NOTES ON C*-ALGEBRAS

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Contents

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1. A first look at C*-algebras

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The main goal in this lecture is proving the Gelfand-Naimark theorem for commutative C*-algebras (Theorem 2.1) and introducing the Functional Calculus (Corollary 2.18).

To that end, we will use without proof all of the results in Section 1. We will introduce the unitization from Section 1, but with more focus on the intuition in Remark 1.16.

From Section 3, we use without proof the correspondence between maximal ideals and characters established in Definition 2.2 - Corollary 2.6. We will also use without proof the fact (Proposition 2.7) that the character space (i.e. spectrum) of a C*-algebra is a weak∗-compact subset of the unit ball of the dual of the C*-algebra.

We will prove Lemma 2.12 and assume its corollary, Lemma 2.13, to complete the proof of Theorem 2.1. However, the proof in the lecture will look a little different from the notes. In particular, we will consider the theory in the unital setting first and then explain how to get to the non-unital setting at the end.

Proposition 2.16 and Corollary 2.17 establish the important fact that the spectrum of an element in a C*-algebra is independent of the ambient unital C*-algebra. However, we will bypass this argument in lecture and go straight for a description of the correspondence in the Functional Calculus (Corollary 2.18).

In a Banach space, there is often additional algebraic structure, in particular multiplication.

Definition 1.1. A *Banach ∗-algebra A* is a multiplicative involutive Banach space whose norm satisfies the following:

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b \in A$.

Ideally, we'd like involution to also be isometric. This and other magical results follow from the additional assumption that the norm $\|\cdot\|$ on A satisfies the C*-*identity*:

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$. It follows from this that

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|,$$

and hence that $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$.

Definition 1.2. A C*-*algebra* is a Banach ∗-algebra whose norm satisfies the C*-identity.

Remark. *Calling these* C*-*algebras is already highly suggestive. In fact, when they were first introduced, they were called B^*-algebras, and the notion of* C*-*algebra was reserved for norm closed ∗-subalgebras of $B(\mathcal{H})$ (hence the "C-*"). In the coming days, we shall justify calling these* C*-*algebras, but for the sake of not encouraging archaic terminology, we take the privilege before we earn it.*

Recall from Exercise 7.32 in the Day 1 lectures that the norm on $B(\mathcal{H})$ satisfies the C*-identity, meaning any closed self-adjoint subspace of $B(\mathcal{H})$ is a C*-algebra. These are known as *concrete* C*-*algebras*.

Example 1.3. Recall the unilaterial shift $S \in B(\ell^2(\mathbb{N}))$ from Example 7.19 in the Prerequisite Notes. The norm closure of the ∗-algebra generated by S in $B(\ell^2(\mathbb{N}))$ is a C*-algebra often called the *Toeplitz algebra*.

Exercise 1.4. Let X be a locally compact Hausdorff topological space. We denote by $C_0(X)$ the space of all continuous functions on X vanishing at infinity. Show this is a C*-algebra with involution given by complex conjugation and norm given by the sup norm.

Example 1.5. Consider the C*-algebra $C(\mathbb{T})$ consisting of all continuous functions on the compact Hausdorff space $\mathbb{T} = \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$ (sometimes denoted S^1). (Why don't we say $C_0(\mathbb{T})$?) It follows from the Stone-Weierstraß approximation theorem ([4, I.5,6]) that Laurent polynomials, i.e. polynomials of the form $\sum_{k=-n}^{n} \alpha_n z^n$, are dense in $C(\mathbb{T})$. So, $C(\mathbb{R})$ is actually the C*-algebra generated by the function $f \in C(\mathbb{R})$ given by $f(z) = z$.

As is often the case, C*-algebras are a little more friendly to work with when they have an identity element (also called a unit). If $1 \in A$ is the identity, then

(1) $1^* = 1^*1 = 11^* = 1$, and
(2) $\|1\| = 1$.

Analogously with elements in $B(\mathcal{H})$ (in fact, we will see soon that it is more than an analogy), we call an element a in a C*-algebra A

- *Normal* if $a^*a = aa^*$,
- *Self-Adjoint* if $a = a^*$,
- a *Projection* if $a = a^* = a^2$,
- a *Unitary* if $a^*a = aa^* = 1$,
- an *Isometry* if $a^*a = 1$,
- a *partial isometry* if $a = aa^*a$.

Note (Check) that for any element a in a C*-algebra is the sum of two self-adjoint operators, its real and imaginary parts:

$$\operatorname{Re}(a) = \frac{1}{2}(a + a^*) \qquad \operatorname{Im}(a) = \frac{1}{2i}(a - a^*). \tag{1.1}$$

This useful decomposition lets us reduce many questions to the case of self-adjoint operators.

Proposition 1.6. *A linear map between* C*-*algebras is $*$-preserving iff it maps self adjoint elements to self adjoint elements.*

Proof. Let $\phi : A \to B$ be a linear map and $a \in A$, and write $a = \operatorname{Re}(a) + i\operatorname{Im}(a)$ and $a^* = \operatorname{Re}(a) - i\operatorname{Im}(a)$. By linearity,

$$\phi(a) = \phi(\operatorname{Re}(a)) + i\phi(\operatorname{Im}(a))$$
$$\phi(a^*) = \phi(\operatorname{Re}(a)) - i\phi(\operatorname{Im}(a)).$$

Since $\operatorname{Re}(a)$ and $\operatorname{Im}(a)$ are self-adjoint, $\phi(\operatorname{Re}(a))$ and $\phi(\operatorname{Im}(a))$ are self adjoint by assumption. So the above computation shows that

$$\phi(a^*) = \phi(\operatorname{Re}(a) + i\operatorname{Im}(a))^*.$$

□

1.1. **Unitizations and Spectra.** Let us briefly recap and expand on some facts about the spectrum of an operator in a Banach algebra– now with C*-algebras.

An element a in a unital algebra is invertible when there exitst another element b in the algebra that acts as a left and right inverse, i.e. $ab = ba = 1$. Sometimes, when you have a left inverse, it is automatically a right inverse. In particular, this is the case for matrix algebras. In fact, a matrix $T \in \mathrm{M}_n(\mathbb{C})$ is invertible if and only if it is injective, i.e. if and only if $\ker(T) = \{0\}$. In infinite dimensions, this is certainly still a necessary condition, but it is no longer sufficient alone.

Exercise 1.7. Give an example of an operator on $B(\ell^2(\mathbb{N}))$ that is injective but not invertible.

Fortunately, the Open Mapping Theorem gives us some guidance on what needs to be satisfied:

Corollary 1.8 (to OMT/Inverse Function Theorem)**.** *For a Hilbert space $\mathcal{H}$, $T \in B(\mathcal{H})$ is invertible iff T is bijective.*

Example 1.9. Unitary operators are important classes of invertible operators. In fact, the group of unitaries in a C*-algebra A forms a subgroup $\mathcal{U}(A)$ of the group of invertible elements, $GL(A)$.

