix} 1 & X \\ X^* & 1 \end{bmatrix}$$
is positive.
2. $\|X\| \leq 1$.

In fact, if we work a little harder, we can show that for $X \in B(H)$ and $Y \in B(H)$ positive, $\begin{bmatrix} 1 & X \\ X^* & Y \end{bmatrix}$ is positive if and only if $X^*X \leq Y$. If $X, Y \in \mathbb{C}$, we have a 2×2 matrix so this is just a determinant calculation.

Proposition

Let $\phi : S \to T$ be a ucp map between operator systems S and T. The map ϕ is contractive.

Proof.

Let $x \in S$ be a contraction. By the previous exercise, the matrix

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}$$

is positive. Since $\phi^{(2)}$ is a positive map,

$$0 \leq \phi^{(2)}\left(\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \phi(x) \\ \phi(x)^* & 1 \end{bmatrix}.$$

By our exercise again, $\phi(x)$ is a contraction. □

Definition

Given operator systems S and T, we say that $\phi : S \to T$ is a complete order embedding if ϕ is unital and for all $n \geq 1$ and for all $x \in M_n(S)$, $x \geq 0$ if and only if $\phi^{(n)}(x) \geq 0$.

By the previous proposition again, for all $x \in S$, $\|x\| \leq 1$ if and only if $\|\phi(x)\| \leq 1$. That is, ϕ must be an isometry.

Proposition

Let $\phi : S \to T$ be a map. It is a complete order embedding if and only if ϕ is an isometry, ϕ is ucp, and $\phi^{-1} : \phi(S) \to S$ is also a ucp map.

There is a Hahn-Banach-type extension Theorem for operator systems, called Arveson's extension theorem.

Theorem (Arveson's extension theorem)

Suppose that S and T are operator systems with a unital complete order embedding $\rho : S \hookrightarrow T$. If $\phi : S \to B(H)$ is a ucp map, then there is a ucp map $\psi : T \to B(H)$ such that the diagram

$$\begin{array}{ccc} T & & \\ {\scriptstyle \rho}\uparrow & \searrow^{\psi} & \\ S & \xrightarrow{\phi} & B(H) \end{array}$$

commutes.

The final topic I wish to talk on in this first part is the notion of a C*-cover. A C*-cover describes a notion of an operator system generating a C*-algebra.

Definition

Let S be an operator system. We say that a pair (A, ρ) is a C*-cover if A is a unital C*-algebra and $\rho : S \hookrightarrow A$ is a unital complete order embedding for which $A = C^*(\rho(S))$.

Example

Let G be a discrete group with generating set $\mathfrak{g}$. Assume that $e \in \mathfrak{g}$ and that $\mathfrak{g}^{-1} = \mathfrak{g}$. For the operator system

$$S_\lambda(\mathfrak{g}) := \text{span}\{\lambda_g \in B(\ell^2(G)) : g \in \mathfrak{g}\} \subseteq B(\ell^2(G)) \, ,$$

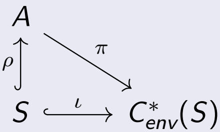
we have the C*-cover $(C^*_\lambda(G), \iota)$, where $\iota : S_\lambda(\mathfrak{g}) \hookrightarrow C^*_\lambda(G)$ is the inclusion map.

The following deep theorem connects the study of operator systems to the study of C*-algebras.

Theorem (Hamana)

Let S be an operator system. There always exists a minimal C-cover of S, denoted $(C^*_{env}(S), \iota)$.*

The C-envelope is minimal in the following sense: if (A, ρ) is a C*-cover of S, then there is a surjective unital *-homomorphism $\pi : A \to C^*_{env}(S)$ such that the diagram*



commutes.

In the next part of this talk, we will explore the C*-envelope in greater detail.

Operator Systems II

Sam Kim

University of Waterloo

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What is a C*-envelope?

Recall that an operator system is a unital, *-closed subspace of $B(H)$.

The notion of morphism between operator systems is given by unital, completely positive (ucp) maps.

From this, we described a notion of a unital complete order embedding as a ucp map with ucp inverse onto its range.

Finally, there was a distinguished C*-cover, called the C*-envelope of S, which is minimal in the sense of quotients preserving S.

Example

Let S be an operator system and suppose that (A, ρ) is a C*-cover for which A is a simple C*-algebra. By the universal property of $(C^*_{\text{env}}(S), \iota)$, there is a quotient *-homomorphism $\pi : A \to C^*_{\text{env}}(S)$ preserving S. Since A is simple, π must be an isomorphism. Therefore, (A, ρ) is the C*-envelope of S.

