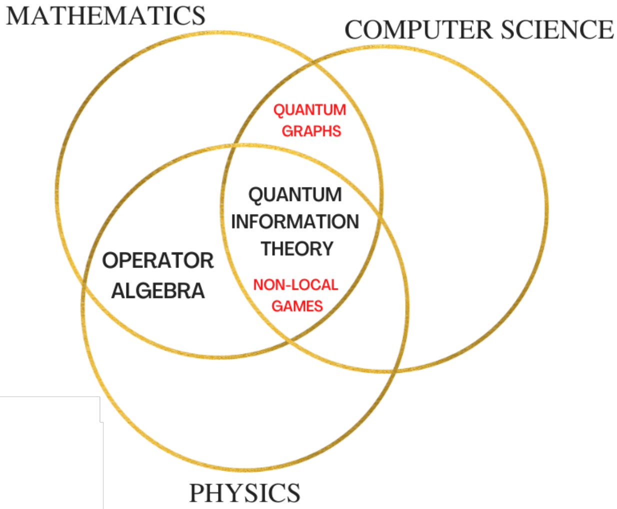
Quantum graphs and colorings

Priyanga Ganesan

University of California, San Diego

GOALS workshop 2022

Research area

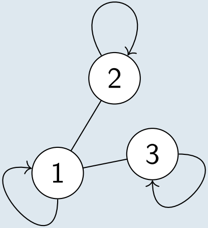


Quantum graphs

G = (Vertex set, Edge set, Adjacency matrix)

Classical reflexive graph

- $V = \{1, 2, 3\}$
- $E = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$
- $A_G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$



$$S_G := \left\{ \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \text{ where } * \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C})$$

Operator generalization of classical graphs

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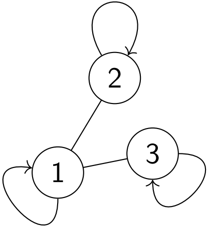

Properties of S_G:

- Linear subspace
- Self-adjointness ($A \in S_G \iff A^* \in S_G$)
- Contains identity

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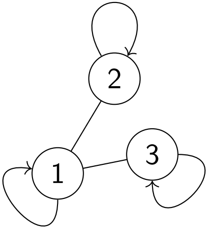

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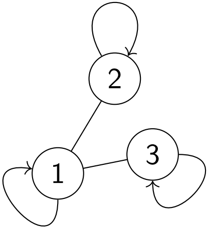

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S_G is an operator system!

Operator System

A subspace $S \subseteq B(H)$ is called an operator system if

- $I \in S$
- $A \in S \implies A^* \in S$

Quantum Graphs

Graph Operator System

Let $G = (V, E)$ be a classical graph on n vertices. The graph operator system S_G associated to G is defined as

$$S_G = \text{ span}\{e_{ij} \ : \ (i,j) \in E \text{ or } i = j, \ \ \forall i,j \in V\} \subseteq \mathbb{M}_n \ ,$$

where e_{ij} are matrix units in $\mathbb{M}_n$.

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Matrix Quantum Graph

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$$S = S_G \iff D_n S D_n \subseteq S,$$

where D_n is the diagonal subalgebra in $\mathbb{M}_n$.

Motivation from information theory

- Generalize the confusability graph of classical channels.

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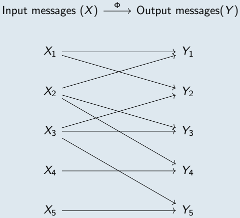
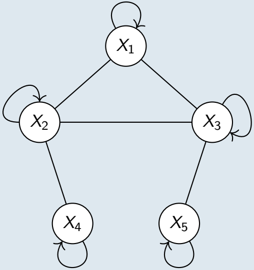
Classical confusability graph

Classical channel $\Phi : \{x_1, x_2, \dots x_m\} \longrightarrow \{y_1, y_2, \dots y_n\}$

$$\Phi = \text{probability transition matrix } [P(y_j|x_i)]$$





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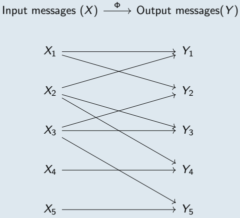