With the notion of invertibility, we can define the spectrum of a given element a in a unital C*-algebra A.

$$\sigma(a) := \{\lambda \in \mathbb{C} \ : \ \lambda 1 - a \notin GL(A)\}$$

Remark 1.10. Unlike when $A = \mathrm{M}_n(\mathbb{C})$, these are not all eigenvalues.

Example 1.11. If A is a unital C*-algebra and $u \in A$ is a unitary, then $\sigma(u) \subset \mathbb{T}$.

Indeed, first note that for any invertible operator $a \in A$, the spectrum of the inverse is the inverse of the spectrum. To see this, fix an invertible a, so that $\lambda = 0$ is not in $\sigma(a)$. For $\lambda \neq 0$, if $\lambda - a$ is invertible, then so is $\lambda^{-1}a^{-1}(\lambda - a) = a^{-1} - \lambda^{-1}$ and vice versa.

Then for any $\lambda \in \sigma(u)$, we have that $\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$. Since u^* is also a unitary, we know $\|u\| = \|u^*\| = 1$, which means $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$, which means $|\lambda| = 1$.

Exercise 1.12. Recall (Example 3.12 in Prerequisite Material) that continuous function f on a X locally compact and Hausdorff space X is invertible if $1/f$ is continuous on X. What is the spectrum of $f(z) = z$ in $C(\mathbb{T})$?

But not all C*-algebras have units. One important example is $\mathcal{K}(\mathcal{H})$, and another important class of examples comes from spaces of continuous functions.

Exercise 1.13. For a locally compact topological Hausdorff space X, when is the C*-algebra $C_0(X)$ unital? What is the unit? Can you think of interesting classes of non-unital algebras? For the C*-algebra $C(\mathbb{T})$, what type of operator is the generator $f(z) = z$?

So, how can we make sense of a spectrum in this setting? We just add a unit! Well, technically, we embed A into a unital C*-algebra.

The "smallest" unital C*-algebra containing A is called its *unitization*, $\tilde{A}$. We define $\tilde{A}$ as follows:

$$\tilde{A} := A \oplus \mathbb{C}$$

with algebraic operations given by

$$\begin{aligned}(a,\alpha)(b,\beta) &= (ab + \alpha b + \beta a, \alpha\beta)\\ (a,\alpha)^* &= (a^*, \bar{\lambda})\\ \|(a,\alpha)\| &= \sup_{b\in A, b\leq 1} \|ab + \alpha b\|\end{aligned}$$

This definition does not feel intuitive the first time around. To get an idea of where this came from, consider the following examples.

Example 1.14.

(1) If $A \subset B(\mathcal{H})$ is a C*-subalgebra of $B(\mathcal{H})$ that does not contain a unit, you can "unitize" it by just taking the C*-algbra generated by A and $1_{\mathcal{H}}$.

$$C^*(A, 1_{\mathcal{H}}) = \{a + \lambda 1_{\mathcal{H}} \ : \ \lambda \in \mathbb{C}, a \in A\}.$$

What would multiplication/ scalar addition look like here? For the norm, it will turn out that $\|(a,\alpha)\| = \|a + \alpha 1_{\mathcal{H}}\|$, but the argument is faster after a little more theory.

(2) Identify

$$C_0((0,1]) := \{f \in C([0,1]) \ : \ f(0) = 0\}.$$

By taking the closure of the algebra generated by $C_0((0,1])$ and the constant function 1, we get its unitization $C([0,1])$. For $f \in C_0((0,1])$ and $a \in \mathbb{C}$, what is the norm of $f + a$ in the sup norm for $C([0,1])$?

Because of the example from $B(\mathcal{H})$, even in an abstract setting, elements of $\tilde{A}$ are often written as $a + \lambda 1_{\tilde{A}}$ as opposed to (a, λ).

Proposition 1.15. *Any* C*-*algebra* A *embeds into the unital* C*-*algebra* $\tilde{A}$ *as an ideal of codimension* 1*, i.e. no other proper ideal of* $\tilde{A}$ *contains* A *and* $\tilde{A}/A = \mathbb{C}$.

Proof. That $\tilde{A}$ is a unital $*$-algebra is readily verified. To see that the norm is a Banach algebra norm, notice that it is exactly the norm induced from $B(A)$ where we identify $a \in A$ with the left multiplication operator $L_a \in B(A)$ given by $L_a(b) = ab$, and we identify (a,α) with $L_a + \alpha \mathrm{id}_A$. In other words, the norm on $\tilde{A}$ is the norm induced from $B(A)$ on the $*$-subalgebra of operators $\{L_a + \alpha \mathrm{id}_a : a \in A, \alpha \in \mathbb{C}\}$. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C*-identity, we have for any nonzero $a \in A$,

$$\|a\| = \|a\left(\frac{a^*}{\|a\|}\right)\| \leq \sup_{\|b\|\leq 1} \|ab\| \leq \|a\|.$$

So, $\|(a,0)\| = \|a\|$, and the embedding of A into $\tilde{A}$ is isometric. Since A is complete, $\{L_a + \alpha \mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$ is complete, and so $\tilde{A}$ is a Banach algebra. By design, A is an ideal of codimension 1.

It remains to show that the given norm satisfies the C*-identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$

$$\begin{aligned}
\|(a,\alpha)\|^2 &= \sup_{\|b\|\le 1} \|ab + \alpha b\|^2 \\
&= \sup_{\|b\|\le 1} \|b^*(a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b)\| \\
&\le \sup_{\|b\|\le 1} \|a^*ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b\| \\
&= \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)^*\|\|(a,\alpha)\|.
\end{aligned}$$

So $\|(a,\alpha)\| \le \|(a,\alpha)^*\|$, and a symmetric argument yields $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Then the above inequality gives

$$\|(a,\alpha)\|^2 \le \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)\|^2.$$

□

Therefore, if a is an element of a non-unital C*-algebra A, then we define its *spectrum* to be the spectrum of a as an element of $\tilde{A}$.

This fits well with what we've already seen in $B(\mathcal{H})$. If $x \in B(\mathcal{H})$, then its spectrum is defined with respect to the unit in $B(\mathcal{H})$, regardless to what closed $*$-subalgebra x belongs to.