Example

Let $S = \text{span}\{1, z\} \subseteq C([0, 1])$, where $z(t) = t$ for all $t \in [0, 1]$. Define

$$\rho : S \to \mathbb{C}^2 : a1 + bz \mapsto (a, a + b) .$$

The map ρ is a complete order embedding. I claim that $(\mathbb{C}^2, \rho)$ must be the C*-envelope. Since there is only one unital quotient of $\mathbb{C}^2$ (namely $\mathbb{C}$), it suffices to show that there cannot be a complete order embedding of S into $\mathbb{C}$. This is because S has dimension 2 while $\mathbb{C}$ has dimension 1.

The following Lemma allows us to multiply unitaries just by using positivity.

Exercise (Walter's Lemma)

Let U, V be unitary operators on a Hilbert space H. Fix an operator $X \in B(H)$. The following are equivalent:

1. *The matrix* $\begin{bmatrix} 1 & U & X \\ U^* & 1 & V \\ X^* & V^* & 1 \end{bmatrix}$ *is positive.*
2. $X = UV$.

Operator Systems generated by unitary operators satisfy the following nice uniqueness result.

Lemma

Suppose that S is an operator system generated by a collection of unitary operators $\mathcal{U} \subseteq U(K)$. If $\pi : C^*(\mathcal{U}) \to B(H)$ is any unital representation and if $\phi : C^*(\mathcal{U}) \to B(H)$ is a ucp map such that $\pi \restriction S = \phi \restriction S$, then $\phi = \pi$.

To show this, first observe that the set

$$\{u_1 u_2 \cdots u_n : u_1, u_2, \ldots, u_n \in \mathcal{U} \cup \mathcal{U}^*\}$$

span a dense subset of $C^*(\mathcal{U})$.

It is then enough to show that for any unitaries $u, v \in C^*(\mathcal{U})$ such that $\phi(u) = \pi(u)$ and $\phi(v) = \pi(v)$, we must have $\phi(uv) = \phi(u)\phi(v)$ since then $\phi(uv) = \phi(u)\phi(v) = \pi(u)\pi(v) = \pi(uv)$.

Proof.

Fix u, v unitaries in $C^*(\mathcal{U})$ for which $\phi(u) = \pi(u)$ and $\phi(v) = \pi(v)$. By Walter's Lemma, the matrix $\begin{bmatrix} 1 & u & uv \\ u^* & 1 & v \\ (uv)^* & v^* & 1 \end{bmatrix}$ is positive.

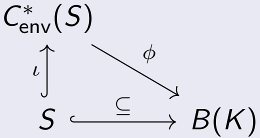
Since $\phi^{(3)}$ is a positive map, the matrix $\begin{bmatrix} 1 & \phi(u) & \phi(uv) \\ \phi(u)^* & 1 & \phi(v) \\ \phi(uv)^* & \phi(v)^* & 1 \end{bmatrix}$ is positive. By Walter's Lemma again, $\phi(uv) = \phi(u)\phi(v)$. □

Proposition

Suppose that S is an operator system generated by a collection of unitary operators $\mathcal{U} \subseteq U(K)$. If $\rho : S \hookrightarrow C^*(\mathcal{U})$ is the inclusion map, then $(C^*(\mathcal{U}), \rho)$ is the C-envelope of S.*

Proof.

There is a quotient map $\pi : C^*(\mathcal{U}) \to C^*_{\text{env}}(S)$ preserving S. By Arveson's extension theorem, we have a ucp map $\phi : C^*_{\text{env}}(S) \to B(K)$ such that the diagram



commutes. The ucp map $\phi \circ \pi$ agrees with the inclusion map $C^*(\mathcal{U}) \subseteq B(K)$ on S. By our Lemma, $\phi \circ \pi$ agrees with the inclusion map on $C^*(\mathcal{U})$. In particular, π is an embedding and hence gives us an isomorphism $C^*(\mathcal{U}) \cong C^*_{\text{env}}(S)$. □

Our rigidity proposition is an example of hyperrigidity of an operator system.

Definition

Let S be an operator system and let (A, ρ) be a C*-cover. We say that a representation

$$\pi : A \to B(H)$$

has the unique extension property if for all ucp maps $\phi : A \to B(H)$ such that $\pi \upharpoonright S = \phi \upharpoonright S$, we must have $\pi = \phi$.
We say that S is hyperrigid in A if all representations of A have the unique extension property.