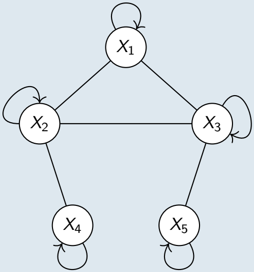

- Classical channels $->$ classical confusability graphs
- Quantum channels $->$ quantum graphs

Quantum graphs as non-commutative confusability graphs

Quantum channel

Quantum communication channel take quantum states to quantum states.

$$\Phi : B(H_A) \xrightarrow{\text{linear}} B(H_B)$$

- Trace preserving (TP): $\mathrm{Tr}(\rho) = \mathrm{Tr}(\Phi(\rho))$.
- Completely positive (CP): Φ is positive and all extensions $\Phi \otimes I_E$ are also positive.

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Krauss representation: $\Phi(\rho) = \sum_{i=1}^{r} K_i \rho K_i^*$, where $K_i \in B(H_A, H_B)$, $\sum_{i=1}^{r} K_i^* K_i = I_A$

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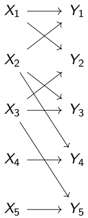
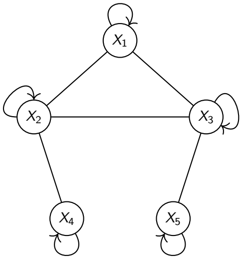
Non-commutative confusability graph [DSW, 2013]

Given a quantum channel $\Phi : \mathbb{M}_m \to \mathbb{M}_n$ with $\Phi(x) = \sum_{i=1}^{r} K_i x K_i^*$, the confusability graph of Φ is the operator system:

$$S_\Phi = \mathrm{span}\{K_i^* K_j : 1 \leq i, j \leq r\} \subseteq \mathbb{M}_m.$$

Significance of quantum graphs





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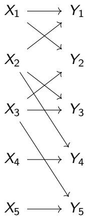

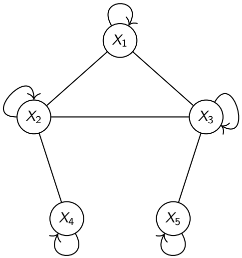

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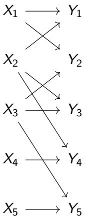

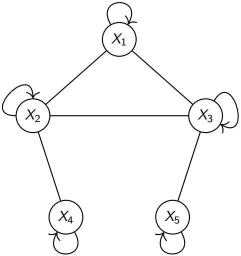

Independence number of confusability graph
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- Input messages x, y are not confusable $\iff |x\rangle\langle y| \perp S_\Phi$
- Useful in zero-error quantum communication

Mathematical interest in quantum graphs

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  - Quantize adjacency matrix [MRV '18, BCE$^+$ '20]
  - Use categorical theory of quantum sets and quantum functions

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Different yet equivalent notions!

Quantum graphs as quantum relations

Quantum set: von-Neumann algebra $\mathcal{M} \subseteq B(H)$.

Quantum relation [Kuperberg-Weaver, 2010]

A quantum relation on $\mathcal{M}$ is a weak*-closed subspace $S \subseteq B(H)$ that is a bi-module over its commutant $\mathcal{M}'$.

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Quantum Graph [Weaver, 2015]

A quantum graph on $\mathcal{M}$ is a weak*-closed operator system $S \subseteq B(H)$ that is a bi-module over $\mathcal{M}'$.

$$_{\mathcal{M}'}S_{\mathcal{M}'} \underset{\text{operator system}}{\subseteq} B(H)$$

The quantum adjacency matrix formalism

Quantum set: finite dimensional C*-algebra with a faithful tracial state.

