Lec 1 //

K-theory for C^*-algebras (GOALS)

Goal: Construct an invariant of C^*-algebras using projections and unitaries.

By "an invariant", we mean a functor from (C^*-algebras, $*$-hom) to (Abelian groups, group hom.) ~~one~~ two groups not just one.

↑ really graded abelian groups.

The K_0-group

Let A be a ~~unital~~ unital C^*-algebra.

$\mathcal{P}(A) = \{p \in A \mid p^* = p \text{ and } p^2 = p\} \leftarrow$ projections in A

Three equivalence relations on $\mathcal{P}(A)$:

1. $p \sim_h q$ if there exists a norm cts map $\mathfrak{p} : [0,1] \to \mathcal{P}(A)$ s.t. $\mathfrak{p}(0) = p$ and $\mathfrak{p}(1) = q$
2. $p \sim_u q$ if there exists $u \in A$ s.t. $q = upu^*$ — if A is unital we can assume $u \in A$.
3. $p \sim q$ if there exists $v \in A$ s.t. $p = v^*v$, $q = vv^*$ — v is a partial isometry.

Issues:

I We have three different relation, so it look like we'll get three different invariants

II Our C^*-algebra might have very few projections ~~&~~: If X is connected
$$P(C(X)) = \{0, 1\}$$

III We want to "add" projections, but $p+q$ is typically not a projection

For I, are these relations really different?

Yes, $p \underset{h}{\sim} q \underset{\not\Leftarrow}{\Rightarrow} p \underset{u}{\sim} q \underset{\not\Leftarrow}{\Rightarrow} p \sim q$.

Ex// ($p \sim q \not\Rightarrow p \underset{u}{\sim} q$)

Let $A = B(H)$

$S: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ $e_i \mapsto e_{i+1}$ (e_i o.n. basis)

Then $S^*S = I$ and $SS^* =$ projection on span $\{e_2, e_3, \ldots\}$

so $S^*S \sim SS^*$ by def'n.

However, $uIu^* = I$ for any unitary,

so $S^*S \underset{u}{\not\sim} SS^*$.

Prop//

1. If $p \sim q$, then $\underbrace{\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}}_{\in M_2(A)} \sim_u \underbrace{\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}}_{\in M_2(A)}$

2. If $p \sim_u q$, then $\underbrace{\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}}_{\in M_2(A)} \sim_h \underbrace{\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}}_{\in M_2(A)}$

Key point: We can "solve" issues I and II by consider projections in matrices over A (rather than just in A).

Let $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$

and $\mathcal{P}_\infty(A) = \underbrace{\bigsqcup_{n \geq 0}}_{\text{disjoint union}} \mathcal{P}_n(A)$.

Def'n// Suppose $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$. We write $p \sim_0 q$ if there exists $v \in M_{m,n}(A)$ s.t. $p = v^* v$ and $q = v v^*$.

We also let $p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ ← this "solves" issue III.

Def^n/Lemma// $\mathcal{D}(A) := P_\infty(A) / \sim_0$

is an abelian semigroup with a zero element (namely $[0_1]$ is the zero element).

To prove this we would prove

(i) $p \sim_0 p \oplus 0_n$ for any n, $0_n \in M_n(A)$

(Idea: Use $\begin{pmatrix} p \\ 0_n \end{pmatrix}$)

(ii) If $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$

(Idea: Use $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$ where v gives $p \sim_0 p'$, w gives $q \sim_0 q'$)

(iii) $p \oplus q \sim_0 q \oplus p$

(Idea: Use $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$)

etc ...

There is a general way of constructing an abelian group from an abelian semigroup.

Denote the abelian semigroup by $(S, +)$

(5)

Define $\sim_G$ on $S \times S$ via
$(x, y) \sim_G (\hat{x}, \hat{y})$ if $\exists z \in S$
when $x + \hat{y} + z = \hat{x} + y + z$

and let $G(S) = S \times S / \sim_G$

with $[(x,y)] + [(\hat{x},\hat{y})] = [(x+\hat{x}, y+\hat{y})]$

Prop// $(G(S), +)$ is an abelian group where
$0_{G(S)} = [(x,x)]$ and
$-[(x,y)] = [(y,x)]$

Proof: Exercise

There is also a map $\gamma_S : S \longrightarrow G(S)$
$x \longmapsto [(x+y, y)]$
indep. of y.

$$K_0(A) := G((\mathcal{D}(A), +))$$

from direct sum.

Ex// $S = \mathbb{N}$ with usual $+$

$$G(\mathbb{N}) = \mathbb{N} \times \mathbb{N} / \sim_G$$

$$= \{[(n,m)] \mid (n,m) \in \mathbb{N} \times \mathbb{N}\}$$

Elements can be written as "$n - m$"

(E.g., $[(2,1)] = [(3,2)] = [(4,3)] = \cdots$
and this corresponds to $2-1 = 1 \in \mathbb{Z}$
Likewise $[(1,2)] \Leftrightarrow 1-2 = -1 \in \mathbb{Z}$)

In the case of $K_0(A)$, we can write
elements as $[p]_0 - [q]_0$ where
$[p]_0$ denotes the equivalence class w.r.t. $\sim_0$
associated to $p \in P_\infty(A)$.

Given $\varphi: A \to B$ a $*$-hom., one can show that there is an induced ~~map~~ group hom. $K_0(\varphi): K_0(A) \to K_0(B)$.

It extends the map $P_\infty(A) \longrightarrow K_0(B)$

$$p \longmapsto [\varphi(p)]_0$$

apply φ to each entry in p

~~Prop~~ Thm $\not{1}$ (K_0 is a functor + more)

(1) $K_0(id_A) = id_{K_0(A)}$

(2) $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$

(3) $K_0(\{0\}) = \{0\}$

(4) $K_0(0: A \to B,\ a \mapsto 0) = 0: K_0(A) \to K_0(B),\ g \mapsto 0$

(5) If $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \to \cdots$ is ~~an inductive system~~ sequence of C^*-algebras with limit A, then $K_0(A)$ is the limit of the inductive ~~system~~ sequence of abelian groups $K_0(A_1) \xrightarrow{K_0(\varphi_1)} K_0(A_2) \xrightarrow{K_0(\varphi_2)} K_0(A_3) \to \cdots$

Ex// $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ for any n. 8

~~4~~ Outline:

1. If $p, q \in M_n(\mathbb{C})$, then TFAE
   - (i) $p \sim q$
   - (ii) $Tr(p) = Tr(q)$ where $Tr([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$
   - (iii) $\dim(p\mathbb{C}^n) = \dim(q\mathbb{C}^n)$
2. Use 1 to show that $P(M_n(\mathbb{C})) \cong \{0, 1, 2, \dots\}$ with usual $+$.
   0: zero projection; 1: projections with trace 1 (also rank 1)
3. We showed that $G(\{0,1,2,\dots\}, +) = (\mathbb{Z}, +)$

Ex// $K_0(B(H)) \cong \{0\}$

Outline:

1. Show $p \sim q \iff \dim(pH) = \dim(qH)$
2. Use 1 to show that $P(B(H)) = \{0, 1, \dots\} \cup \{+\infty\}$

   where $n + \infty = \infty + n = \infty$
3. $G(P(B(H))) = \{0\}$ ~~If $g \in G(P(B(H)))$~~ ~~then~~ (since $(n,m) \sim (0,0)$ since $n + 0 + \infty = 0 + m + \infty$