Corollary

Let S be generated by unitary operators $\mathcal{U}$. Then S is hyperrigid in $C^*(\mathcal{U})$.

The following open question is currently one of the biggest problems in the theory of operator systems.

Question (Arveson's Hyperrigidty Conjecture)

Suppose that S is an operator system with C-cover (A, ρ). If all irreducible representations of A have the unique extension property, then is S hyperrigid in A?*

Example

Let G be a discrete group with discrete generating set $\mathfrak{g}$. The operator system $S_\lambda(\mathfrak{g}) \subseteq C^*_\lambda(G)$ has C*-envelope $C^*_\lambda(G)$.

Example

Let $S = \text{span}\{1, z, \bar{z}\} \subseteq C(\mathbb{T})$, where $z(t) = t$ for all $t \in \mathbb{T}$. The C*-envelope of S is $C(\mathbb{T})$.

Unlike C*-algebras, where all the finite dimensional C*-algebras embed into M_n, operator systems are not as nice.

Proposition

The operator system $S := \text{span}\{1, z, \bar{z}\} \subseteq C(\mathbb{T})$ *does not have a unital, complete order embedding into* M_n *for any* n.

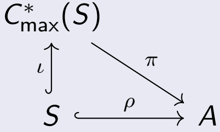
Proof.

Suppose there is a unital complete order embedding $\rho : S \hookrightarrow M_n$. By the universal property of the C*-envelope, there is a quotient map $\pi : C^*(\rho(S)) \to C(\mathbb{T})$ preserving S. Since $C^*(\rho(S))$ has dimension at most n^2 but $C(\mathbb{T})$ has dimension $\aleph_0$, this is a contradiction. □

One might think that operator subsystems of M_n are nicer, but our next goal is to show that this is not the case.

Definition

Given an operator system S, there exists a maximal C*-cover of S, denoted $(C^*_{\max}(S), \iota)$. It is maximal in the following sense: if (A, ρ) is a C*-cover of S, then there is a quotient *-homomorphism $\pi : C^*_{\max}(S) \to A$ such that the diagram



commutes.

We can then define two tensor products for operator systems.

Definition

Let S and T be operator systems.

1. The minimal tensor product of S and T, denoted $S \otimes_{\min} T$ is the operator system given by the algebraic tensor product $S \otimes T$ in $C^*_{\text{env}}(S) \otimes_{\min} C^*_{\text{env}}(T)$.
2. The commuting tensor product of S and T, denoted $S \otimes_c T$ is the operator system given by the algebraic tensor product $S \otimes T$ in $C^*_{\max}(S) \otimes_{\max} C^*_{\max}(T)$.

If A and B are unital C*-algebras, then $A \otimes_c B$ agrees with $A \otimes_{\max} B$. As well, since $C^*_{\text{env}}(A) = A$ and $C^*_{\text{env}}(B) = B$, the minimal tensor product agree as well.

A remarkable result of A. Kavruk states the following:

Theorem (Kavruk)

There is an operator system S such that for all unital C-algebras A, the following are equivalent:*

1. $S \otimes_{\min} A = S \otimes_c A$.
2. *The C*-algebra A is nuclear.*

Remember that nuclearity of A means $A \otimes_{\min} B = A \otimes_c B$ holds for any C*-algebra B.

The existence of S means that if you check that $S \otimes_{\min} A = S \otimes_c A$ holds, then $B \otimes_{\min} A = B \otimes_c A$ holds for all C*-algebras B.

Because of this, operator systems with the above property are known as nuclearity detectors.

Here is a nuclearity detector.

$$S := \left\{ \left[\begin{array}{cc|cc|cc} a & b & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & c & a & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & d & a \end{array} \right] : a, b, c, d \in \mathbb{C} \right\} \subseteq M_2 \oplus M_2 \oplus M_2 \ .$$

If we conjugate S by the unitary $U = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]^{\oplus 3}$, then we get that

$$S \cong \{ \mathrm{diag}(a+b, a-b, a+c, a-c, a+d, a-d) : a, b, c, d \in \mathbb{C} \} \ .$$

In fact, $C^*_{\mathrm{env}}(S) = \mathbb{C}^6$!

Why not end with an open question?

Question

Does there exist a separable C-algebra that detects nuclearity?*

If we allow for non-separable, the following example of Pop exists:.

Theorem

The C-algebra*

$$C^*(\mathbb{F}_\infty) \otimes_{\min} B(H) \otimes_{\min} (B(H)/K(H))$$

is a nuclearity detector.