Model theory and von Neumann algebras

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Introduction

- Hi, my name is Isaac and I'm a model theorist.
- What's a model theorist you ask?
- And why am I speaking in a summer program on operator algebras?
- It turns out we have a common love interest: **ultraproducts**!

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Ultrafilters and ultralimits

Definition

If I is a set, a **ultrafilter on** I is a finitely additive, $\{0,1\}$-valued probability measure on I. The ultrafilter is called **nonprincipal** if finite sets get measure 0.

We often identify an ultrafilter with its set of measure 1 sets and when doing so use letters like $\mathcal{U}$ and $\mathcal{V}$ to denote ultrafilters, writing $A \in \mathcal{U}$ to mean that the $\mathcal{U}$-measure of A is 1.

Theorem/Definition

Given any compact Hausdorff space X, any set I, any sequence $(a_i)_{i\in I}$ from X, and any ultrafilter $\mathcal{U}$ on I, there is a unique element $a \in X$ with the property: for every open neighborhood U of a, we have $\{i \in I \ : \ a_i \in U\} \in \mathcal{U}$. We call a the $\mathcal{U}$**-ultralimit of** $(a_i)_{i\in I}$ and denote it by $\lim_{\mathcal{U}} a_i$.

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Tracial ultraproduct

- Suppose that $\mathcal{M} = (M_i)_{i\in I}$ is a family of tracial von Neumann algebras and $\mathcal{U}$ is an ultrafilter on I.
- We set $\ell^\infty(I,\mathcal{M}) := \{(a_i) \in \prod_{i\in I} M_i \ : \sup_{i\in I} \|a_i\| < \infty\}$.
- We also set $c_{\mathcal{U}}(\mathcal{M}) := \{(a_i) \in \ell^\infty(I,\mathcal{M}) \ : \ \lim_{\mathcal{U}} \|a_i\|_2 = 0\}$.
- The quotient C*-algebra $\ell^\infty(I,\mathcal{M})/c_{\mathcal{U}}(\mathcal{M})$ is a von Neumann algebra again, called the **tracial ultraproduct** of the family $\mathcal{M}$ with respect to the ultrafilter $\mathcal{U}$, denoted $\prod_{\mathcal{U}} M_i$.
- We denote the coset of (a_i) by $(a_i)_{\mathcal{U}}$.
- $\prod_{\mathcal{U}} M_i$ has a natural trace: $\tau((a_i)_{\mathcal{U}}) := \lim_{\mathcal{U}} \tau_{M_i}(a_i)$.
- If each $M_i = M$, we write $M^{\mathcal{U}}$, and call this the **ultrapower of** M **with respect to the ultrafilter** $\mathcal{U}$.
- There is a natural **diagonal embedding** $M \hookrightarrow M^{\mathcal{U}}$ given by $a \mapsto (a, a, a, \ldots)_{\mathcal{U}}$.
- $M^{\mathcal{U}}$ is nonseparable as soon as $\mathcal{U}$ is sufficiently incomplete and M is infinite-dimensional.

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Property Gamma and McDuff

- The tracial ultraproduct construction is a useful way to succintly express some important properties that a tracial von Neumann algebra may or may not have.
- We set $M' \cap M^{\mathcal{U}} := \{(a_i)_{\mathcal{U}} \in M^{\mathcal{U}} \ : \ [b, (a_i)_{\mathcal{U}}] = 0 \text{ for all } b \in M\}$.

Definition

A II_1 factor M has:

- **property Gamma** if $M' \cap M^{\mathcal{U}} \neq \mathbb{C}$;
- the **McDuff property** if $M' \cap M^{\mathcal{U}}$ is not abelian.

- Does not depend on the choice of nonprincipal ultrapower.
- $\mathcal{R}$ is McDuff.
- $L(\mathbb{F}_2)$ does not have property Gamma.
- Dixmier and Lance constructed algebras with property Gamma but that are not McDuff.

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Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\| \leq n} \varphi$ and $\sup_{\|x\| \leq n} \varphi$ are formulae. (Operator norm balls)
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\| \leq 1} \max(\|x - x^*\|_2, \|x - x^2\|_2, |\mathrm{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

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- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
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Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
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Elementary equivalence and elementary embeddings

Definition

II_1 factors M and N are **elementarily equivalent**, denoted $M \equiv N$, if $\sigma^M = \sigma^N$ for every sentence σ.

Definition

An embedding $i : M \hookrightarrow N$ is **elementary** if $\varphi(a)^M = \varphi(i(a))^N$ for all formulae $\varphi(x)$ and all $a \in M$. If M is a subalgebra of N and the inclusion is an elementary embedding, we say that M is an **elementary substructure** of N, denoted $M \preceq N$.

Theorem (Downward Löwenheim-Skolem)

Given any II_1 factor N and separable $X \subseteq N$, there is a separable $M \preceq N$ with $X \subseteq M$.

