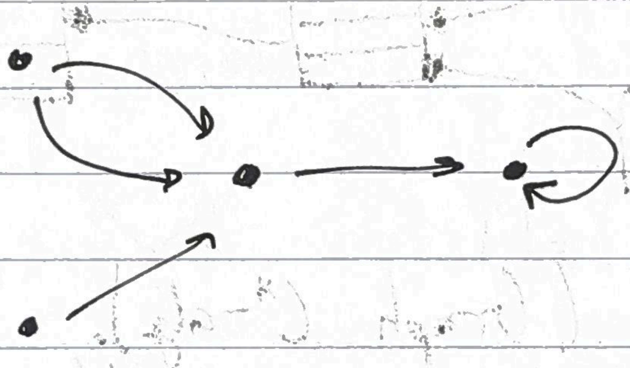
GOALS '26 - Graph Algebras

Intro to graph algebras,

By a directed graph we mean a 4-tuple $E = (E^0, E^1, r, s)$

vertices edges range source

e.g.



say

in general, practically anything is allowed - multiple edges, self-loops, E^0 and/or E^1 infinite, vertices can emit / receive infinitely many edges.

write

We will focus on row-finite graphs - $|r^{-1}(v)| < \infty$.

skip

v is a source $\iff r^{-1}(v) = \phi$

v is a sink $\iff s^{-1}(v) = \phi$

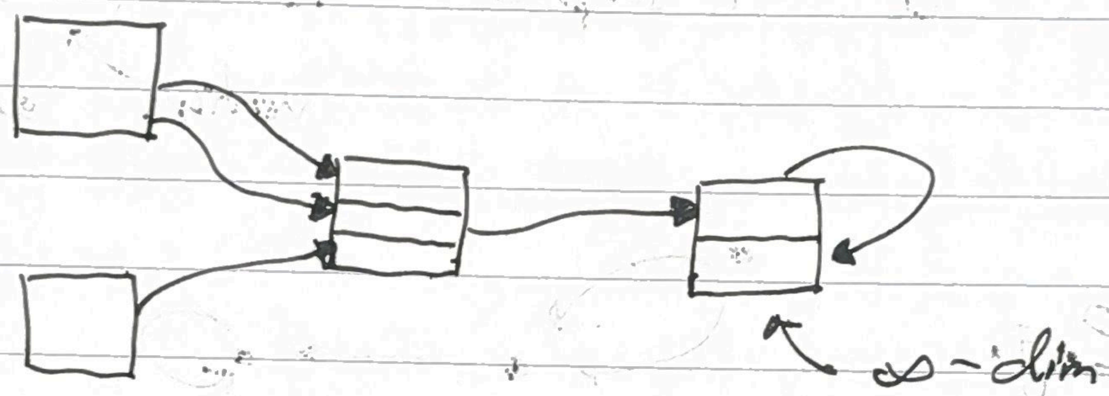
paths are written right to left;

e.g. $\mu = \mu_1 \mu_2 \mu_3$ $\quad \bullet \xleftarrow{\mu_1} \bullet \xleftarrow{\mu_2} \bullet \xleftarrow{\mu_3} \bullet$

idea of graph algebra:

vertices $\rightsquigarrow$ Hilbert spaces

edges $\rightsquigarrow$ (partial) isometries.



Def'n (Graph C^*-alg). The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by a family of orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ s.t.

(CK 1) $s_e^* s_e = p_{s(e)} \quad \forall e \in E^1$

(CK 2) $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ if v is not a source

Warning: reverse convention in literature (exchanging roles of r and s)

A family $(T, Q) \subseteq B$ satisfying CK1,2 is called a CK E-family

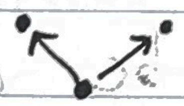
for a path $\mu = \mu_1 \mu_2 \cdots \mu_n$ (written right to left)

$$S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$$

three basic facts:

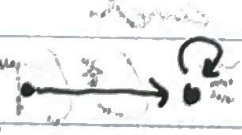
Say: any word in P_v's and S_e's may be simplified to

