talk 2

Hilbert C^*-modules, Cuntz-Pimsner Algebras, + Topological Graphs

<u>Def</u> A Hilbert C^*-module X over A is a right A-module so that there is an inner product $\langle \cdot, \cdot \rangle_A : X \times X \to A$ satisfying: $\forall x, y, z \in X$, $a \in A$, $\lambda \in \mathbb{C}$

1) $\langle x, \lambda y + z \rangle_A = \lambda \langle x, y \rangle_A + \langle x, z \rangle_A$

2) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$

3) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$

4) $\langle x, x \rangle_A \geq 0$, and $\langle x, x \rangle_A = 0$ iff $x = 0$

Furthermore, X is complete in

$$\|x\|_X = \|\langle x, x \rangle_A\|^{1/2}$$

"Hilbert spaces where $\mathbb{C} \rightsquigarrow A$"

Some things are similar, but there are differences (e.g. Cauchy-Schwarz-like identity)

Ex 1) Hilbert spaces

2) A_A where $\langle a,b\rangle_A = a^*b$

3) Let E be a directed graph.
$A = c_0(E^0)$ (E^0 as discrete set)
small outside finite set

$c_c(E^1)$ (finitely supported ftns)

for $x \in c_c(E^1)$, $a \in c_0(E_0)$

$(x \cdot a)(e) = x(e)\,a(s(e))$

$\langle x,y\rangle_A (v) = \sum_{\{e : s(e)=v\}} \overline{x(e)}\, y(e)$

this makes $c_c(E^1)$ a semi-inner-product $c_0(E^0)$ module; quotienting by length zero vectors + completing gives a Hilbert $c_0(E^0)$ module $X(E)$

→ probably just say.

operators on Hilbert modules

bounded $\not\Rightarrow$ adjointable

Let $\mathcal{L}(X) := \left\{ T : X \to X \text{ s.t. } \exists\, T^* : X \to X \text{ with } \langle Tx, y \rangle = \langle x, T^* y \rangle \ \forall x, y \in X \right\}$

these are automatically bdded so $\mathcal{L}(X)$ is a C^*-alg.

Rmk $M(A) \cong \mathcal{L}(A_A)$

"compact" operators $\mathcal{K}(X) = \overline{\operatorname{span}} \{ \theta_{x,y} : x, y \in X \}$

where $\theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$

is an ideal of $\mathcal{L}(X)$

Def A C^*-correspondence over A is a Hilbert A-module X with a $*$-hom $\varphi: A \to \mathcal{L}(X)$

i.e. X has a left A-action

$$a \triangleright x = \varphi(a)x$$

Ex: $\alpha: A \to A$ a $*$-hom, then $a \triangleright b = \alpha(a)b$ makes A_A into a C^*-correspondence.

Ex define $\varphi: c_0(E^0) \to \mathcal{L}(X(E))$ by $(\varphi(a)x)(e) = a(r(e))x(e)$ for $x \in C_c(E^1)$

(need to verify well-definedness, cty, and adjoints)

Correspondence ⇝ C^*-algebra

<u>Def</u> A Toeplitz rep (ψ, π) of ${}_A X_A$ in a C^*-alg B is

a linear map $\psi: X \to B$ & $*$-hom $\pi: A \to B$ s.t.

1) $\psi(x \cdot a) = \psi(x)\pi(a)$

(2) ⇒ (1), actually 2) $\psi(x)^* \psi(y) = \pi(\langle x, y\rangle_A)$

3) $\psi(a \cdot x) = \pi(a)\psi(x)$

The Toeplitz algebra of X is the C^*-algebra $\mathcal{T}_X$ generated by a universal Toeplitz rep (i_X, i_A).

Concretely represent on Fock Space

$$\mathcal{F}(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n} \quad (\otimes \text{ over } A, \; X^{\otimes 0} = A)$$

$\mathcal{T}_X$ is generated by

$$T_x(y_1 \otimes \cdots \otimes y_n) = x \otimes y_1 \otimes \cdots \otimes y_n$$

$$a \cdot (y_1 \otimes \cdots \otimes y_n) = (a \cdot y_1) \otimes y_2 \otimes \cdots \otimes y_n$$

Given the ^linear^ map $\psi: X \to B$ in a Toeplitz rep, (ψ,π) there is an induced $*$-hom $\mathcal{K}(X) \to B$. If this map agrees with π (after precomp w/ $\varphi: A \to \mathcal{L}(X)$) this is called a covariant Toeplitz rep. The ~~universal~~ C^*-alg generated by a universal covariant rep is the Cuntz-Pimsner alg $\mathcal{O}_X$ and is a quotient of $\mathcal{T}_X$ by the Katsura ideal.

<u>Fact</u> $\mathcal{O}_{X(E)} \cong C^*(E)$. also groupoid picture

<u>Topological graphs</u>

<u>Def</u> $E = (E^0, E^1, r, s)$ is a top graph if E^0, E^1 locally compact top. spaces, $r: E^1 \to E^0$ cts, $s: E^1 \to E^0$ is a local homeo.

graph correspondence $X(E)$

$$C_c(E^1) \qquad C_0(E_0) \qquad C_0(E^0)$$

$$(x \triangleleft a)(e) = x(e)\,a(s(e))$$

$$\langle x, y\rangle_A(v) = \sum_{\{e : s(e)=v\}} \overline{x(e)}\,y(e)$$

$$\cancel{}\,(a\,\cancel{\triangleright x})(e) = \cancel{x(e)}\,a(r(e))x(e)$$

topological graph algebra is ~~$\mathcal{O}(X(E))$~~. $\mathcal{O}_{X(E)}$

<u>other graph-algebras</u>

k-graphs, separated graphs, directed graphs of groups, ...