GOALS Expository Talk: The Group Measure Space Construction

The Group Measure Space Construction

Let (X, μ) be a probability space. Let Γ be a countable discrete group. Suppose there exists a homomorphism

$$\alpha : \Gamma \to \mathrm{Aut}(L^\infty(X, \mu))$$

$\mathrm{Aut}(L^\infty(X, \mu))$ is the group of normal *-isomorphisms of the von Neumann algebra $L^\infty(X, \mu)$. We usually denote $\alpha(g)$ by α_g.

In this case, we call α an **action** of Γ on $L^\infty(X, \mu)$ and we write $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$.

Example 1. $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ acting on $(\{0, 1\}, \text{counting measure}/2)$ by $(\alpha_0(f))(x) = f(x)$ and $(\alpha_1(f))(x) = f(x - 1)$.

Example 2 (Irrational Rotation)**.** $\mathbb{Z}$ acts on $(\mathbb{T}, \text{arc length measure}/2\pi)$ by $(\alpha_n(f))(x) = f(e^{-in}x)$.

We say the action α is:

- **probability measure preserving** (p.m.p.) if $\int_X \alpha_g(f)d\mu = \int_X f d\mu$ for all $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.
- **free** if given $f \in L^\infty(X, \mu)$ and $g \in \Gamma \setminus \{e\}$, we have $f = 0$ whenever $f\alpha_g(h) = fh$ for all $h \in L^\infty(X, \mu)$.
- **ergodic** if there exists $f \in L^\infty(X, \mu)$ is such that $\alpha_g(f) = f$ for all $g \in \Gamma$, then $f \in \mathbb{C}1$.

For $f \in L^\infty(X, \mu)$, define a linear operator $\pi_\alpha(f)$ on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ by

$$\pi_\alpha(f)\left(\sum_{g\in\Gamma} \delta_g \otimes f_g\right) = \sum_{g\in\Gamma} \delta_g \otimes (\alpha_g^{-1}(f)f_g)$$

One can show $\pi_\alpha(f)$ is bounded and

$$\pi_\alpha\colon L^\infty(X, \mu) \to B(\ell^2\Gamma) \otimes L^2(X, \mu)$$

is a normal unital injective *-homomorphism. (Problem session question 1a,1b)

For $g \in \Gamma$, define $\lambda(g)$ on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ by

$$\lambda(g)\left(\sum_{h\in\Gamma} \delta_h \otimes f_h\right) = \sum_{h\in\Gamma} \delta_{gh} \otimes f_h$$

One can show $\lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha_g(f))$ (Problem session question 1c). This implies that the *-algebra generated by $\pi_\alpha(L^\infty(X, \mu))$ and $\lambda(\Gamma)$ is the set

$$\mathbb{C}\left\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma)\right\rangle := \left\{\sum_{j=1}^{d} \pi_\alpha(f_j)\lambda(g_j)\colon d \in \mathbb{N},\ f_1, \cdots, f_d \in L^\infty(X, \mu),\ g_1, \cdots, g_d \in \Gamma\right\}$$

Since this *-algebra is unital, we can define the von Neumann algebra:

Definition 3. Given an action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$,

$$L^\infty(X, \mu) \rtimes_\alpha \Gamma := \mathbb{C}\left\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma)\right\rangle''$$

is called the **crossed product** of $L^\infty(X, \mu)$ by Γ.

We will show that if the action α is free and ergodic, then $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is a factor. Our strategy will be to utilize the commutant.

Definition 4. Define $\phi_\alpha(f) \in B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ by:

$$\phi_\alpha(f)\left(\sum_{g\in\Gamma} \delta_g \otimes f_g\right) = \sum_{g\in\Gamma} \delta_g \otimes f_g f$$

One can show $\phi_\alpha(f) \subset (L^\infty(X, \mu) \rtimes_\alpha \Gamma)'$ (Problem session exercise 2 part 1).

Lemma 5. *Let P be the projection onto the subspace $\mathbb{C}\delta_e \otimes L^2(X,\mu) \cong L^2(X,\mu)$. For $x \in B(\ell^2(\Gamma)\otimes L^2(X,\mu))$ and $g \in \Gamma$, define an operator x_g on $L^2(X,\mu)$ by:*

$$x_g(h) = Px(\delta_{g^{-1}} \otimes h)$$

If $x \in L^\infty(X,\mu) \rtimes_\alpha \Gamma$, then $x_g \in L^\infty(X,\mu)$. The significance of this lemma is that the x_g we define is a multiplication operator for x in the cross product.