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Quantum graphs [Musto-Reutter-Verdon, 2018]

A quantum graph is a pair $(\mathcal{M}, A_G)$ containing

- Quantum set $(\mathcal{M}, \psi)$
- Quantum adjacency matrix $A_G : \mathcal{M} \xrightarrow{\text{linear}} \mathcal{M}$ with
  - Schur Idempotency: $m(A_G \otimes A_G)m^* = A_G$
  - Reflexivity: $m(A_G \otimes I)m^* = I$
  - Symmetry: $(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

$m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is the multiplication map, $\eta : \mathbb{C} \to \mathcal{M}$ is the unit map.
m^*, η^* are their duals w.r.t Hilbert space structure on $L^2(\mathcal{M}, \psi)$.

Translating different notions of quantum graphs

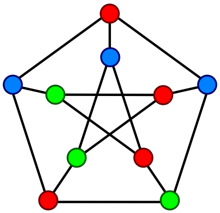
Quantum set M: finite dimensional C*-algebra with faithful tracial state ψ.

CLASSICAL GRAPH	MATRIX Q.GRAPH	QUANTUM RELATIONS	PROJECTION	ADJACENCY MATRIX
$G = (V, E, A_G)$ $A_G \in \mathbb{M}_n\{0,1\}$	$S \subseteq \mathbb{M}_n$ is an operator system.	$(M, {}_{M'}S_{M'})$	(M, p) $p \in M \otimes M^{op}$	(M, A_G) $A_G : M \to M$
Idempotency: $A_G \odot A_G = A_G$	$A_G \odot \mathbb{M}_n = S$	$M'SM' \subseteq S$	$p = p^2$	$m(A_G \otimes A_G)m^* = A_G$
Reflexivity: 1s on the diagonal	$1 \in S$	$M' \subseteq S$	$m(p) = 1_M$	$m(A_G \otimes I)m^* = I$
Irreflexivity: 0s on the diagonal	$\mathrm{Tr}(S) = 0$	$M' \perp S$	$m(p) = 0$	$m(A_G \otimes I)m^* = 0$
Undirected: $A_G = A_G^T$	$S = S^*$	$S = S^*$	$\sigma(p) = p$	$(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

Graph coloring

Coloring problem

Assign colors to vertices of graph such that no adjacent vertices get same color.



Chromatic number

Least number of colors required to color that graph.

Quantum graph coloring

	Classical Graph	**Quantum** Graph
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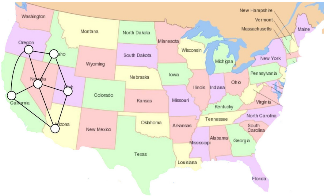
Classical Chromatic No.		
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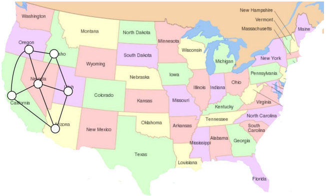
Classical Chromatic No.		
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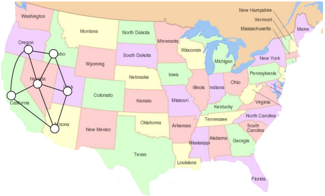
Classical Chromatic No.		Partition orthonormal basis into independent subsets
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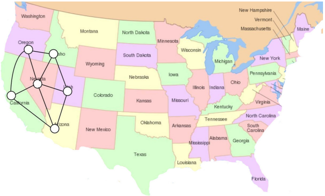
Classical Chromatic No.		**Partition orthonormal basis into independent subsets**
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- Winning condition: $\lambda(v, w, a, b) = 1$
  - Adjacency rule: $(v, w) \in E \implies a \neq b$
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Players can use different strategies (local (loc), quantum (q), quantum approximate (qa) and quantum commuting (qc)) to win the game.

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- *Outputs*: *colors* $\{1, 2, \dots k\}$.
- *Winning Criteria*:
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The t-chromatic number $\chi_t(\mathcal{G})$ is the least k needed to win the game with strategy $t \in \{loc, q, qa, qs, qc\}$.