Remark 1.16. Suppose A is a non-unital C*-subalgebra of a unital C*-algebra B. Then there is a clear $*$-preserving bijective homomorphism between $\tilde{A}$ and $C^*(A,1)$ given by $(a,\alpha) \mapsto a + \alpha$. By appealing to the same subspace $\{L_a + \alpha \mathrm{id}_A \ : \ a \in A, \alpha \in \mathbb{C}\} \subset B(A)$, one can show that this is isometric. That means that, when a unit is available in an ambient C*-algebra, the unitization of A is just adjoining that unit. Of course, there is now the problem that for any $a \in A$, its spectrum in A might be larger than its spectrum in B (an element has more potential inverses in B). We will see later that this is not the case.

Remark 1.17. There are two conventions you will see in the literature for $\tilde{A}$ when A is already unital. The first is to assume that $A = \tilde{A}$ when A is unital, and the second is to have a "forced unitization" where A is still embedded as a maximal ideal in $A \oplus \mathbb{C}$, and the unit of A becomes just the projection $1_A \oplus 0$. The choice in a given paper is often due to technical considerations (e.g. when you just want to make sure your C*-algebra has a unit vs. when you want to control where a map sends the unit) and is (hopefully) addressed somewhere in the preliminaries.

One thing that makes unitizations nice to work with is that a $*$-homomorphism always has a unique and natural extension to the unitization.

Proposition 1.18. *Let A, B be C*-algebras with B unital and A non-unital and $\pi : A \to B$ a $*$-homomorphism. Then there is a unique extension of π to a unital $*$-homomorphism $\tilde{\pi} : \tilde{A} \to B$ given by $\tilde{\pi}(a + \lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_B$.*

Note that this works also when we have $\pi : A \to B$ with B non-unital but identified with its copy inside $\tilde{B}$.

Proof. We just need to check that this is a $*$-homomorphism. Linearity and $*$-preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned}
\tilde{\pi}(a + \lambda 1_{\tilde{A}})\tilde{\pi}(b + \eta 1_{\tilde{A}}) &= (\pi(a) + \lambda 1_B)(\pi(b) + \eta 1_B) \\
&= \pi(ab) + \lambda \pi(b) + \eta \pi(a) + \lambda\eta 1_B = \tilde{\pi}(ab + \lambda b + \eta a + \lambda \eta 1_{\tilde{A}}).
\end{aligned}$$

The uniqueness is forced by the fact that we require $\tilde{\pi}$ to be linear and $1_{\tilde{A}} \mapsto 1_B$. Indeed, if $\psi : \tilde{A} \to B$ is another unital extension of π, then for each $a + \lambda 1_{\tilde{A}} \in \tilde{A}$, we have

$$\psi(a + \lambda 1_{\tilde{A}}) = \psi(a) + \psi(\lambda 1_{\tilde{A}}) = \pi(a) + \lambda 1_B = \tilde{\pi}(a + \lambda 1_{\tilde{A}}).$$

□

Now that we have a notion of spectra for unital and nonunital C*-algebras, we are ready to see two consequences of the C*-identity that are, quite frankly, magic.

First we recall Theorems 3.16 and 3.20 from the pre-requisite material:

Theorem. *For any element a in Banach algebra A, $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, the spectrum of a is contained in the closed ball $\{x \in A \ : \ \|x\| \leq \|a\|\}$. In particular, this means that $r(a) \leq \|a\|$ where $r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|$ is the spectral radius of a.*

Remark 1.19. This implies the very useful fact that for any element a in a unital Banach algebra with $\|a\| < 1$, the element $1 - a$ is invertible with inverse $\sum_{n\geq 0} a^n$.

Theorem. *For any element a in Banach algebra A,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

When our Banach algebra A is a C*-algebra, it turns out the norm of any normal element *is* its spectral radius.

Lemma 1.20. *For any normal element a in a C*-algebra A,*

$$\|a\| = r(a).$$

Proof. First, we assume that $a = a^*$. Then repeated use of the C*-identity for a, i.e. $\|a\|^2 = \|a^2\|$, tells us that

$$r(a) = \lim_n \|a^{2^n}\|^{2^{-n}} = \|a\|.$$

Now, suppose a is normal. Then a^*a is self-adjoint, and so

$$\begin{aligned} r(a)^2 &= \|a\|^2 = \|a^*a\| = r(a^*a) \\ &= \lim_n \|(a^*a)^n\|^{1/n} = \lim_n \|(a^n)^*a^n\|^{1/n} = \lim_n \|a^n\|^{2/n} \\ &= r(a)^2. \end{aligned}$$

□

As a Banach $*$-algebras, we consider C*-algebras "the same" when they are $*$-isomorphic, i.e. there exists a $*$-preserving homomorphism between them. Normally, for a Banach space, you'd also request that the bijective linear map be isometric. For $*$-isomorphisms between C*-algebras, this will be automatic, thanks again to the C*-identity.

Proposition 1.21. *A $*$-homomorphism $\pi : A \to B$ between C*-algebras is contractive (i.e. $\|\pi\| \leq 1$) and hence continuous. A $*$-isomorphism between C*-algebras is isometric.*

Proof. Suppose $\pi : A \to B$ is a $*$-isomorphism. Let $a \in A$. Then a^*a is a normal element in A, which means $\|a^*a\| = r(a^*a)$. Since homomorphisms preserve invertibility, $r(\pi(a^*a)) \leq r(a^*a)$. This is where the C*-norm comes in:

$$\|a\|^2 = \|a^*a\| = r(a^*a) \geq r(\pi(a^*a)) = r(\pi(a)^*\pi(a)) = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

Now, assume π is injective. If π is a $*$-isomorphism, then the inequality above is an equality. □

So, in C*-algebras, the algebraic structure determines the norm:

$$\|x\| = \sqrt{\|x^*x\|} = \sqrt{r(x^*x)}$$

(Compare with the same fact for matrices.) It follows from this that a C*-algebra carries a unique norm making it a C*-algebra.

Remark 1.22. What this is saying is that if $(A, \|\cdot\|)$ is a C*-algebra and $\|\cdot\|'$ is another C*-norm on A (without assuming A is complete with respect to $\|\cdot\|'$), then $\|\cdot\| = \|\cdot\|'$.

There's a subtlety here that can sometimes be a little tricky. If B is just a $*$-algebra, then we can often define multiple distinct C*-norms on B so that the completion of B with respect to these norms becomes a C*-algebra.

We will be able to say more about $*$-homomorphisms once we have established more on C*-ideals.

2. Commutative C*-algebras

Some of you may have heard of the study of C*-algebras described as "non-commutative topology" or "non-commutative continuous functions". This perspective is really what jump-started the interest in C*-algebras in the first place, and it comes from the following theorem, which is the focal point of this section:

Theorem 2.1 (Gelfand Naimark Theorem)**.** *Any commutative* C**-algebra A is $*$-isomorphic to the* C**-algebra $C_0(X)$ for some locally compact Hausdorff space X. Moreover, when A is unital, X is compact.*

Definition 2.2. A nonzero homomorphism into the base field of an algebra is called a *character*. The *spectrum* of a commutative Banach algebra A, denoted $\hat{A}$, is the set of all nonzero characters from A into $\mathbb{C}$. Hence $\hat{A}$ is often called the *character space* for $\hat{A}$.

