Groupoid C*-algebras

Robin Deeley

University of Colorado Boulder

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The C*-game

Input \rightarrow C^*-algebra \rightarrow K\text{-theory}

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Remarks:

1. Going from a $C^*$-algebra to its $K$-theory is not a technical issue but it almost certainly a computational issue (see Mark Tomforde’s talk).
2. Going from an input to a $C^*$-algebra is a technical issue (e.g., given an input how do we construct a $C^*$-algebra, in the second line should we take the reduced or full group $C^*$-algebra?).
Big Picture: Our approach will be based on the fact that many classes of inputs naturally lead to groupoids and will discuss a method for constructing C*-algebras from (certain) groupoids.

More concrete plan: We will play the C*-game when the input is an equivalence relation.
Let $\sim$ denote an equivalence relation on a nonempty set $X$, so that

1. for each $x \in X$, $x \sim x$;
2. if $x \sim y$, then $y \sim x$;
3. if $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of $x$ is denote by $[x]$ and is the set

$$\{ y \in X \mid x \sim y \}.$$ 

We view an equivalence relation as a subset of $X \times X$ via

$$R = \{ (x, y) \in X \times X \mid x \sim y \}$$

$$X/ \sim = \{ [x] \mid x \in X \}$$
Examples when $X$ is a finite set

Consider the following equivalence relations:

Let $X_1 = \{x_1, x_2\}$ and $R_1 = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)\}$.

Let $X_2 = \{x_1, x_2, x_3, x_4\}$ and $R_2$ be the equivalence relation with all elements equivalent to each other.

Let $X_3 = \{x_1, x_2, x_3, x_4\}$ and $R_3$ be the equivalence relation generated by $x_1 \sim x_2$ and $x_3 \sim x_4$.

Let $X_4 = \{x_1, x_2\}$ and $R_4 = \{(x_1, x_1), (x_2, x_2)\}$.
First attempt at making a $C^*$-algebra: 

For each $R_i$ we have that $X_i/\sim_i$ is a compact, Hausdorff space so we can consider $C(X_i/\sim_i)$.

Why is this not a great choice?

It forgets a lot of information about the input. For example,

$$C(X_1/\sim_1) = C(X_2/\sim_2) \cong \mathbb{C}$$

Likewise $C(X_3/\sim_3)$ and $C(X_4/\sim_4)$ are the same.
Given a finite set $X$ and an equivalence relation $R \subset X \times X$, we let

$$C^*(R) = \{ f : R \to \mathbb{C} \}$$

with

$$(\lambda \cdot f)(x, y) = \lambda f(x, y)$$

$$(f + g)(x, y) = f(x, y) + g(x, y)$$
The star operation when $X$ is finite

Define $(f^*)(x, y) = f(y, x)$.

Exercise: Prove that the star operation is well-defined.

Exercise: Would it be well-defined if $R$ was an arbitrary subset of $X \times X$ rather than an equivalence relation?
Multiplication in $C^*(R)$ when $X$ is finite

The convolution product on $C^*(R)$ is defined via

$$(f \ast g)(x, y) = \sum_{z \sim x} f(x, z)g(z, y)$$

Exercise: Prove that $\ast$ is well-defined. Would it be well-defined if $R$ was an arbitrary subset of $X \times X$ rather than an equivalence relation?

Exercise: Prove $C^*(R)$ is a $\ast$-algebra.

Somewhat involved exercise: Define a norm on $C^*(R)$ (still with $X$ finite) so that it becomes a $C^*$-algebra. (Hint: the examples on the next slide should be helpful).
Back to $R_1$, which was defined by taking $X_1 = \{x_1, x_2\}$ and

$$R_1 = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)\}$$

and

$$C^*(R_1) = \{f : R_1 \rightarrow \mathbb{C}\}$$

As a vector space $C^*(R_1)$ is four dimensional; it has basis:

$$f_{ij}(x_l, x_m) = \begin{cases} 1 & l = i \text{ and } m = j \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq i \leq 2$ and $1 \leq j \leq 2$. 
Exercise: Prove that $C^*(R_1) \cong M_2(\mathbb{C})$ as $\ast$-algebras via the map defined on the above basis by

$$
\begin{align*}
    f_{11} & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
    f_{12} & \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
    f_{21} & \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
    f_{22} & \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
$$

Exercise: Proved that $C^*(R_2) \cong M_4(\mathbb{C})$, $C^*(R_3) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $C^*(R_4) \cong \mathbb{C}^2$. 
The C*-game in this case

**Definition**

A C*-algebra $A$ is finite dimensional if it is finite dimensional as a vector space.

