Ⓘ Completely Positive Maps on C*-alg's

Exercises are in purple

GNS: C*-alg's have an abundance of positive linear functionals, but we shouldn't expect lots of *-h'isms.

↳ $\phi : A \xrightarrow{C^*\text{-alg}} \mathbb{C}$

One question/motivation: What happens when $\mathbb{C}$ is a different C*-alg?

i.e., how do we extend our theory for positive linear functionals on C*-algs to positive linear maps between C*-alg's?

$\phi : A^{C^*} \to B^{C^*}$ linear, ~~positive~~ completely positive

Setup: Let A, B be unital C*-alg's. — much of the following extends to non-unital C*-alg's w/o much fuss. See Vern Paulsen's Textbook

Recall: For $n \in \mathbb{N}$,

$M_n(A)$ = n×n matrices w/ entries in A

is called the "n^{th} amplification/inflation/ampliation" of A

- $M_n(A)$ is a $\mathbb{C}$-alg w/ the usual op's
- Elizabeth showed $M_n(A)$ is a C*-algebra via GNS (using uniqueness of C*-norm)

Def'n: Let $\phi: A \to B$ be linear.

Define $\phi_n : M_n(A) \to M_n(B)$ by

$$\phi_n((a_{ij})) = (\phi(a_{ij}))$$ — apply ϕ entry wise

ϕ_n is called the "n^{th} amplification" of ϕ

Notation: ϕ_n "=" $\phi_{(n)}$ "=" $\phi^{(n)}$ — different authors use different notation ... use context

Remark: C^*-algs carry (as Paulsen says) "baggage" of norms & order on its amplifications

$$\boxed{M_n(A_+) \neq M_n(A)_+}$$

Def'n: A linear map $\phi: A \to B$ is

1. *positive* if $\phi(a) \geq 0 \quad \forall a \in A_+$ = positive cone of positive elts in A
2. *n-positive* if $\phi_n : M_n(A) \to M_n(B)$ is positive
3. *completely positive* (cp) if ϕ is n-positive $\forall n \in \mathbb{N}$

Remark: completely contractive (cc) / bounded (cb) are defined similarly

- cc — all amplifications of ϕ are contractive
- cb — $\|\phi\|_{cb} := \sup_n \|\phi_n\| < \infty$

Ex's

① $\boxed{*\text{-homomorphisms are cp}}$ — exercise

② if $\phi : A \to \mathbb{C}$ is positive, linear, then ϕ is cp.

proof: let $(a_{ij}) = (b_{ij})^*(b_{ij}) \geq 0$ in $M_n(A)$, $h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathbb{C}^n$

$$\langle \phi_n(a_{ij})h, h\rangle = h^*(\phi(a_{ij}))h$$

$$= \sum_{i,j=1}^{n} \overline{h_i} h_j \phi(a_{ij})$$

$$= \phi\Big(\sum_{ij} \overline{h_i} h_j a_{ij}\Big)$$

$$= \phi\big((1_A \otimes h)^*(a_{ij})(1_A \otimes h)\big)$$

$1_A \otimes h = \begin{bmatrix} h_1 1_A \\ \vdots \\ h_n 1_A \end{bmatrix} \in A^n$ here

each in $M_n(A)$

(1,1)-entry of matrix

But $\begin{bmatrix} 1_A \otimes h & 0 & \cdots & 0 \end{bmatrix}^* (a_{ij}) \begin{bmatrix} 1_A \otimes h & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} (1_A \otimes h)^*(a_{ij})(1_A \otimes h) & 0 \cdots 0 \\ 0 & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix}$

$\parallel$

$$\Big((b_{ij})\begin{bmatrix} 1_A \otimes h & 0 & \cdots & 0 \end{bmatrix}\Big)^*\Big((b_{ij})\begin{bmatrix} 1_A \otimes h & 0 & \cdots & 0 \end{bmatrix}\Big) \geq 0$$

ex — $\boxed{\text{if } \operatorname{diag}(a_1, \dots, a_n) \geq 0 \text{ in } M_n(A),\ a_i \geq 0\ \forall i}$

∎

(3) Fix $x \in A$.

$\phi : A \to A$ by

$\phi(a) = x^* a x$ — conj by fixed elt

is cp

pf: Let $(b_{ij})^*(b_{ij}) = (a_{ij}) \geq 0$ in $M_n(A)$

$$\phi_n((a_{ij})) = (x^* a_{ij} x)$$

$$= \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}^* (a_{ij}) \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}$$

$$= \left((b_{ij}) \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \right)^* \left((b_{ij}) \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \right) \geq 0$$

(4) Let $\pi : A \to B$ be a $*$-h'ism.
Fix $b \in B$.