Remark 2.3. We assume for now that these are just homomorphisms. In fact, much of the theory we develop on our way to the Gelfand Naimark theorem holds in general for Banach algebras. A consequence of the Gelfand Naimark theorem for commutative C*-algebras will show that characters on a commutative C*-algebra are automatically $*$-preserving.

Notice that the kernel of a character is a closed ideal in A of co-dimension 1, and so it is automatically a maximal ideal, i.e. it is not contained in any other proper ideal. It turns out there is a one-to-one correspondence between maximal ideals in A and ideals of co-dimension 1 (and hence characters).

Exercise 2.4. A maximal ideal in a unital C*-algebra is automatically closed.
(Hint: If $J \subset A$ is a proper ideal, consider $\overline{J} \cap B(1_A, 1)$.)

Exercise 2.5 (Gelfand-Mazur)**.** If A is a simple, unital, abelian Banach algebra, then $A = \mathbb{C}$.

Corollary 2.6. *If A is a unital abelian Banach algebra, then any maximal ideal in A has co-dimension* 1*, i.e. if $J \subset A$ is a maximal ideal, then $A/J \simeq \mathbb{C}$.*

Proof. If $J \subset A$ is a maximal ideal, then A/J is simple. The rest follows from Gelfand-Mazur. □

From Theorem 3.8 in the Prerequisite notes, we have for each maximal ideal $J \triangleleft A$, a continuous homomorphism $\phi_J : A \to \mathbb{C}$.

Proposition 2.7. *Let A be a commutative* C**-algebra. Then $\hat{A} \cup \{0\}$ is a weak-$*$ compact subset of the unit ball of A^*. When A is unital, $\hat{A}$ is weak-$*$ compact.*

In particular, $\hat{A}$ is a locally compact Hausdorff space, which is compact when A is unital.

Proof. Let $\phi \in \hat{A}$. Suppose $\|\phi\| > 1$ and $a \in A$ with $\|a\| < 1$ and $\phi(a) = 1$. Since $\|a\| < 1$, its spectrum is in the unit ball, meaning $1 - a$ is invertible. So, we compute

$$1 = \phi((1-a)(1-a)^{-1}) = (\phi(1) - \phi(a))\phi((1-a)^{-1}) = (0)\phi((1-a)^{-1}) = 0,$$

which is an obvious contradiction.

Now, since $\hat{A} \cup \{0\}$ is contained in the unit ball of A^*, by Alaoglu's theorem (Theorem 2.20 in the Prereqs), all we need to show is that it is weak-$*$ closed. To that end, suppose we have a net $(\phi_i)_{i \in I}$ of characters (multiplicative linear functionals) that converges weak-$*$ to some bounded linear functional $\phi \in A^*$. We need to check that ϕ is multiplicative, but this follows from the fact that pointwise multiplication is continuous. Indeed, for any $a, b \in A$, we have

$$\phi(ab) = \lim_i \phi_i(ab) = \lim_i \phi_i(a)\phi_i(b) = \lim_i \phi_i(a) \lim_i \phi_i(b) = \phi(a)\phi(b).$$

It follows that $\hat{A} \cup \{0\}$ is a compact Hausdorff space (with respect to the weak-$*$ topology).

Note that if A is unital, then for any $\phi \in \hat{A}$, we have $\phi(1) = 1$, and so $\|\phi\| \geq 1$. It follows by the preceeding argument that $\hat{A}$ is itself a weak-$*$ closed subset of the unit ball in A^*. □

Recall that when A is communtative but not unital, it embeds into $\tilde{A}$ as an ideal with co-dimension 1, which means it's the kernel of a character $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$. Notice that when restricted to A, this is exactly the 0 homomorphism. It turns out there is a one-to-one correspondence between $\hat{A}$ and $\hat{\tilde{A}} \backslash \{\phi_0\}$. In particular, $\hat{\tilde{A}}$ is (also) the one-point compactification of $\hat{A}$.

Proposition 2.8. *Suppose A is a non-unital commutative* C**-algebra, and let* $\phi_0 : \tilde{A} \to \tilde{A}/A = \mathbb{C}$*. Then, there is a one-to-one correspondence between* $\hat{A}$ *and* $\hat{\tilde{A}} \backslash \{\phi_0\}$*.*

Proof. Suppose $\tilde{\phi} \in \hat{\tilde{A}} \backslash \{\phi_0\}$. Since $\tilde{A}/\ker(\tilde{\phi}) = \mathbb{C}$, $\ker(\tilde{\phi})$ is a maximal ideal in $\tilde{A}$. Similarly, A is also a maximal ideal, and so $\ker(\tilde{\phi}) \cap A \subsetneq A$. Then $\ker(\tilde{\phi}) \cap A$ is an ideal of co-dimension 1 in A, which means the map $\phi : A \mapsto A/(A \cap \ker(\tilde{\phi}))$ gives a character in $\hat{A}$.

On the other hand, if $\phi \in \hat{A}$, define $\tilde{\phi} : \tilde{A} \to \mathbb{C}$ by $\tilde{\phi}(a, \lambda) = \phi(a) + \lambda$. Then (as per Proposition 1.18) $\tilde{\phi} \in \hat{\tilde{A}}$ is the unique extension of ϕ to a character on $\tilde{A}$. With that, we have established the desired bijective correspondence. □

Definition 2.9. For a commutative C*-algebra A, we define the *Gelfand transform* $\Gamma : A \to C_0(\hat{A})$ by $\Gamma(a)(\phi) = \phi(a)$, i.e. $\Gamma(a)$ is the point evaluation at a.

Exercise 2.10. Here's an exercise to build intuition:

(1) Show that all maximal ideals in $C([0,1])$ are of the form $\{f \in C([0,1]) \ : \ f(t) = 0\}$ for some $t \in [0,1]$.
(2) For each $t \in [0,1]$, define the map $ev_t : C([0,1]) \to \mathbb{C}$ by $ev_t(f) = f(t)$. Show that $\widehat{C([0,1])} = \{ev_t \ : \ t \in [0,1]\}$.
(3) Recall that for $A = C_0((0,1])$, its unitization is $\tilde{A} := C([0,1])$. That means we can identify $C_0((0,1])$ with a maximal ideal inside $C([0,1])$. To which character $\phi \in \hat{\tilde{A}}$ does this ideal correspond?
Show that this character agrees with the functional $\phi_0 : \tilde{A} \to \mathbb{C}$ given by $\phi_0(f + \lambda 1) = \lambda$ for all $f \in A$.