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Theorem (Łos' theorem or the Fundamental Theorem of Ultraproducts)

Fix a family $(M_i)_{i\in I}$ of tracial von Neumann algebras, an ultrafilter $\mathcal{U}$ on I, a formula $\varphi(x)$, and $(a_i)_{\mathcal{U}} \in \prod_{\mathcal{U}} M_i$. Then

$$\varphi((a_i)_{\mathcal{U}})^{\prod_{\mathcal{U}} M_i} = \lim_{\mathcal{U}} \varphi(a_i)^{M_i}.$$

The ultraproduct is democratic!

Corollary

The diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$ is an elementary embedding. In particular, if $M^{\mathcal{U}} \cong N^{\mathcal{V}}$, then $M \equiv N$.

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We just saw: if $M^{\mathcal{U}} \cong N^{\mathcal{V}}$, then $M \equiv N$. Amazingly, the converse holds, giving a "logic-free" characterization of elementary equivalence:

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How many ultrapowers of a given factor? (CH)

Question

If M is a separable II_1 factor, do there exist nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ (on $\mathbb{N}$) such that $M^{\mathcal{U}} \not\cong M^{\mathcal{V}}$?

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- The model-theoretic explanation: $M^{\mathcal{U}}$ has density character $2^{\aleph_0}$ and is $\aleph_1$-**saturated.**
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- II_1 factors have a model-theoretically nasty property called the **order property**, which roughly means that one can encode something resembling an order in a II_1 factor.
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Suppose that M and N are separable and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ such that $M^{\mathcal{U}} \cong N^{\mathcal{U}}$. Must it be the case that $M \cong N$?

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For any separable II_1 factor M, there are continuum many separable II_1 factors N such that $M \equiv N$.

- Nicoara, Popa, and Sasyk constructed a family (M_α) of separable II_1 factors indexed by 2^ω, each of which embeds into $\mathcal{R}^{\mathcal{U}}$, such that only countably many can embed into any given separable II_1 factor.
- Given M, consider $M_\alpha \hookrightarrow \mathcal{R}^{\mathcal{U}} \hookrightarrow M^{\mathcal{U}}$.
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Progress on the number of nonisomorphic separable II_1 factors was slow. A major breakthrough was the following:

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There is a family of countable groups Γ_α indexed by 2^ω such that, setting $M_\alpha := L(\Gamma_\alpha)$, one has $M_\alpha \not\cong M_\beta$ for all $\alpha < \beta < 2^\omega$.

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There is a family of countable groups Γ_α indexed by 2^ω such that, setting $M_\alpha := L(\Gamma_\alpha)$, one has $M_\alpha \not\cong M_\beta$ for all $\alpha < \beta < 2^\omega$.

Progress on the number of nonisomorphic *ultrapowers* of separable II_1 factors was also slow, until:

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CEP

- In his seminal paper on the classification of injective factors from 1976, Connes proved that $L(\mathbb{F}_2)$ embeds into $\mathcal{R}^{\mathcal{U}}$ (we simply say **embeddable**).
- He then "suggested" that all II_1 factors are embeddable; this is known as the **Connes embedding problem (CEP)**.
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Universal theories

- A sentence σ is **universal** if it is of the form $\sup_{\vec{x}} \varphi(\vec{x})$, where φ has no quantifiers in it.
- If $M \hookrightarrow N$ and σ is universal, then $\sigma^M \leq \sigma^N$.

Proposition

If $\sigma^M \leq \sigma^N$ for all universal sentences σ, then $M \hookrightarrow N^{\mathcal{U}}$ for some $\mathcal{U}$.

Corollary

CEP is equivalent to the statement that $\sigma^M = \sigma^{\mathcal{R}}$ for all II_1 factors M.

- So the failure of CEP tells us that there are at least two distinct *universal theories* of II_1 factors.
- We believe there should be $2^{\aleph_0}$ many such distinct universal theories.

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CEP and computability of the universal theory of $\mathcal{R}$

- Suppose that σ is a universal sentence.
- By plugging in elements from $M_n(\mathbb{C})$ for larger n, we can start "effectively" enumerating lower bounds for $\sigma^{\mathcal{R}}$.
- But what effectively enumerating upper bounds for $\sigma^{\mathcal{R}}$?
- There is a **proof system** for continuous logic and the **Completeness Theorem** tells us that if something is a (first-order) **theorem** about II_1 factors, then there will be a **formal proof** of it from the axioms of II_1 factors.
- If $\sigma^{\mathcal{R}} \leq r$, then by CEP, "$\sigma \leq r$" is a theorem about II_1 factors, so by running our "proof machine" we will eventually know this fact. This effectively enumerates upper bounds for $\sigma^{\mathcal{R}}$. (Soundness makes sure we make no mistakes.)
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Locally universal factors

Definition

A II_1 factor M is called **locally universal** if every II_1 factor embeds into an ultrapower of M.

So CEP asks if $\mathcal{R}$ is locally universal.

Theorem (Farah-Hart-Sherman)

A separable locally universal II_1 factor exists.

Clearly any factor extending a locally universal factor is also locally universal. Using a technique known as **model-theoretic forcing**, one can construct locally universal factors with a wide variety of extra properties.