**Theorem**

If $X$ a finite nonempty set and $R \subseteq X \times X$ is an equivalence relation, then $C^*(R)$ is finite dimensional.

**Theorem**

If $A$ is a C*-algebra that is finite dimensional, then there exists $X$ finite and an equivalence relation $R \subseteq X \times X$ such that $A \cong C^*(R)$.

Exercise: (Still with $X$ finite) compute the $K$-theory of $C^*(R)$ in terms of the equivalence relation $R$. 
Generalizing to the case when $X$ is infinite

To generalize to the case when $X$ is infinite, we need more structure on $R$. One can work more generally but we will assume $R$ is a locally compact, Hausdorff space.

Let $r : R \to X$ be defined via $(x, y) \mapsto x$ and $s : R \to X$ be defined via $(x, y) \mapsto y$.

**Definition**

A topological equivalence relation $R$ is an equivalence relation over $X$ with a locally compact Hausdorff topology such that

$$(x, y) \mapsto (y, x) \text{ and } ((x, y), (y, z)) \mapsto (x, z)$$

are continuous. This implies that $r$ and $s$ are both continuous (see the next slide).
The topology on $R$ determines the topology on $X$ (rather than the opposite). How?

Using the identification of sets: $X \cong \{(x, x) \mid x \in X\} \subseteq R$.

Based on this, $r$ is more correctly defined as $r : R \to R$ via

$$(x, y) \mapsto (x, x)$$

Exercise: Based on the definition of $r$, define $s$.

Exercise: Prove that $r$ and $s$ are continuous.
Étale equivalence relations

**Definition**

A topological equivalence relation $R$ is étale if $r$ and $s$ are local homeomorphisms.

**Theorem**

*If $R$ is étale, then for each $x \in X$

\[
\{(x, y) \mid y \sim x\} \text{ and } \{(y, x) \mid y \sim x\}
\]

*are discrete subsets of $R$.***
Examples

1. If $X$ is nonempty and finite, then any equivalence relation $R$ on $X$ is étale.

2. Suppose $\pi : X \to Y$ is a local homeomorphism and

$$R_\pi := \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}.$$ 

Then $R_\pi \subseteq X \times X$ is étale.

3. As specific case of the previous example, take $\pi : \mathbb{R} \to S^1$ the standard covering map.

Exercise: What is missing from the above examples to make the statements “is étale” precise?
An example from dynamical systems

Definition

A dynamical system is a pair \((X, \varphi)\) where \(X\) is a compact Hausdorff space and \(\varphi : X \to X\) is a homeomorphism.

We say that \((X, \varphi)\) is free if the following property holds: \(\varphi^n(x) = x\) for some \(x \in X\) if and only if \(n = 0\).

Definition

Suppose \((X, \varphi)\) is free. Define

\[
R_{\text{orbit}} = \{(x_1, x_2) \in X \times X \mid \varphi^n(x_1) = x_2 \text{ for some } n \in \mathbb{Z}\}
\]

Exercise: Prove \(R_{\text{orbit}}\) is an equivalence relation.
Étale topology on $R_{\text{orbit}}$

Using the assumption that $(X, \varphi)$ is free, we can (as sets) identify $R_{\text{orbit}}$ with $X \times \mathbb{Z}$ via

$$(x, n) \in X \times \mathbb{Z} \mapsto (\varphi^n(x), x) \in R_{\text{orbit}}.$$  

Since $X \times \mathbb{Z}$ has a natural topology this gives $R_{\text{orbit}}$ a topology.

Facts:

1. The original topology on $X$ agrees with the topology on $X \times \{0\} \subseteq X \times \mathbb{Z}$.

2. $X$ is compact but $R_{\text{orbit}}$ is not compact.