Here is our goal theorem:

Theorem 2.11 (Gelfand-Naimark)**.** *For any commutative* C**-algebra A, the Gelfand transform is an isometric $*$-isomorphism*[1] *of A onto* $C_0(\hat{A})$*.*

Notice that if A is unital, then $C_0(\hat{A}) = C(\hat{A})$. If A is not unital, then the one point compactification of $\hat{A}$ is $\hat{\tilde{A}} = \hat{A} \cup \{\phi_0\}$, which means $C_0(\hat{A})$ is exactly the continuous functions on $\hat{\tilde{A}}$ that vanish at ϕ_0.

Before we prove the Gelfand-Naimark theorem, we will establish a few lemmas, which are interesting in their own right.

Lemma 2.12. *For any commutative* C**-algebra A, the Gelfand transform is a contractive (and hence continuous) homomorphism. Moreover, if A is unital, then for any $a \in A$,*

$$\sigma(a) = \sigma(\Gamma(a)) = \{\phi(a) \ : \ \phi \in \hat{A}\} = ran(\Gamma(a)),$$

and Γ is isometric.

Proof. Multiplicativity follows from multiplicativity of characters. Notice that $\Gamma(a)$ is automatically continuous because the topology on $\hat{A}$ is the weak-$*$ topology. When A is nonunital, $\Gamma(a)(\phi_0) = \phi_0(a) = 0$ for each $a \in A$, which, by the above remarks, means $\Gamma(A) \subset C_0(\hat{A})$.

Since each character is contractive and the norm on $C_0(\hat{A})$ is the sup norm, it follows that Γ is contractive.

Now, suppose A is unital. First, we show that $a \in A$ is invertible iff $\Gamma(a) \in C(\hat{A})$ is invertible. The forward direction follows immediately from the fact that Γ is a homomorphism. On the other hand, if $a \in A$ is not invertible, then it lives in some maximal ideal, meaning it is in the kernel of some nonzero character $\phi \in \hat{A}$. Then $\Gamma(a)(\phi) = \phi(a) = 0$, meaning $\Gamma(a)$ is not invertible. It follows that $\sigma(a) = \sigma(\Gamma(a))$ for all $a \in A$.

Now, suppose $\lambda \in \sigma(a)$. Then there exists $\phi \in \hat{A}$ such that $\Gamma(\lambda 1 - a)(\phi) = 0$, i.e. $\Gamma(a)(\phi) = \lambda$. It follows that $\|\Gamma(a)\|_\infty = r(a)$.

Since A is commutative, all elements of A are normal. Hence it follows from Lemma 1.20 that for any $a \in A$,

$$\|a\| = r(a) = \|\Gamma(a)\|_\infty.$$

So, Γ is isometric.

□

[1] $*$-preserving isomorphism

Notice that the above argument shows that when A is not unital, its Gelfand transform extends to the Gelfand transform on its unitization.

Lemma 2.13. *Let A be a commutative* C*-*algebra. If $a \in A$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.*

Proof. Suppose $a \in A$ is self-adjoint, and assume $A \subset \tilde{A}$. For each $t \in \mathbb{R}$, the power series

$$\sum_{n \geq 0} \frac{(ita)^n}{n!}$$

converges to some element $\exp(ita)$ in $\tilde{A}$. One checks that

$$\exp(ita)^* = \sum_{n \geq 0} \frac{(-ita)^n}{n!} = \exp(-ita) = \exp(ita)^{-1},$$

which means $\exp(ita)$ is a unitary in $\tilde{A}$. Now, consider the Gelfand map $\Gamma : \tilde{A} \to C(\hat{\tilde{A}})$. By the preceeding lemma, we know $\sigma(a) = \mathrm{ran}(\Gamma(a)) = \{\phi(a) : \phi \in \hat{A}\}$. So, it suffices to show that $\phi(a) \in \mathbb{R}$ for each $\phi \in \hat{A}$. Fix $\phi \in \hat{A}$. Since ϕ is a character (i.e. continuous, linear, multiplicative), it follows that for any $t \in \mathbb{R}$,

$$\phi(\exp(ita)) = \phi(\sum_{n \geq 0} \frac{(ita)^n}{n!}) = \sum_{n \geq 0} \frac{(it\phi(a))^n}{n!} = e^{it\phi(a)}.$$

Since $\exp(ita)$ is a unitary, we know from Example 1.11 that $e^{it\phi(a)} \in \mathbb{T}$ for all $t \in \mathbb{R}$. It follows that $\phi(a) \in \mathbb{R}$ as desired. □

Remark 2.14. We shall see soon that we did not need to assume A was commutative in Lemma 2.13. The same argument would work by just considering the Gelfand transform on $\mathrm{C}^*(a, 1)$. However, we will need to first establish that the spectrum of a in $\mathrm{C}^*(a, 1)$ is the same as its spectrum in A.

Now we are ready to prove the theorem.

Proof of Gelfand Naimark Theorem. First, we assume that A is unital. We know from Lemma 2.12 that Γ is isometric, which means its image in $C(\hat{A})$ is closed.

For any self-adjoint $a \in A$, we have $ran(\Gamma(a)) \subset \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. So Proposition 1.6, tells us Γ is $*$-preserving.

So, altogether, $\Gamma(A)$ is a unital, norm closed self-adjoint subalgebra of $C(\hat{A})$ where $\hat{A}$ is compact and Hausdorff. Then the Stone-Weierstrass Theorem ([Conway, I.5,6]) says that $\Gamma(A) = C(\hat{A})$ provided that it separates the points of $\hat{A}$. But if ϕ and ψ are distinct points in $\hat{A}$, then they have distinct kernels, and so $\Gamma(A)$ separates the points of $\hat{A}$.

Now suppose that A is not unital. Then Γ extends to the isometric $*$-isomorphism $\tilde{\Gamma} : A \to C(\hat{A})$. Since A is an ideal of co-dimension one, $\tilde{\Gamma}(A)$ is a maximal ideal in $C(\hat{A})$ contained in the maximal ideal $\{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$. Then $\tilde{\Gamma}(A) = \{f \in C(\hat{\tilde{A}}) : f(\phi_0) = 0\}$, and it follows that $\Gamma(A) = C_0(\hat{A})$ from the aforementioned identifications. □

Corollary 2.15. *Characters on commutative* C*-*algebras are $*$-homomorphisms.*

Proof. By Proposition 1.6 suffices to prove that they map self-adjoint elements to real numbers. For any $\phi \in \hat{A}$, and $a \in A$ self-adjoint, we have $\Gamma(a)(\phi) = \phi(a) \in \mathbb{R}$. □

For any element a in a C*-algebra A, we write $\mathrm{C}^*(a)$ for the C*-algebra generated by a. When A is unital, $\mathrm{C}^*(a, 1)$ can be identified with the closure of the set of all polynomials on $a, a^*, 1$ (aka $*$-polynomials on a).

