II Non-C^*-Substructures of C^*-alg's

Exercises are in purple

A, B unital C^*-alg

$\phi : A \to B$ linear

Q: What structure on A did we actually need for most of our results for positive/cp maps?

① A subspace of a C^*-alg
② $A^* \subseteq A$
③ $1 \in A$

ensures lots of positive elements – why?

Def'n: A self-adjoint ②, unital ③ $1_e \in S$, subspace ① S of a C^*-algebra $\mathcal{C}$ is called an <u>operator system</u>

Ex's:

① C^*-alg's

② Let $z \in C(\mathbb{T})$ be the identity map $z(\lambda) = \lambda$.

$S := \operatorname{span}\{1, z, \bar{z}\}$ is an operator system

<u>NOT an algebra</u>

Let $1_e \in S^{\text{op sys}} \subseteq \mathcal{C}^{C^*\text{-alg}}$.

Thm (Krein '37)

If $\phi : S \to \mathbb{C}$ is positive, $\exists$ a positive map $\tilde{\phi} : \mathcal{C} \to \mathbb{C}$ s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Q: What if this is a larger C^*-alg?

Ex:

$S = \operatorname{span}\{1, z, \bar{z}\} \subseteq C(\mathbb{T})$ (ex 2 above)

Define $\phi : S \to M_2(\mathbb{C})$

$$a + bz + c\bar{z} \longmapsto \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}$$

One can show that ϕ is positive.

Suppose $\exists \tilde{\phi} : C(\mathbb{T}) \to M_2(\mathbb{C})$ positive s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Then

by a previous lemma using positivity

$$\|\tilde{\phi}\| \overset{\text{by a previous lemma using positivity}}{=} \|\tilde{\phi}(1_{C(\mathbb{T})})\| = \|\phi(1_{C(\mathbb{T})})\| = 1$$

$$\shortparallel$$

$$\|\phi\| = \sup_{\|f\|=1} \|\phi(f)\| \geq \|\phi(z)\| = \left\| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\| = 2 \quad \lightning$$

Thm (Arveson's Extension Thm '69)

If $\phi : S \to B(\mathcal{H})$ is cp, $\exists$ a cp map $\tilde{\phi} : C \to \mathbb{C}$ s.t. $\tilde{\phi}|_S = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Remark: This says $B(\mathcal{H})$ is injective in the category of operator systems (w/ cp maps)

$$\begin{tikzcd}
C \arrow[r, dashed, "\tilde{\phi}"] & B(\mathcal{H}) \\
S \arrow[u] \arrow[ur, "\phi"'] \arrow[ur, phantom, "\circlearrowright", pos=0.3, shift left=3] &
\end{tikzcd}$$

Categories

	Objects	Morphisms	
	C^*-alg's	$*$-h'isms	
no $*$ —	operator algebras	cc algebra h'isms	ucc = ucp in these contexts
no mult —	operator systems	cp linear maps	
no fun —	operator spaces	cb linear maps	

Q: How are these related to C*-algebras?

$\rightsquigarrow$ Switch focus to operator algebras...

<u>Def'n</u> A (unital) <u>operator algebra</u> $\mathcal{A}$ is a norm closed subalgebra of a C*-alg $\mathcal{C}$ (w/ $1_{\mathcal{C}} \in \mathcal{A}$).

<u>Ex's</u>:

① T_2 = upper 2×2 triangular matrices

$$= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

is a <u>non-self-adjoint</u> operator algebra

not $*$-closed

② $A(\mathbb{D}) = \{ f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic} \}$

$$= \overline{\mathbb{C}[z]}^{\|\cdot\|} \subseteq C(\overline{\mathbb{D}})$$

is a <u>non-self-adjoint</u> operator algebra called the "disc algebra"

Def'n A *C*-cover* for a (unital) operator alg $A \subseteq \mathbb{C}^{C^*\text{-alg}}$ is a pair (D, j) s.t.

① $j : A \to D^{C^*\text{-alg}}$ is a ucis (unital, completely isometric) alg h'ism

② $D = C^*(j(A))$

Ex's

① $(M_2(\mathbb{C}), \text{incl})$ is a C^*-cover for T_2

② $(C(\overline{\mathbb{D}}), \text{incl})$ is a C^*-cover for $A(\mathbb{D})$

③ $(C(\mathbb{T}), (\cdot)|_{\mathbb{T}})$ is a C^*-cover for $A(\mathbb{D})$

Theorem (Arveson '69, Hamana '79, Dritschel-McCullough '05, Arveson '08, Davidson-Kennedy '13)

There exists a minimal C^*-cover $C^*_{env}(A) \equiv (C^*_{env}(A), i_{min})$ for A called the *C^*-envelope for A*.

Minimality: for any C^*-cover (D, j) for A, $\exists$ $*$-h'ism φ that makes the following diagram commute:

$$\begin{tikzcd} & D \arrow[d, dashed, "\varphi"] \\ A \arrow[ur, "j"] \arrow[r, "i_{min}"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & C^*_{env}(A) \end{tikzcd}$$

Ex's

① $C^*_{env}(T_2) = M_2(\mathbb{C})$

② $C^*_{env}(A(\mathbb{D})) = C(\mathbb{T})$

Topological Dynamics

Def'n: (X, τ) is a <u>(topological) dynamical system</u>

if ⓘ X is a compact, Hausdorff space

ⓘⓘ $\tau : X \to X$ is a homeomorphism

This induces an action of $\mathbb{Z}$ on $C(X)$...

Thm (Arveson-Josephson '69, Peters '84, Hadwin-Hoover '88, Power '92, Davidson-Katsoulis '08)

(X_1, τ_1) and (X_2, τ_2) are <u>conjugate</u> $\left(\exists \text{ homeo } \varphi : X_1 \to X_2 \text{ s.t. } \varphi \circ \tau_1 = \tau_2 \circ \varphi\right)$

if and only if

$$\underbrace{C(X_1) \rtimes_{\tau_1} \mathbb{Z}^+}_{} \cong \underbrace{C(X_2) \rtimes_{\tau_2} \mathbb{Z}^+}_{}$$

non-self-adjoint operator alg's called <u>semicrossed products</u>

FACTS:

① $C^*_{env}(C(X_i) \rtimes_{\tau_i} \mathbb{Z}^+) \cong C(X_i) \rtimes_{\tau_i} \mathbb{Z}$ C*- crossed product

② This theorem does NOT hold for C*- crossed products