3. The topology on $R_{\text{orbit}}$ is not the subspace topology from $R_{\text{orbit}} \subseteq X \times X$.

4. Using the topology above, $R_{\text{orbit}}$ is étale (but with the subspace topology is not).
Example of an equivalence relation that is not étale

Example

Let $X = [0, 1]$ and define an equivalence relation

$$R = \{(x, x) \mid x \in X\} \cup \{(0, 1), (1, 0)\}.$$ 

where we give $R \subseteq X \times X$ the subspace topology.

Exercise: Prove that for each $x \in X$

$$\{(x, y) \mid (x, y) \in R\} \text{ and } \{(y, x) \mid (y, x) \in R\}$$

are finite (and hence discrete) subsets of $R$.

Exercise: Prove that $R$ is not étale.
Functions of compact support

Suppose $R$ is a locally compact Hausdorff étale equivalence relation. Let

$$C_c(R) = \{ f : R \to \mathbb{C} \mid f \text{ is continuous and has compact support} \}$$

We have the following algebraic operations:

$$(\lambda \cdot f)(x, y) = \lambda f(x, y)$$
$$(f + g)(x, y) = f(x, y) + g(x, y)$$
$$(f^*)(x, y) = \overline{f(y, x)}$$
$$(f \ast g)(x, y) = \sum_{z \sim x} f(x, z)g(z, y)$$

Very involved exercise: Prove that the convolution product is well-defined.

Exercise: Prove that the convolution product is **not** well-defined for the non-étale equivalence relation from the previous slide.
Plan: Complete $C_c(R)$ to get a $C^*$-algebra.

The issue is what norm should we complete with respect to?

There is no canonical choice.

In general, given a faithful representation $\phi : C_c(R) \to B(\mathcal{H})$, we can form

$$C^*_\phi(R) := \overline{C_c(R)}||\cdot||$$

where $||\cdot||$ is the operator norm in $B(\mathcal{H})$.

We will discuss representations in more detail later in the talk.

If $(X, \varphi)$ is free, then $C^*(R_{orbit}) \cong C(X) \rtimes \mathbb{Z}$. 
An explicit example

Take $X = \{0, 1\}^\mathbb{N}$ with the product topology ($X$ is the Cantor set) and $\varphi$ the odometer homeomorphism (i.e., $\varphi$ acts via add one to the first coordinate and then carry over). That is,

$$
\varphi(0, x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)
$$

$$
\varphi(1, 0, x_2, \ldots) = (0, 1, x_2, \ldots)
$$

$$
\varphi(1, 1, 0, x_3, \ldots) = (0, 0, 1, x_3, \ldots)
$$

and

$$
\varphi(1, 1, 1, \ldots) = (0, 0, 0, \ldots)
$$

Goal: Understand the orbit relation and the $C^*$-algebra associated to the orbit relation for this dynamical system.
Example of an orbit

The orbit of $x \in X$ is the set

$$\{ y \in X \mid y = \varphi^n(x) \text{ for some } n \in \mathbb{Z} \}$$

By the definition of the orbit relation, $[x]_{\text{orbit}}$ is exactly the orbit of $x$.

For example, when $x = (0, 1, 0, 1, 0, \ldots)$, we have

$$\varphi(x) = (1, 1, 0, 1, 0, \ldots)$$

$$\varphi^2(x) = (0, 0, 1, 1, 0, \ldots)$$

$$\varphi^{-1}(x) = (1, 0, 0, 1, 0, \ldots)$$
Another relation on $X$

Recall that $X = \{0, 1\}^\mathbb{N}$ with the product topology.

**Definition**

We say that $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ are tail equivalent and write

$$x \sim_{tail} y$$

if there exists $N \in \mathbb{N}$ such that $x_i = y_i$ for all $i \geq N$.

Q: Is orbit equivalence relation from the odometer action the same as the tail equivalence relation?

