Lec 2 // K_1 for C^*-algebras (GOALS) ①

Let A be a unital C^*-algebra and
~~$\tilde{A}$ its minimal unitization.~~

$$\mathcal{U}(\tilde{A}) = \{ u \in \tilde{A} \mid u^*u = uu^* = 1_{\tilde{A}} \}$$

$$\mathcal{U}_n(\tilde{A}) = \mathcal{U}(M_n(\tilde{A}))$$

$$\mathcal{U}_\infty(\tilde{A}) = \bigcup_{n \geq 1} \mathcal{U}_n(\tilde{A})$$

We also define $u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$

Two u, v unitaries are homotopy (written $u \underset{h}{\sim} v$) if there exists a ~~cts~~ norm cts (within the relevant C^*-algebra so A) ~~such that~~ ~~u and~~ from u to v.

For $u, v \in \mathcal{U}_\infty(\tilde{A})$, $u \sim_1 v$ if there exists $k \geq \max\{m, n\}$ s.t.

$$u \oplus 1_{k-n} \underset{h}{\sim} v \oplus 1_{k-m}$$

← min. unitization of A.

Def'n: $K_1(A) = \mathcal{U}_\infty(\tilde{A}) / \sim_1$

This is an abelian group and given $\varphi : A \to B$ a $*$-hom, we set

$$K_1(\varphi) : K_1(A) \longrightarrow K_1(B)$$
$$[u]_1 \longmapsto [\tilde{\varphi}(u)]_1$$

where $\tilde{\varphi} : \hat{A} \longrightarrow \hat{B}$
$$(a, \lambda) \longmapsto (\varphi(a), \lambda)$$

Facts: (I) $K_1(M_n(\mathbb{C})) = K_1(\mathcal{B}(H)) = \{0\}$.
(II) K_1 is a functor.

~~Thm~~ <u>Remark</u>

The def'n of $K_0(A)$ for A non-unital is not $K_0(\hat{A})$ and is also not just apply the construction used in the unital case.

By def'n, if A is non-unital, then

$$K_0(A) = \operatorname{Ker}(K_0(\pi))$$

where $0 \to A \xrightarrow{\iota} \tilde{A} \underset{\pi}{\overset{\lambda}{\rightleftarrows}} \mathbb{C} \to 0$

Thm: Suppose $0 \to I \xrightarrow{\varphi} A \underset{\lambda}{\overset{\psi}{\rightleftarrows}} B \to 0$

is a split exact sequence. Then

$$0 \to K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(B) \to 0$$

and $0 \to K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \underset{K_1(\lambda)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(B) \to 0$

are each split exact.

Idea of proof:

~~Harder~~ Involved part is $K_*(\varphi)$ injective.

Split is automatic: $K_*(\psi \circ \lambda) = K_*(id_B) = id_{K_*(B)}$

$K_*(\psi) \circ K_*(\lambda)$

Cor: $K_*(A \oplus B) \cong K_*(A) \oplus K_*(B)$

Thm: Suppose $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ is exact. Then there is a ~~six~~ six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\varphi)} & K_0(A) & \xrightarrow{K_0(\psi)} & K_0(B) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(B) & \xleftarrow{K_1(\psi)} & K_1(A) & \xleftarrow{K_1(\varphi)} & K_1(I) \end{array}$$

Main Example

Let $X = \cancel{}\ \{0,1\}^{\mathbb{N}}$ (the Cantor set)

Define $\varphi : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{N}}$

$$
\begin{aligned}
(0, \text{———}) &\longmapsto (1, \text{———})\\
(1, 0, \text{———}) &\longmapsto (0, 1, \text{———})\\
(1,1,0, \text{——}) &\longmapsto (0,0,1, \text{——})\\
(1,1,1,\ldots) &\longmapsto (0,0,0, \text{--})
\end{aligned}
$$

Fact: φ is a homeomorphism called the odometer
(Show it is cts, injective, and surjective and use X compact)

We get a $\mathbb{Z}$-action on X via
$$n \cdot x := \varphi^n(x).$$

Q: What are $K_0(C(X) \rtimes \mathbb{Z})$
$K_1(C(X) \rtimes \mathbb{Z})$?

There are two six term exact sequences that can be used.

Method 1: there exists six term exact sequence (Ian Putnam, ~~1990~~ 1989)

~~There is an~~

$$\begin{tikzcd}
K_0(\mathbb{C}) \arrow[r] & K_0(\text{CAR algebra}) \arrow[r] & K_0(C(X) \rtimes_\varphi \mathbb{Z}) \arrow[d] \\
K_1(C(X) \rtimes_\varphi \mathbb{Z}) \arrow[u] & K_1(\text{CAR-algebra}) \arrow[l] & K_1(\mathbb{C}) \arrow[l]
\end{tikzcd}$$

What are $K_0(\mathbb{C})$, $K_0(\text{CAR algebra})$?

What are $K_1(\mathbb{C})$, $K_1(\text{CAR algebra})$?

We get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{Z}[\tfrac{1}{2}] \arrow[r] & K_0(C(X) \rtimes_\varphi \mathbb{Z}) \arrow[d] \\
K_1(C(X) \rtimes_\varphi \mathbb{Z}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

with work one shows that this map is onto

$[u]_1 \longmapsto 1 \in \mathbb{Z}$

Method 2:

Thm (Pimsner – Voiculescu 1980)

If $\alpha \in \mathrm{Aut}(A)$, then

$$\begin{tikzcd}
K_0(A) \arrow[r, "id - K_0(\alpha)_*"] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "K_1(\iota)"'] & K_1(A) \arrow[l, "id - K_1(\alpha)_*"']
\end{tikzcd}$$

is exact $\left(\begin{array}{l} \text{where} \\ \iota : A \hookrightarrow A \rtimes_\alpha \mathbb{Z} \\ \alpha : A \to A \end{array} \right)$

In our special case, $A = C(X)$ ← Cantor set

α is induced from φ

and we get

$$\begin{tikzcd}
K_0(C(X)) \arrow[r, "id - K_0(\alpha)"] & K_0(C(X)) \arrow[r] & K_0(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(C(X) \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(C(X)) \arrow[l] & K_1(C(X)) \arrow[l, "id - K_1(\alpha)"']
\end{tikzcd}$$

Leading to $\bigoplus \mathbb{Z} \to \bigoplus \mathbb{Z} \to ?? \to 0$

$0 \to ??$ ↗ $\quad \ker(\quad) \quad \mathrm{coker}(\quad)$

Q: What to do next?

A: Read / Do exercises of

"An introduction to K-theory for C^*-algebras"

by Rørdam, Larsen, Laustsen.

Some of the many important things I missed:

1. $K_1(A) \cong K_0(SA)$

   where $SA = \{ f: [0,1] \to A \mid f(0) = f(1) = 0 \}$

2. $K_0(A) \cong K_1(SA)$ (Bott)

   (One can define $K_n(A) = K_{n-1}(SA)$ using ①. Then ② becomes $K_0(A) \cong K_2(A)$.)

3. Additional structure on $K_0(A)$

4. Connection to index theory

etc ---