Proof. Suppose $x \in L^\infty(X,\mu) \rtimes_\alpha \Gamma$. Then, x commutes with $\phi_\alpha(f)$ for all $f \in L^\infty(X,\mu)$. Notice

$$\big(x\phi_\alpha(f)\big)_g(h) = Px\phi_\alpha(f)(\delta_{g^{-1}} \otimes h) = Px\delta_{g^{-1}} \otimes hf = Px(\delta_{g^{-1}} \otimes hf) = x_g(hf) = x_g(fh)$$

and noticing $P\phi_\alpha(f)(\cdot) = fP(\cdot)$ we have

$$(\phi_\alpha(f)x)_g(h) = P\phi_\alpha(f)x(\delta_{g^{-1}} \otimes h) = fPx\big(\delta_{g^{-1}} \otimes h\big) = fx_g(h)$$

We get $x_g(fh) = fx_g(h)$. (We've shown that x_g commutes with all multiplication operators: $M_f \circ x_g = x_g \circ M_f$) Therefore, $x_g \in B(L^2(X,\mu)) \cap L^\infty(X,\mu)' = L^\infty(X,\mu)$. □

Proposition 6. *Suppose the action α is free. Then $L^\infty(X,\mu) \rtimes_\alpha \Gamma \cap L^\infty(X,\mu)' \subset L^\infty(X,\mu)$.*

Proof. Let $x \in L^\infty(X,\mu) \rtimes_\alpha \Gamma \cap L^\infty(X,\mu)'$. Compute for $f, h \in L^\infty(X,\mu)$ and $g \in \Gamma \setminus \{e\}$:

$$(x\pi_\alpha(f))_g(h) = Px(\delta_{g^{-1}} \otimes \alpha_g(f)h) = x_g(\alpha_g(f)h)$$

also

$$(\pi_\alpha(f)x)_g(h) = P\pi_\alpha(f)x(\delta_{g^{-1}} \otimes h)) = fx_g(h)$$

Since x_g is multiplication by some $f_{xg} \in L^\infty(X,\mu)$, we get $f_{xg}\alpha_g(f)h = ff_{xg}h$. Rewriting:

$$(f_{xg}h)\alpha_g(f) = (f_{xg}h)f$$

By freeness of α, we get $f_{xg}h = 0$ and by arbitrariness of h, we have $f_{xg} = 0$ and hence $x_g = 0$. This means

$$\langle x^*(\delta_e \otimes 1), \delta_g \otimes 1\rangle = \langle \delta_e \otimes 1, x(\delta_g \otimes 1)\rangle = \langle P\delta_e \otimes 1, x(\delta_g \otimes 1)\rangle = \langle \delta_e \otimes 1, Px(\delta_g \otimes 1)\rangle = \langle \delta_e \otimes 1, x_g(1)\rangle = 0$$

for $g \neq e$. Therefore $x^*(\delta_e \otimes 1) = \delta_e \otimes f$ for some $f \in L^\infty(X,\mu)$. Notice also $\pi_\alpha(f)(\delta_e \otimes 1) = \delta_e \otimes f$. Since $\delta_e \otimes 1$ is separating by (Problem Session 3 part 2) (We're invoking the full power of the commutant with a separating vector), $x = (x^*)^* = \pi_\alpha(f)^* = \pi_\alpha(\overline{f})$.

□

We've obtained that, denoting by Z the center,

$$Z(L^\infty(X,\mu) \rtimes_\alpha \Gamma) = L^\infty(X,\mu) \rtimes_\alpha \Gamma \cap (L^\infty(X,\mu) \rtimes_\alpha \Gamma)' \subset L^\infty(X,\mu) \rtimes_\alpha \Gamma \cap L^\infty(X,\mu)' \subset L^\infty(X,\mu)$$

Theorem 7. *Assume α is free and ergodic. Then $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is a factor.*

Proof. Let $\pi_\alpha(f) \in Z(L^\infty(X,\mu) \rtimes_\alpha \Gamma)$. Then for $g \in \Gamma$ we have

$$\pi_\alpha(f) = \lambda(g)\lambda(g^{-1})\pi_\alpha(f) = \lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha_g(f))$$

Therefore $f = \alpha_g(f)$ for all $g \in \Gamma$. By ergodicity of α we obtain $f = \lambda 1$ for some $\lambda \in \mathbb{C}$. Hence the center $Z(L^\infty(X,\mu) \rtimes_\alpha \Gamma) = \mathbb{C}$ i.e $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is a factor. □