A: Not quite

$(1, 1, 1, \ldots) \sim_{orbit} (0, 0, 0, \ldots)$ (since $\varphi(1, 1, 1, \ldots) = (0, 0, 0, \ldots)$) but

$(1, 1, 1, \ldots) \not\sim_{tail} (0, 0, 0, \ldots)$
The relationship between these two equivalence relations is the following:

1. If $x \in X$ and $x \not\sim_{\text{orbit}} (0, 0, 0, \ldots)$, then

$$[x]_{\text{tail}} = [x]_{\text{orbit}}$$

2. The orbit of $(0, 0, 0, \ldots)$ has been broken into its forward and backward parts:

$$[(1, 1, 1, \ldots)]_{\text{tail}} \cup [(0, 0, 0, \ldots)]_{\text{tail}} = [(0, 0, 0, \ldots)]_{\text{orbit}}$$

3. $R_{\text{tail}} \subseteq R_{\text{orbit}}$ as an open subrelation.

Exercise: Prove that if $\hat{R}$ is an open subrelation of $R$ and $R$ is étale, then $\hat{R}$ (with the subspace topology) is also étale.
Define $\iota : C_c(R_{tail}) \to C_c(R_{orbit})$ by

$$\iota(f)(x_1, x_2) = \begin{cases} f(x_1, x_2) & (x_1, x_2) \in R_{tail} \\ 0 & (x_1, x_2) \notin R_{tail} \end{cases}$$

Fact: $\iota$ can be extended to a $\ast$-homomorphism $C^*(R_{tail}) \to C^*(R_{orbit}).$

Exercise: Prove that $C^*(R_{tail}) \to C^*(R_{orbit})$ is injective (so that we can view $C^*(R_{tail})$ as a subalgebra of $C^*(R_{orbit})$).

Fact: $C^*(R_{tail})$ is a large subalgebra of $C^*(R_{orbit}) \cong C(X) \rtimes \mathbb{Z}.$ (see Dawn Archey’s talk)
The structure of $C^*({R}_{\text{tail}})$

$C^*({R}_{\text{tail}})$ is the CAR algebra!

Outline of the ideas of the proof:

Define $\mathbb{C} \to C^*({R}_{\text{tail}})$ via $\lambda \mapsto \lambda I$ where $I \in C_c({R}_{\text{tail}})$ is defined by

$$I(x, \hat{x}) = \begin{cases} 1 & x = \hat{x} \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Prove that $\mathbb{C} \to C^*({R}_{\text{tail}})$ is a well-defined injective $\ast$-homomorphism.

Next we want to define $M_2(\mathbb{C}) \to C^*({R}_{\text{tail}})$. 
The map: $M_2(\mathbb{C}) \to C^*(R_{tail})$

For $i, j \in \{0, 1\}$ define

$$E_{ij} = \{(x, \hat{x}) \mid \pi_0(x) = i, \pi_0(\hat{x}) = j \text{ and for } k > 0, \pi_k(x) = \pi_k(\hat{x})\}$$

where $\pi_k : \{0, 1\}^\mathbb{N} \to \{0, 1\}$ is the projection onto the $k$-coordinate.

For example

$$((0, 1, 1, \ldots), (1, 1, 1, \ldots)) \in E_{01}$$
$$((0, 1, 1, \ldots), (0, 1, 1, \ldots)) \notin E_{01}$$
$$((0, 1, 1, \ldots), (1, 0, 0, \ldots)) \notin E_{01}$$
$$((0, 1, 0, \ldots), (1, 0, 0, \ldots)) \in E_{01}$$
$$((0, 1, 0, 1, 0, \ldots), (1, 1, 0, 1, 0, \ldots)) \in E_{01}$$

Note: One of these ordered pairs is not even in $R_{tail}$. 
The map: \( M_2(\mathbb{C}) \rightarrow C^*(R_{tail}) \)

\[ E_{ij} = \{(x, \hat{x}) \mid \pi_0(x) = i, \pi_0(\hat{x}) = j \text{ and for } k > 0, \pi_k(x) = \pi_k(\hat{x})\} \] where \( \pi_k : \{0,1\}^\mathbb{N} \rightarrow \{0,1\} \) is the projection onto the \( k \)-coordinate.