When a is a normal, $B := \mathrm{C}^*(a)$ is a commutative C*-algebra, and so it is $*$-isomorphic to $C_0(\hat{B}) \subset C(\hat{\tilde{B}})$. Moreover, any character $\phi \in \hat{B}$ is determined by where it maps a. So, the map $\hat{\tilde{B}} \to \mathbb{C}$ given by $\phi \mapsto \phi(a)$ is a homeomorphism onto $\Gamma(a)(\hat{\tilde{B}})$, which we know is equal to $\sigma(a)$. Moreover, the Gelfand map then identifies a with the identity function $z \mapsto z$ on $C(\sigma(a))$. When a is not invertible, $C_0(\hat{B})$ corresponds to the ideal consisting of functions that vanish at 0. If a is invertible, then $0 \notin \sigma(a)$, so either way, we can say

$$\mathrm{C}^*(a) \simeq C_0(\sigma(a) \backslash \{0\}).$$

Problem: What do we mean by $\sigma(a)$ here? By design, this must be the set of $\lambda \in \mathbb{C}$ such that $\lambda 1 - a$ is not invertible **in (the unitization of)** B, i.e. this is $\sigma_B(a)$, not $\sigma_A(a)$. In general, $\sigma_A(a)$ is smaller (there are more potential inverses for $a - \lambda 1$ in $A \supseteq B$), and we have no reason to suspect that these are the same set. But for C*-algebras, they are.

For now, we just establish the following.

Proposition 2.16. *Let a be a normal element of a* C^**-algebra A and $B = \mathrm{C}^*(a)$. Then $B \simeq C_0(\sigma_A(a)\backslash\{0\})$ and $\sigma_A(a) = \sigma_B(a)$.*

Proof. We have already established that $B \simeq C_0(\sigma_B(a)\backslash\{0\})$.

Suppose $\lambda \in \sigma_B(a)\backslash\{0\}$. Then for each $\varepsilon > 0$, there exists $b \in B$ with $\|\Gamma(b)\| = 1$ and $\|\lambda\Gamma(b) - \Gamma(a)\Gamma(b)\| < \varepsilon$. That means $\|b\| = 1$ and $\|\lambda b - ab\| < \varepsilon$, which means $\lambda 1 - a$ is not invertible in $\tilde{A}$. (Indeed, if $c(\lambda 1 - x) = 1$, then $1 = \|b\| = \|c(\lambda 1 - x)b\| < \|c\|\varepsilon$ for all ε.) ☐

This justifies the terminology "spectrum" for the space of characters on a commutative C*-algebra.

Before moving too far away from Proposition 2.16, we remark that it yields a more general corollary.

Corollary 2.17. *If a is a normal element in a unital* C^**-algebra A and B is any unital* C^**-subalgebra of A containing a, then $\sigma_A(a) = \sigma_B(a)$.*

Now we come to an incredibly powerful tool, with which we conclude the section: *The Functional Calculus*. Let A be a unital C*-algebra, $a \in A$ a normal element, and $f \in C(\sigma(a))$. We denote by $f(a)$ the inverse image of f under the Gelfand transform of $\mathrm{C}^*(a, 1)$ (the isometric $*$-isomorphism between $\mathrm{C}^*(a, 1)$ and $C(\sigma(a))$).

Corollary 2.18 (The Functional Calculus)**.** *Let a be a normal element of a unital* C^**-algebra A and $f, g \in C(\sigma(a))$. Then*

(1) $f(\sigma(a)) = \sigma(f(a))$,
(2) $g(f(\sigma(a)) = (g \circ f)(a)$, and
(3) if $0 \in \sigma(a)$ and $f(0) = 0$, then $f(a)$ is in the non-unital C^**-algebra,* $\mathrm{C}^*(a)$.

Proof. Since $f(a) \in \mathrm{C}^*(a, 1)$, we have

$$\sigma(f(a)) = \sigma(\Gamma(f(a))) = \sigma(f) = f(\sigma(a)).$$

Since Γ is a homomorphism, the second claim holds immediately when g is a Laurent polynomial (i.e. a polynomial in z and $\bar{z}$). Then the general case follows by approximating g uniformly with Laurent polynomials.

The third claim follows immediately from Proposition 2.16. ☐

Exercise 2.19. If $a \in A$ is a normal element in a unital C*-algebra and $\Gamma : \mathrm{C}^*(a) \to C_0(\sigma(a)\backslash\{0\})$ the Gelfand transform,

(1) What is its image $\Gamma(a) \in C_0(\sigma(a)\backslash\{0\})$?
(2) If a is invertible, is $a^{-1} \in C^*(a)$?

We will see this applied repeatedly in the section on positive elements.

Exercise 2.20. Suppose A and B are commutative unital C*-algebras and $\phi : A \to B$ a unital $*$-homomorphism. Then for any $a \in A$ and $f \in C(\sigma(a))$, we have $\phi(f(a)) = f(\phi(a))$.

Exercise 2.21. Let $\pi : A \to B$ be a surjective $*$-homomorphism between C*-algebras and $b \in B$ a self-adjoint element. Show that b lifts to a self-adjoint element $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.

Exercise 2.22. Suppose $A = C_0(X)$. Write down an explicit formula for the Gelfand transform $\Gamma : A \to C_0(\widehat{A})$ in this case.

3. Positive elements

Preview of Lecture:

- Exercise 3.3 is a cornerstone of the theory of C*-algebras; see if you can figure out why it's true before lecture!
- In lecture, we will prove Proposition 3.6 and Example 3.9.
- We will not prove Corollary 3.7 in lecture.
- Theorem 3.10 is really important, and the proof uses all of the exercises that precede it in this section, but it's otherwise pretty straightforward. We won't discuss the proof.
- We will discuss the proofs of Proposition 3.12 and Corollary 3.13 in lecture.

The Functional Calculus is an incredibly powerful tool for handling normal elements. Of course, not every element in a C*-algebra is normal. Nonetheless, by associating to each element $a \in A$ the self-adjoint element $a^*a \in A$, we have been able to spread the influence of the functional calculus to an entire non-commutative C*-algebra. It turns out that elements of the form a^*a take on an even more important structural role in C*-algebras, which we will explore now.

Definition 3.1. A self-adjoint element a in a C*-algebra A is *positive* if $\sigma(a) \subset [0, \infty)$. We denote this by $a \geq 0$.

This allows us to define a partial ordering on the self-adjoint elements of A: for a and b self-adjoint, we say $a \leq b$ if $b - a \geq 0$.

Example 3.2. The positive elements in $C_0((0, 1])$ are exactly the ones whose range (i.e. spectrum) lies in $[0, \infty)$.