$\phi : A \to B$ by $\phi(a) = b^* \pi(a) b$

is cp b/c compositions of cp are cp

$\boxed{\text{GOAL}}$ every cp map has this form — $\circledast$ ish

⑤ A non-example

$\phi : M_2 \to M_2$ by $\phi(a) = a^T$

- ϕ is positive
- Consider

$$x = \left[\begin{array}{cc|cc} 1 & & & 1 \\ & & & \\ \hline & & & \\ 1 & & & 1 \end{array}\right] \in M_2(M_2) \cong M_4 \quad \text{but} \quad \phi_2(x) = \left[\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right] \text{ has}$$

$\sigma(x) = \{0, 2\}$ so $x \geq 0$

spectrum $\{-1, 1\}$

i.e., $\phi_2(x)$ not positive

So ϕ_2 is not 2-positive

$\Rightarrow \phi = (\cdot)^T$ is <u>NOT</u> cp

<u>Lemma</u>: If A, B are unital C^*-alg's and $\phi : A \to B$ is positive, then

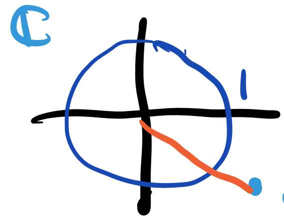
$$\|\phi\| = \|\phi(1_A)\|$$

<u>Remark</u>: Still true if A, B not necessarily unital but A <u>or</u> B is commutative

<u>Proof of ($B = \mathbb{C}$ case)</u>

$$\|\phi\| = \sup_{\|a\|=1} |\phi(a)| \geq \underbrace{\phi(1_A) \geq 0}_{\text{since } \phi \text{ is positive}}$$

WTS: $\|\phi\| \leq \phi(1)$



$\phi(a) = |\phi(a)|e^{it}$ some t

Let $a \in A$ s.t. $\|a\| = 1$.
Choose θ s.t. for $\lambda := e^{i\theta}$

$$|\phi(a)| = \lambda\phi(a) = \phi(\lambda a)$$

Then

$$0 \leq |\phi(a)| = \phi(\lambda a)$$
$$= \underbrace{\phi(Re(\lambda a))}_{\in \mathbb{R}} + i\,\underbrace{\phi(Im(\lambda a))}_{\in \mathbb{R}}$$

since ϕ $*$-preserving

$$\Rightarrow \phi(Im(\lambda a)) = 0$$

Since $Re(\lambda a) \leq \|Re(\lambda a)\| 1_A$, (since $Re(\lambda a) \in A_{sa}$)

we get $0 \leq |\phi(a)| = \phi(Re(\lambda a))$

$$\leq \|Re(\lambda a)\|\,\phi(1_A) \quad \text{since } \phi \text{ positive}$$
$$\leq \|\lambda a\|\,\phi(1_A)$$
$$= \phi(1_A)$$

$$\boxed{\|Re(x)\| = \left\|\frac{x+x^*}{2}\right\| \leq \|x\|}$$

$\blacksquare$

MAIN EVENT

Theorem (Stinespring Dilation Thm '55)

Let A be (unital) & $\phi : A \to B(\mathcal{H})$ cp.

Then $\exists$ ① a Hil space K

② a (unital) $*$-h'ism $\pi : A \to B(K)$

③ $V \in B(\mathcal{H}, K)$ w/ $\|V\|^2 = \|\phi\| \underset{\text{by lemma}}{=} \|\phi(1_A)\|$

s.t.

$$\phi(a) = V^* \pi(a) V$$

Remarks:

① Stinespring's Dilation Thm says cp maps are "almost $*$-h'isms" in the following sense:

If $\phi(1_A) = 1_{B(\mathcal{H})}$, then V is an isometry

so $\mathcal{H} \cong V\mathcal{H} \subseteq K$ as Hilbert spaces.

Identifying $\mathcal{H}$ w/ $V\mathcal{H}$, we write

$$\phi(a) = V^*\pi(a)V = \underbrace{P_{\mathcal{H}}}_{V^*} \pi(a)\big|_{\mathcal{H}}$$

Think

$$\pi(a) = \begin{array}{cc} \left[\begin{array}{c|c} \phi(a) & * \\ \hline * & * \end{array}\right] & \begin{array}{c} V\mathcal{H} \\ \oplus \\ (V\mathcal{H})^{\perp} \end{array} \end{array}$$

② • The triple (π, V, K) is called a **Stinespring dilation** of ϕ

• (π, V, K) is **minimal** if $K = \overline{\operatorname{span}}(\pi(A)V\mathcal{H})$

(this dilation is unique up to unitary equiv.)

proof of Stinespring

Motivation: Mimic the GNS construction

Steps:

① Build a positive, sesquilinear form $[\cdot,\cdot]: A \odot \mathcal{H} \to \mathbb{C}$

② Quotient to get an inner product, then complete to get $\mathcal{K}$

③ Rep A as multipliers on $A \odot \mathcal{H}$ that factors through quotient and yields a representation of A on $\mathcal{K}$.

④ Build V by embedding $\mathcal{H}$ into $\mathcal{K}$.