Define \( M_2(\mathbb{C}) \rightarrow C^*(R_{tail}) \) via

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \mapsto a\chi_{00} + b\chi_{01} + c\chi_{10} + d\chi_{11}
\]

where \( \chi_{ij} : R_{tail} \rightarrow \mathbb{C} \) is defined via

\[
\chi_{ij}(x, \hat{x}) = \begin{cases} 
1 & (x, \hat{x}) \in E_{ij} \\
0 & \text{otherwise}
\end{cases}
\]

Exercise: Prove that \( M_2(\mathbb{C}) \rightarrow C^*(R_{tail}) \) is a well-defined injective \(*\)-homomorphism.
The structure of $C^*(R_{tail})$

Exercise: How are the maps $\mathbb{C} \to C^*(R_{tail})$ and $M_2(\mathbb{C}) \to C^*(R_{tail})$ related? (Hint: What is $\chi_{00} + \chi_{11}$?)

Involved exercise: For each $n \in \mathbb{N}$, define the relevant injective $\ast$-homomorphism $M_{2n}(\mathbb{C}) \to C^*(R_{tail})$.

Very involved exercise: Prove that $\bigcup_{n \in \mathbb{N}} M_{2n}(\mathbb{C})$ is dense in $C^*(R_{tail})$.

Exercise: Is $\bigcup_{n \in \mathbb{N}} M_{2n}(\mathbb{C}) = C_c(R_{tail})$?

**Summary:** Using the fact that $C^*(R_{tail})$ is large inside $C^*(R_{orbit})$ reduces many questions about $C^*(R_{orbit})$ to $C^*(R_{tail})$ and we know “everything” about $C^*(R_{tail})$ because it is the CAR algebra.
What is a groupoid?

Definition

A groupoid is a nonempty set \( \mathcal{G} \) with the following additional structure:

1. a fixed subset of \( \mathcal{G} \times \mathcal{G} \) denoted by \( \mathcal{G}^2 \);
2. a map \( \mathcal{G}^2 \rightarrow \mathcal{G} \) denoted by \( (g, h) \mapsto gh \);
3. an involution \( \mathcal{G} \rightarrow \mathcal{G} \) denoted by \( g \mapsto g^{-1} \).

such that

1. if \( g, h \) and \( k \) are in \( \mathcal{G} \) with \( (g, h), (h, k) \) both in \( \mathcal{G}^2 \), then \( (gh, k) \), \( (g, hk) \) are both in \( \mathcal{G}^2 \) and \( (gh)k = g(hk) \) (so that we can simply write \( ghk \));
2. for each \( g \in \mathcal{G} \), both \( (g, g^{-1}) \) and \( (g^{-1}, g) \) are in \( \mathcal{G}^2 \) and moreover if \( (g, h) \in \mathcal{G}^2 \), then \( g^{-1}gh = h \) and if \( (h, g) \in \mathcal{G}^2 \), then \( hgg^{-1} = h \).
An equivalence relation is a groupoid

Let $X$ be a nonempty set and $R \subseteq X \times X$ be an equivalence relation. We take

1. $G = R$
2. $G^2 = \{(x, y), (a, z) \in R \times R \mid y = a\}$
3. $G^2 \to G$ defined via $((x, y), (y, z)) \mapsto (x, z)$
4. $G \to G$ defined via $(x, y) \mapsto (y, x)$

Exercise: Prove that the above defines a groupoid.
More structure

**Definition**

Let $\mathcal{G}$ be a groupoid. The set of units of $\mathcal{G}$ is

$$\mathcal{G}^0 = \{g^{-1}g \mid g \in G\}$$

Define the range map, $r : \mathcal{G} \to \mathcal{G}^{(0)}$ via $g \mapsto gg^{-1}$ and the source map, $s : \mathcal{G} \to \mathcal{G}^{(0)}$ via $g \mapsto g^{-1}g$.

For the groupoid associated to an equivalence relation, we have

$$\mathcal{G}^0 = \{(y, x)(x, y) \mid (x, y) \in \mathcal{G}\} = \{(y, y) \mid y \in X\} \cong X$$

Moreover,

$$r(x, y) = (x, y)(y, x) = (x, x)$$

and

$$s(x, y) = (y, x)(x, y) = (y, y)$$
More examples

Example

Given a nonempty set $X$, we can define a groupoid by taking

1. $G = X$,
2. $G^2 = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$,
3. $G^2 \to G$ defined via $(x, x) \mapsto x$ and
4. $G \to G$ defined via $x \mapsto x$.