Let's start with a few observations using the functional calculus:

Exercise 3.3. Each positive element in a C*-algebra has a unique positive square root.

Exercise 3.4. If $a \in A$ is a self-adjoint element, then there exist positive elements a_+ and a_- such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.

Exercise 3.5. Let $a \in A$ be self-adjoint, a_+ and a_- its positive and negative parts as in Exercise 3.4, and $\sqrt{a_+}$ and $\sqrt{a_-}$ their respective unique positive square roots. Show that $a_+\sqrt{a_-} = 0$ and $\sqrt{a_+}\sqrt{a_-} = 0$.

The following proposition is mostly technically useful.

Proposition 3.6. *Let a be a self-adjoint element in a unital* C*-*algebra A. Then the following are equivalent.*

(1) $a \geq 0$*;*
(2) $a = b^2$ *for some self-adjoint* $b \in A$*;*
(3) $\|\alpha 1 - a\| \leq \alpha$ *for all* $\alpha \geq \|a\|$*;*
(4) $\|\alpha 1 - a\| \leq \alpha$ *for some* $\alpha \geq \|a\|$.

Proof. We assume A is unital or pass to its unitization.

That (1) $\Rightarrow$ (2) follows from the functional calculus, and that (3) $\Rightarrow$ (4) is clear.

Assume (2). Let $f \in C(\sigma(b))$ be given by $f(z) = z^2$. Then

$$\|f\|_{\sup} = \|b^2\| = \|a\|,$$

and so (since $\sigma(b) \subseteq \mathbb{R}$ by Lemma 2.13) $0 \leq f \leq \|a\|$. Then $0 \leq \alpha - f \leq \alpha$ for any $\alpha \geq \|a\|$. Then (identifying α with the constant function on $\sigma(b)$ when appropriate), we compute

$$\|\alpha 1 - a\| = \|\alpha(b) - f(b)\| = \|(\alpha - f)(b)\| = \|\alpha - f\|_{\sup} \leq \alpha.$$

It remains to show (4) $\Rightarrow$ (1). Suppose $\alpha \geq \|a\|$ is such that $\|\alpha 1 - a\| \leq \alpha$. Let $h(z) = z$ denote the identity function on $\sigma(a)$. Then we have

$$\alpha \geq \|\alpha 1 - a\| = \|(\alpha - h)(a)\| = \|\alpha - h\|_{\sup} = \sup_{\lambda \in \sigma(a)} |\alpha - \lambda|.$$

It follows that $\sigma(a) \subset [0, \infty)$. Since a was assumed to be self-adjoint, this means $a \geq 0$. $\square$

Some concluding notation: The collection of positive elements in a C*-algebra A is denoted by A_+, and the self-adjoints are often denoted by $A_{s.a.}$

Corollary 3.7. *For a* C*-*algebra* A*, the sets* $A_{s.a.}$ *and* A_+ *are both closed.*

Proof. Suppose x_n is a sequence in $A_{s.a.}$ converging to $x \in A$. Then

$$\|x_n^* - x^*\| = \|x_n - x\| \to 0,$$

and so $x_n = x_n^* \to x^*$. Hence $x^* = x$. Now, suppose $(a_n) \in A_+$ converges to $a \in A$. Then we know $a = a^*$ and $\|a_n\| \to \|a\|$. Assume A is unital or unitize. Let $\alpha = \sup_n \|a_n\| \geq \|a\|$. Then $\alpha 1 - a_n \to \alpha 1 - a$, and $\|\alpha 1 - a_n\| \leq \alpha$ for all n by Proposition 3.6. It follows that $\|\alpha 1 - a\| \leq \alpha$, which again by Proposition 3.6 implies that a is positive. □

Exercise 3.8. If $a, b \in A$ are positive, then so is $a + b$. (Note that we are not assuming they commute – use the previous exercise.) If a and b moreover commute, then $ab \geq 0$. Can you think of two positive elements in a C*-algebra whose product is not positive? (Hint: For the first part, you can assume A is unital or work in $\tilde{A}$ (why?). Then use Proposition 3.6. For the second part consider operators in $M_2(\mathbb{C})$.)

Example 3.9. The positive operators in $B(\mathcal{H})$ are exactly the positive semi-definite operators.

Suppose $T \in B(\mathcal{H})$. By the preceeding proposition, if $T \geq 0$, then there exists a self-adjoint $S \in B(\mathcal{H})$ such that $T = S^2 = S^*S$. Then for any $x \in \mathcal{H}$, we have

$$\langle Tx, x\rangle = \langle S^*Sx, x\rangle = \langle Sx, Sx\rangle = \|Sx\|^2 \geq 0.$$

Now, suppose $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. By Exercise 7.42 from Day 1 lecture notes, $T = T^*$ and so $\sigma(T) \subset \mathbb{R}$. So, given $\lambda < 0$ we want to show that $T - \lambda I$ is invertible. If $\lambda < 0$, then for every nonzero $x \in \mathcal{H}$,

$$\begin{aligned}
\|(T - \lambda I)x\|^2 &= |\langle (T - \lambda I)x, (T - \lambda I)x\rangle| \\
&= |\|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2| \\
&= \|Tx\|^2 + 2|\lambda|\langle Tx, x\rangle + |\lambda|^2\|x\|^2 \\
&\geq |\lambda|^2\|x\|^2.
\end{aligned}$$

That means that for every $x \in \mathcal{H}$, $\|(T - \lambda I)x\| \geq |\lambda|\|x\|$. In other words, the operator $T - \lambda I$ is bounded below, which means it is injective (Exercise 7.46 from Day 1 lecture notes). So, by the Open Mapping Theorem, to show that $T - \lambda I$ is invertible, it remains to show that it is surjective.

For any operator $S \in B(\mathcal{H})$, $\ker(S) = (S^*(\mathcal{H}))^\perp$ (Exercise 7.44 from Day 1 lecture notes). Since $(T - \lambda I) = (T - \lambda I)^*$, the above argument shows that $\ker(T - \lambda I) = 0 = ((T - \lambda I)(\mathcal{H}))^\perp$, which means $T - \lambda$ is surjective and thus invertible.

Theorem 3.10. *For any* $a \in A$*, the element* a^*a *is positive.*

Proof. Suppose $b = a^*a \in A$. Then b is self-adjoint, and hence by Exercise 3.4, we can write it as $b = b_+ - b_-$ for some $b_+, b_- \geq 0$ with $b_+b_- = 0$. We want to show that $b_- = 0$. Since it is self-adjoint, we know $\|b_-\| = r(\sigma(b_-))$, and so it suffices to show that $\sigma(b_-) = \{0\}$. Now, for notational ease, we write $c = a\sqrt{b_-}$, where $\sqrt{b_-}$ is its unique positive square root. By Exercise 3.5, we have that $\sqrt{b_-}b_+ = 0$, and so we compute

$$-c^*c = -\sqrt{b_-}a^*a\sqrt{b_-} = -\sqrt{b_-}b\sqrt{b_-} = -\sqrt{b_-}(b_+ - b_-)\sqrt{b_-} = b_-^2.$$

Then $-c^*c = b_-^2 \geq 0$, which means $\sigma(-c^*c) \subset [0, \infty)$.

