

II) Non- C^* -Substructures of C^* -alg's

A, B unital C^* -alg

$\phi: A \rightarrow B$ linear

Exercises are in purple

Q: What structure on A did we actually need for most of our results for positive/cp maps?

① A subspace of a C^* -alg

② $A^* \subseteq A$

③ $1 \in A$

} ensures lots of positive elements - why?

Def'n: A self-adjoint^②, unital^① subspace S of a C^* -algebra C is called an operator system if $1 \in S$ ^③

Ex's:

① C^* -alg's

② Let $z \in C(\mathbb{T})$ be the identity map $z(\lambda) = \lambda$.

$S := \text{span}\{1, z, \bar{z}\}$ is an operator system

NOT an algebra

Let $1_S \in S^{\text{op sys}} \subseteq C^{C^*-\text{alg}}$.

Thm (Krein '37)

If $\phi: S \rightarrow \mathbb{C}$ is positive, \exists a positive map $\tilde{\phi}: C \rightarrow C$ s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Q: What if this is a larger C^* -alg?

Ex:

$S = \text{span}\{1, z, \bar{z}\} \subseteq C(\pi)$ (ex 2 above)

Define $\phi: S \rightarrow M_2(\mathbb{C})$

$$a + bz + c\bar{z} \mapsto \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}$$

One can show that ϕ is positive.

Suppose $\exists \tilde{\phi}: C(\pi) \rightarrow M_2(\mathbb{C})$ positive

s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Then

$$\|\tilde{\phi}\| = \|\tilde{\phi}(1_{C(\pi)})\| = \|\phi(1_{C(\pi)})\| = 1$$

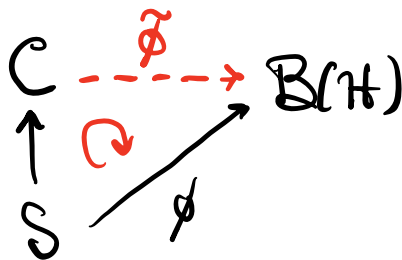
↑
by a previous lemma using positivity

$$\|\phi\| = \sup_{\|f\|=1} \|\phi(f)\| \geq \|\phi(z)\| = \left\| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\| = 2$$

Thm (Arveson's Extension Thm '69)

If $\phi: S \rightarrow B(\mathcal{H})$ is **cp**, \exists a **cp**
^{map} $\tilde{\phi}: C \rightarrow B$ s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Remark: This says $B(\mathcal{H})$ is injective in the category of operator systems (w/ cp maps)



Categories

	Objects	Morphisms	
	C^* -algs	$*$ -h'isms	
no $*$	operator algebras	cc algebra h'isms	} $ucc = ucp$ in these contexts
no mult	operator systems	cp linear maps	
no fun	operator spaces	cb linear maps	

Q: How are these related to C^* -algebras?

→ Switch focus to operator algebras...

Def'n A (unital) operator algebra A is a norm closed subalgebra of a C^* -alg \mathcal{C} (w/ $1_{\mathcal{C}} \in A$).

Ex's:

① $T_2 =$ upper 2×2 triangular matrices

$$= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

is a non-self-adjoint operator algebra
↑ not $*$ -closed

② $A(\mathbb{D}) = \{ f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic} \}$

$$= \overline{C[z]}^{\|\cdot\|} \subseteq C(\bar{\mathbb{D}})$$

is a non-self-adjoint operator algebra
called the "disc algebra"

Def'n A C^* -cover for a (unital) operator alg

$A \in \mathcal{C}^{C^*-alg}$ is a pair (D, j) s.t.

① $j: A \rightarrow D^{C^*-alg}$ is a
ucis (unital, completely isometric) alg h'ism

② $D = C^*(j(A))$

Ex's

① $(M_2(\mathbb{C}), \text{incl})$ is a C^* -cover for T_2

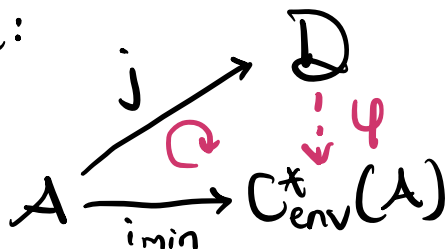
② $(C(\overline{D}), \text{incl})$ is a C^* -cover for $A(D)$

③ $(C(\mathbb{T}), (\cdot)|_{\mathbb{T}})$ is a C^* -cover for $A(D)$

Theorem (Arveson '69, Hamana '79, Dritschell-McCullough '05, Arveson '08, Davidson-Kennedy '13)

There exists a minimal C^* -cover $C_{env}^*(A) \equiv (C_{env}^*(A), i_{min})$ for A
called the C^* -envelope for A .

Minimality: for any C^* -cover (D, j) for A ,
 \exists $*$ -h'ism ψ that makes the following
diagram commute:



Ex's

$$\textcircled{1} C_{\text{env}}^*(T_2) = M_2(\mathbb{C})$$

$$\textcircled{2} C_{\text{env}}^*(A(\mathbb{D})) = C(\mathbb{T})$$

Topological Dynamics

Def'n: (X, τ) is a (topological) dynamical system

if \textcircled{i} X is a compact, Hausdorff space

\textcircled{ii} $\tau: X \rightarrow X$ is a homeomorphism

This induces an action of \mathbb{Z} on $C(X)$...

Thm (Arveson-Josephson '69, Peters '84, Hadwin-Hoover '88, Power '92, Davidson-Katsoulis '08)

(X_1, τ_1) and (X_2, τ_2) are conjugate \rightarrow $(\exists \text{ homeo } \psi: X_1 \rightarrow X_2 \text{ s.t.})$
 $\psi \circ \tau_1 = \tau_2 \circ \psi$

if and only if

$$\underbrace{C(X_1) \rtimes_{\tau_1} \mathbb{Z}^+}_{\text{non-self-adjoint operator alg's called}} \cong \underbrace{C(X_2) \rtimes_{\tau_2} \mathbb{Z}^+}_{\text{semicrossed products}}$$

non-self-adjoint
operator alg's called
semicrossed products

FACTS:

$$\textcircled{1} C_{\text{env}}^*(C(X_i) \rtimes_{\tau_i} \mathbb{Z}^+) \cong C(X_i) \rtimes_{\tau_i} \mathbb{Z} \quad \leftarrow C^* \text{-crossed product}$$

$\textcircled{2}$ This theorem does NOT hold for C^* -crossed products