$S_\mu S_\nu^*$ where μ, ν are paths with

Examples:  $M_3(\mathbb{C})$

 $C(\mathbb{T})$

 $\mathcal{O}_2$

 $\mathcal{T}$

 $\mathcal{K}$

Gauge action and uniqueness theorems

$C^*(E)$ carries a gauge action γ of $\mathbb{T}$ given by

$$\gamma_z(p_v) = p_v \quad \gamma_z(s_e) = z s_e \quad \forall v \in E^0, e \in E^1.$$

Thm (Gauge-invariant uniqueness)

Let (T, Q) be a Cuntz-Krieger E-family in a C^*-algebra B such that each $Q_v \neq 0$ and B has an action β of $\mathbb{T}$ s.t. ~~$\beta(P_v) = P_v$ and~~

$$\beta(Q_v) = Q_v \text{ and } \beta(T_e) = z T_e \quad \forall v \in E^0 \text{ and } e \in E^1.$$

Then $C^*(E) \cong C^*(T, Q)$

Thm (Cuntz-Krieger uniqueness).

Let (T, Q) be a Cuntz-Krieger E-family w/ $Q_v \neq 0$ $\forall v \in E^0$. If every cycle in E has an entrance, then $C^*(E) \cong C^*(T, Q)$

first is condition on images

2^{nd} is on E itself

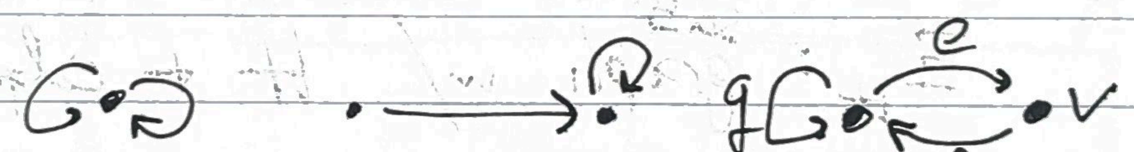
Condition (K) and ideals

<u>Def</u> A return path for $v \in E^0$ is a path μ with $s(\mu) = r(\mu) = v$, which otherwise does not visit v.

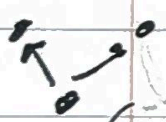
<u>Note</u>: not necessarily a cycle.

<u>Def</u> A graph E satisfies Condition (K) if every vertex on a cycle has at least 2 return paths

<u>Ex</u>



✓ X ✓ $ef + egf$



✓

every ideal is generated by vertex projections with Condition (K)
(really, gauge-invariant)

Thm Let E be a graph satisfying Condition (K). Then there is a bijection $H \to I_H$ (ideal generated by $\{p_v : v \in H\}$) between hereditary, saturated subsets $H \subseteq E^0$ and (closed, 2-sided) ideals of $C^*(E)$. Furthermore, $I_H \otimes \mathcal{K} \cong C^*(E_H) \otimes \mathcal{K}$, and

$$0 \longrightarrow I_H \to C^*(E) \to C^*(E \setminus H) \to 0$$

is a s.e.s.

E_H: subgraph w/ H + edges between

$E \setminus H$: subgraph w/ H removed

Def H is hereditary if $v \xrightarrow{e} w$ and $w \in H \Rightarrow v \in H$

is saturated if

$$s(r^{-1}(w)) \left\{ \begin{array}{l} v_1 \to w \\ v_2 \to w \\ \vdots \\ v_n \to w \end{array} \right.$$

and $v_i \in H \ \forall i$, then $w \in H$

Ex $v \longrightarrow w$

only non-trivial H is $\{v\}$

so $0 \to I_{\{v\}} \to C^*(E) \to C^*(E \setminus \{v\}) \to 0$

$E_{\{v\}} = \overset{\bullet}{v}$ $\quad I_{\{v\}} \otimes K \cong C^*(E_{\{v\}}) \otimes K \cong \mathbb{C} \otimes K$

$E \setminus \{v\} = w$ $\quad \Longrightarrow I_{\{v\}} \cong K$

$C^*(E \setminus \{v\}) \cong \mathcal{O}_2$

Relating graphical properties to C^*-properties

Graph	Graph algebra
$\lvert E^0 \rvert < \infty$	unital
no cycles	AF
Cond (K) + cofinal	Simple
Cond (K)	real rank zero
no cycle has an entrance	AF-embeddable

lots of projections

can reach any vertex from every infinite path

if read more: geometric classification