Exercise: How are the constructions of a groupoid from a set and an equivalence relation related? (Hint: One is a special case of the other).

Exercise: What is $G^0$ in this case?
More examples

Example

Given a group \( G \), we can define a groupoid by taking

1. \( \mathcal{G} = G \),
2. \( \mathcal{G}^2 = G \times G \),
3. \( \mathcal{G}^2 \to \mathcal{G} \) defined by group multiplication and
4. \( \mathcal{G} \to \mathcal{G} \) defined by taking the inverse.

Exercise: What is \( \mathcal{G}^0 \) in this case?
A topological groupoid is a groupoid with a (locally compact, Hausdorff) topology on $\mathcal{G}$ such that

1. $\mathcal{G}^2$ is given the subspace topology from $\mathcal{G} \times \mathcal{G}$;
2. $\mathcal{G}^2$ is closed;
3. $\mathcal{G}^2 \to \mathcal{G}$ is continuous;
4. $\mathcal{G} \to \mathcal{G}$ is continuous.

Examples? Same as above but with topologies, so topological spaces (rather than sets), topological groups (note: discrete groups are topological groups), and topological equivalence relations.

Exercise: Prove that given a topological groupoid the range and source maps are continuous.
Definition

A topological groupoid is étale if $r$ and $s$ are local homeomorphisms.

Example

The groupoid associated to a topological group, $G$, is étale if and only if $G$ is discrete.

Example

If $X$ is a locally compact and Hausdorff space, then the associated groupoid is étale.
Suppose that $\mathcal{G}$ is a locally compact, Hausdorff, étale groupoid. Let

$$C_c(\mathcal{G}) = \{ a : \mathcal{G} \to \mathbb{C} \mid a \text{ is continuous and has compact support} \}.$$ 

Algebraic operations:

\[(\lambda \cdot a)(g) = \lambda a(g)\]
\[(a + b)(g) = a(g) + b(g)\]
\[(a^*)(g) = a(g^{-1})\]
\[(a \ast b)(g) = \sum_{r(h) = r(g)} a(h)b(h^{-1}g)\]
Theorem

Let \( u \) be a unit in a locally compact, Hausdorff étale groupoid \( G \) (i.e., \( u \in \mathcal{G}^0 \)). For each \( a \in C_c(G) \) and \( \xi \in \ell^2(s^{-1}(u)) \) the expression

\[
(\pi_u^u(a)\xi)(g) = \sum_{r(h) = r(g)} a(h)\xi(h^{-1}g)
\]

defines an element in \( \ell^2(s^{-1}(u)) \). Moreover, \( \pi_u^u(a) \) is in \( \mathcal{B}(\ell^2(s^{-1}(u))) \) with \( \|\pi_u^u(a)\| \) bounded by a constant that is independent of \( u \) (it is does depend on \( a \)), and \( \pi_u^u : C_c(G) \to \mathcal{B}(\ell^2(s^{-1}(u))) \) is a \(*\)-representation.
**Theorem**

Suppose that $\mathcal{G}$ is a locally compact, Hausdorff, étale groupoid. Then

$$ ||a|| := \sup \{ ||\pi(a)|| \mid \pi \text{ a representation of } C_c(\mathcal{G}) \} $$

and

$$ ||a||_\lambda := \sup \{ ||\pi^u_\lambda(a)|| \mid u \in \mathcal{G}^0 \} $$

define (non-complete) $C^*$-norms on $C_c(\mathcal{G})$.

**Definition**

The full groupoid $C^*$-algebra of $\mathcal{G}$ is $C^*(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{||\cdot||}$.

**Definition**

The reduced $C^*$-algebra of $\mathcal{G}$ is $C^*_\lambda(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{||\cdot||\lambda}$. 
What’s next?

For more, see

1. Ian Putnam’s lecture notes on $C^*$-algebra (available on his website, see Chapter 3);


3. Karen Strung “An introduction to $C^*$-algebras and the Classification Programme” (available on her website, see Exercises 9.6.12 and 9.6.13 for more on the odometer action and its $C^*$-algebra);


Thank you!