[Brannan-Ganesan-Harris, 2020]

A quantum graph $\mathcal{G} = (\mathcal{M}, S, M_n)$ has a k-coloring if there exists a finite von-Neumann algebra $\mathcal{N}$ with a faithful normal trace and projections $\{P_a\}_{a=1}^{k} \subseteq \mathcal{M} \otimes \mathcal{N}$ such that

1. $P_a^2 = P_a = P_a^*, \ \forall a$,
2. $\sum_{a=1}^{k} P_a = I_{\mathcal{M}} \otimes I_{\mathcal{N}}$,

which satisfy $\boxed{P_a(X \otimes I_{\mathcal{N}})P_a = 0}$, for all $X \in S$ and $1 \leq a \leq k$.

Combinatorial characterization of quantum graph coloring

[Brannan-Ganesan-Harris, 2020]

A quantum graph $\mathcal{G} = (\mathcal{M}, S, M_n)$ has a k-coloring if there exists a finite von-Neumann algebra $\mathcal{N}$ with a faithful normal trace and projections $\{P_a\}_{a=1}^{k} \subseteq \mathcal{M} \otimes \mathcal{N}$ such that

1. $P_a^2 = P_a = P_a^*, \quad \forall a,$
2. $\sum_{a=1}^{k} P_a = I_{\mathcal{M}} \otimes I_{\mathcal{N}},$

which satisfy $\boxed{P_a(X \otimes I_{\mathcal{N}})P_a = 0}$, for all $X \in S$ and $1 \leq a \leq k$.

Chromatic number of quantum graphs:

- $\chi_{loc}(\mathcal{G})$: least k with $\dim(\mathcal{N}) = 1$
- $\chi_q(\mathcal{G})$: least k with $\dim(\mathcal{N}) < \infty$
- $\chi_{qc}(\mathcal{G})$: least k with finite $\mathcal{N}$ (possibly infinite dimensional)

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- $\chi_{alg}(\mathcal{G}) \leq \chi_{qc}(\mathcal{G}) \leq \chi_{qa}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi_{loc}(\mathcal{G})$.

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Tool

If $\{P_a\}_{a=1}^k \subseteq \mathcal{M} \otimes \mathcal{N}$ is a quantum coloring of $\mathcal{G}$, then

$$P_a(A \otimes I_{\mathcal{N}})P_a = 0, \ \ \forall a,$$

where A is the unique self-adjoint quantum adjacency operator of $\mathcal{G}$.

Hoffman's bound

Let $\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_{min}$ be all the eigenvalues of A.

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For a classical graph $G = (V, E, A)$,

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Example: For a quantum complete graph, $\lambda_{max} = dim(\mathcal{M}) - 1, \ \ \lambda_{\min} = -1$:

$$\chi_q(\mathcal{G}) \leq \dim(\mathcal{M})$$

More bounds

Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph

Spectral bounds [Ganesan, 2021]

$$1 + \max\left\{\frac{\lambda_{\max}}{|\lambda_{\min}|}, \frac{2m}{2m - n\gamma_{\min}}, \frac{s^{\pm}}{s^{\mp}}, \frac{n^{\pm}}{n^{\mp}}, \frac{\lambda_{\max}}{\lambda_{\max} - \gamma_{\max} + \theta_{max}}\right\} \leq \chi_q(\mathcal{G}),$$

where

- $\lambda_{\max}, \lambda_{\min}$ are the maximum and minimum eigenvalues of A
- s^{+}, s^{-} are the sum of the squares of the positive and negative eigenvalues of A respectively
- n^{+}, n^{-} are the number of positive and negative eigenvalues of A including multiplicities
- $\gamma_{\max}, \gamma_{\min}$ are the maximum and minimum eigenvalues of the signless Laplacian operator
- $\theta_{\max}$ is the maximum eigenvalue of the Laplacian operator.

THANK YOU FOR YOUR ATTENTION