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Popa's Factorial Commutant Embedding Problem

In connection with the CEP, Popa asked the following question:

Popa's FCEP

Suppose that M is embeddable. Must there exist an embedding $i : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$ such that $i(M)' \cap \mathcal{R}^{\mathcal{U}}$ is a factor?

- $\mathcal{R}$ satisfies the FCEP. (Dixmier and Lance)
- $L(\mathrm{SL}_3(\mathbb{Z}))$ satisfies the FCEP. (Popa)
- Brown showed that embeddings i as above are the *extreme points* in a convex-like space of embeddings from M to $\mathcal{R}^{\mathcal{U}}$.

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Recent progress on the FCEP

Theorem (Atkinson-G.-Your friend Sri, 2020)

If $M \equiv \mathcal{R}$, then M satisfies the FCEP.

The proof uses the model-theoretic notion of **heir** along with the above work of Nate Brown.

Theorem (G.)

There is a locally universal II_1 factor M such that, for all property (T) factors N, there is an embedding $i : N \hookrightarrow M^{\mathcal{U}}$ such that $i(N)' \cap M^{\mathcal{U}}$ is a factor.

- The proof uses the model-theoretic notion of **infinitely generic factor** as well as a **model-theoretic bicommutant theorem.**
- One can identify two precise hurdles from adapting this argument to establishing the original FCEP for property (T) factors.

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The Jung property

Theorem (Jung)

If M is a separable embeddable factor, then any two embeddings $\pi, \rho : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$ are ***unitarily conjugate*** *if and only if $M \cong \mathcal{R}$.*

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Definable sets

Exercise

Suppose that $p \in \prod_{\mathcal{U}} M_i$ is a projection. Then there are projections $p_i \in M_i$ such that $p = (p_i)_{\mathcal{U}}$. Ditto for unitaries.

For a formula $\varphi(\vec{x})$ and M, set $Z(\varphi^M) := \{\vec{a} \in M \ : \ \varphi(\vec{a})^M = 0\}$.

Theorem/Definition

Fix a formula $\varphi(\vec{x})$. The following are equivalent:

1. $Z(\varphi^{\prod_{\mathcal{U}} M_i}) = \prod_{\mathcal{U}} Z(\varphi^{M_i})$.
2. For any formula $\psi(\vec{x}, \vec{y})$, $\sup_{\vec{x} \in Z(\varphi)} \psi(\vec{x}, \vec{y})$ and $\inf_{\vec{x} \in Z(\varphi)} \psi(\vec{x}, \vec{y})$ "are" formulae again.

In this case, we call $Z(\varphi)$ a **definable set**.

So projections and unitaries form definable sets.

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Property (T) and definability

Theorem (G., Hart, and Sinclair)

Given a II_1 factor N, we can treat N-N bimodules (see Corey's talk) as structures in an appropriate language just like we have been doing for tracial von Neumann algebras.

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Existentially closed tracial von Neumann algebras

Theorem/Definition

Given an inclusion $M \subseteq N$ of tracial von Neumann algebras, the following are equivalent:

- for any quantifier-free formula $\varphi(\vec{x}, \vec{y})$ and any $\vec{a} \in M$, we have:

$$(\inf_{\vec{x}} \varphi(\vec{x}, \vec{a}))^M = (\inf_{\vec{x}} \varphi(\vec{x}, \vec{a}))^N.$$

- There is an embedding $N \hookrightarrow M^{\mathcal{U}}$ such that the restriction $M \hookrightarrow M^{\mathcal{U}}$ is the diagonal embedding.

In this case, we say that M is **existentially closed** (e.c.) in N. M is existentially closed if it is e.c. in all extensions. Can also relativize to the embeddable case.

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Some facts about e.c. tracial von Neumann algebras

- Every tracial von Neumann algebra embeds into an e.c. tracial von Neumann algebra (of the same density character).
- E.c. tracial von Neumann algebras are locally universal McDuff II_1 factors.
- (G., Hart, Sinclair) You cannot axiomatize the e.c. factors.
- (G.) Suppose that N has property (T) and $N \subseteq M$ with M e.c. Then $(N' \cap M)' \cap M = N$. (Model-theoretic bicommutant theorem.)
- CEP is equivalent to the statement that $\mathcal{R}$ is e.c.

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Thanks for your attention!

Suggestions for future reading

- S. Atkinson, I. Goldbring, and S. Kunnawalkam Elayavalli, *Factorial commutants and II_1 factors with the generalized Jung property*.
- I. Farah, I. Goldbring, B. Hart, and D. Sherman, *Existentially closed II_1 factors*.
- I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I, II, and III*.
- I. Goldbring, *Spectral gap and definability*.
- I. Goldbring, *Enforceable operator algebras*.
- I. Goldbring and B. Hart, *The universal theory of the hyperfinite II_1 factor is not computable*.
- I. Goldbring, B. Hart, and T. Sinclair, *The theory of tracial von Neumann algebras does not have a model companion*.
- I. Goldbring, B. Hart, and H. Towsner, *Explicit sentences distinguishing McDuff's II_1 factors*.