⑤ Verify (i) $\phi(a) = V^* \pi(a) V$ and (ii) $\|V\|^2 = \|\phi\|$

Proof:

① Define $[\cdot,\cdot]: A \odot \mathcal{H} \to \mathbb{C}$

by $[a \odot h, b \odot h'] = \langle \phi(b^* a) h, h' \rangle_{\mathcal{H}}$

- $[\cdot,\cdot]$ is positive since ϕ is cp

<u>Sketch</u>: $\left[\sum_c a_c \odot h_c, \sum_r a_r \odot h_r\right] = \sum_{r,c} \langle \phi(a_r^* a_c) h_c, h_r \rangle_{\mathcal{H}}$

$$= \left\langle \phi_n\left((a_r^* a_c)_{r,c=1}^n \right) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\rangle_{\mathcal{H}^n}$$

$$(a_r^* a_c) = \begin{bmatrix} a_1 & \cdots & a_n \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} a_1 & \cdots & a_n \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \geq 0 \text{ in } M_n(A)$$

(2) $\mathcal{N} := \{ x \in A \odot \mathcal{H} : [x,x] = 0 \}$ is a subspace of $A \odot \mathcal{H}$

Build quotient space $A \odot \mathcal{H} / \mathcal{N}$ and define

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle_{\frac{A \odot \mathcal{H}}{\mathcal{N}}} := [x, y]_{A \odot \mathcal{H}}$$

$\langle \cdot, \cdot \rangle$ is a positive <u>definite</u> sesquilinear form

$K :=$ completion of $A \odot \mathcal{H}/\mathcal{N}$ w.r.t. $\langle \cdot, \cdot \rangle$ (note K is separable if $\mathcal{H}$ is)

(3) Fix $a \in A$.

Define $\pi_a : A \odot \mathcal{H} \to A \odot \mathcal{H}$ by

$$\pi_a := L_a \odot \mathrm{id}_{\mathcal{H}}$$

i.e., $\pi_a(b \odot h) = ab \odot h$

- $\pi_a(\mathcal{N}) \subseteq \mathcal{N}$

Sketch Suppose $\sum_{i=1}^{n} a_i \odot h_i \in$

$$[\pi_a(\sum a_c \odot h_c), \pi_a(\sum a_r \odot h_r)]$$
$$= \sum_{r,c} \langle \phi((a a_r)^* a a_c) h_c, h_r \rangle_{\mathcal{H}}$$
$$= \left\langle \phi_n([a_r^* a^* a a_c]_{r,c=1}^{n}) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\rangle_{\mathcal{H}^n}$$

Exercise: show $[a_r^* a^* a a_c]_{r,c=1}^{n} \leq \|a^* a\| [a_r^* a_c]_{r,c=1}^{n}$ in $M_n(A)$

Hint: Factor $[a_r^* a^* a a_c]$ and use $a^* a \leq \|a\| 1_A$ to show $[a^* a]_{r,c=1}^{n} \leq \|a^* a\| [1_A]_{r,c=1}^{n}$ in $M_n(A)$

So

by exercise above & ϕ being cp

$$[\pi_a(\sum a_c \odot h_c), \pi_a(\sum a_r \odot h_r)] \leq \left\langle \phi_n(\|a^* a\| [a_r^* a_c]_{r,c=1}^{n}) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\rangle$$
$$= \|a\|^2 [\sum_c a_c \odot h_c, \sum_r a_r \odot h_r] \quad \text{—} \; (*)$$

This implies $\pi_a(\mathcal{N}) \subseteq \mathcal{N}$ so

π_a factors through the quotient to yield

$$\pi_a : A \odot \mathcal{H} / \mathcal{N} \longrightarrow A \odot \mathcal{H} / \mathcal{N} .$$

$(*)$ implies π_a is continuous on a dense subset of K so π_a extends to a map in $B(K)$.

Exercise: $\pi : A \to B(\mathcal{H})$ is a $*$-h'ism
$a \longmapsto \pi_a$

(4) Define $V : \mathcal{H} \to \mathcal{K}$ by
$$h \mapsto (1_A \otimes h) + \mathcal{N}$$

(use an approximate unital in non-unital case)

<u>Note</u>: $\|Vh\|_{\mathcal{K}}^2 = \|(1 \otimes h) + \mathcal{N}\|^2$

$= \langle \phi(1_A)h, h \rangle_{\mathcal{H}}$ $= \|h\|^2$ if ϕ is unital

$\Rightarrow V$ is an isometry

$\leq \|\phi(1)\| \cdot \|h\|^2$

$\Rightarrow V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\|V\|^2 \leq \|\phi(1)\| = \|\phi\|$

(5)

(i) <u>Exercise</u>: Check $V^* \pi(a) V = \phi(a)$ using $\mathcal{H}$

(ii) <u>Exercise</u>: Finish checking $\|V\|^2 = \|\phi(1_A)\| = \|\phi\|$

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