Write $c = \mathrm{Re}(c) + i\mathrm{Im}(c)$ as in (1.1). Then we compute

$$\begin{aligned}
cc^* &= [c^*c + cc^*] - c^*c \\
&= [(\mathrm{Re}(c) + i\mathrm{Im}(c))^*(\mathrm{Re}(c) + i\mathrm{Im}(c)) + (\mathrm{Re}(c) + i\mathrm{Im}(c))(\mathrm{Re}(c) + i\mathrm{Im}(c))^*] - c^*c \\
&= 2(\mathrm{Re}(c)^2 + \mathrm{Im}(c)^2) + b_-^2.
\end{aligned}$$

Then cc^* is the sum of positive elements, and hence is positive. Since[2] $\sigma(cc^*) \cup \{0\} = \sigma(cc^*) \cup \{0\}$, it follows that both cc^* and c^*c have non-negative spectra, which means both are positive. But then we've shown that

[2]This is a more general ring theoretic fact that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ for any x, y in a complex unital ring. Indeed, if $0 \neq \lambda \notin \sigma(xy)$, then there exists z such that $z(\lambda - xy) = 1 = (\lambda - xy)z$. Then $\lambda^{-1}(\lambda + yzx)$ is the inverse of $\lambda - yx$. Check this if you haven't seen it before!

$\pm c^*c$ are both positive. It follows that $\sigma(c^*c) = \{0\}$. Since c^*c is self-adjoint, its norm is its spectral radius, and so

$$0 = \|c^*c\| = \|-c^*c\| = \|b_-^2\| = \|b_-\|^2,$$

and we are done. □

Exercise 3.11. Let A be a C*-algebra. Show the following:

(1) If $a, b \in A$ are self-adjoint with $a \leq b$ and $c \in A$, then $c^*ac \leq c^*bc$. (Hint: Take a square root and use the previous theorem.)
(2) Assuming A is a unital C*-algebra and $a \in A$ positive, show that $a \leq \|a\|1$. Moreover, $\|a\| \leq 1$ iff $a \leq 1$. In this case we also have $1 - a \leq 1$ and $\|1 - a\| \leq 1$.
(3) If A is unital and $a \in A$ is invertible, then so is a^*, a^*a, and $\sqrt{a^*a}$. Moreover, the inverses are in $\mathrm{C}^*(a)$.

3.1. **Polar decomposition.** For each $a \in A$, we define the positive operator $|a|$ to be the unique positive square root of a^*a, i.e.

$$|a| = \sqrt{a^*a}.$$

Proposition 3.12. *For each operator $T \in B(\mathcal{H})$, there is a unique partial isometry $U \in B(\mathcal{H})$ with* $\ker(U) = \ker(T)$ *and $U|T| = T$. Moreover $|T| \in \mathrm{C}^*(T)$ and $U \in \mathrm{C}^*(T)''$. If T is invertible, then U is a unitary.*

The description $T = U|T|$ is called the *polar decomposition* of T, in analogy with the fact that every complex number z can be written as a norm-1 element e^{it}, times a non-negative real number r. U is sometimes called the *polar part* of T and $|T|$ is the *positive part*.

Proof. Note that for all $\xi \in H$, we have

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle |T|^2\xi, \xi\rangle = \langle |T|\xi, |T|\xi\rangle = \||T|\xi\|^2. \tag{3.1}$$

It follows that the linear map $U_0 : |T|\mathcal{H} \to T\mathcal{H}$ given by $|T|x \mapsto Tx$ is isometric, and hence extends to an isometry $\overline{|T|\mathcal{H}} \to \overline{T\mathcal{H}}$ (also denoted U_0). We define $U \in B(\mathcal{H})$ to be U_0 on $\overline{|T|\mathcal{H}}$ and 0 on $(|T|\mathcal{H})^\perp$. It follows from Exercise 7.39 from the Day 1 Lecture Notes that U is a partial isometry with $U^*|_{\overline{T\mathcal{H}}} = U_0^{-1}$ and $\ker(U^*) = (T\mathcal{H})^\perp$, and by definition $U|T| = T$. Moreover, we have from (3.1) and Exercise 7.44 from the Day 1 Lecture Notes that $\ker(U) = |T|(\mathcal{H})^\perp = \ker(|T|) = \ker(T)$. For uniqueness, suppose $V \in B(\mathcal{H})$ is another partial isometry with $\ker(V) = \ker(T)$ and $V|T| = T$. Since $V|_{|T|\mathcal{H}} = U|_{|T|\mathcal{H}}$, it follows from continuity that they also agree on $\overline{|T|\mathcal{H}}$. As $\ker(V) = \ker(T) = \ker(U) = (|T|(\mathcal{H}))^\perp$ by construction, the fact that $\mathcal{H} = \overline{|T|\mathcal{H}} \oplus (|T|\mathcal{H})^\perp$ implies that $V\xi = U\xi$ for any $\xi \in \mathcal{H}$.

It follows from the functional calculus that $|T| \in \mathrm{C}^*(T)$. Now, suppose $S \in \mathrm{C}^*(T)'$. If $\xi \in \ker(T) = \ker(U)$, then $TS\xi = ST\xi = 0$ and so $S\xi \in \ker(T) = \ker(U)$. Then $US\xi = 0 = SU\xi$ for every $\xi \in \ker(T) = (|T|\mathcal{H})^\perp$. For $\xi = |T|\eta \in |T|\mathcal{H}$, we have

$$US\xi = US|T|\eta = U|T|S\eta = TS\eta = ST\eta = SU|T|\eta = SU\xi.$$

Since $|T|\mathcal{H}$ is dense in $\overline{|T|\mathcal{H}}$, it follows that $US = SU$ on $\overline{|T|\mathcal{H}}$ and on $(|T|\mathcal{H})^\perp$. Then it follows by a linearity argument as above that S and U commute. Hence $U \in \mathrm{C}^*(T)''$.

Finally, if T is invertible, then so is $\sqrt{T^*T}$. Then we have

$$U = T(T^*T)^{-1/2},$$

and one checks that $U^*U = UU^* = I$. □

As the range space of U is $\overline{T\mathcal{H}}$, and U is a partial isometry, it follows that $UU^* = \mathrm{proj}_{\overline{T\mathcal{H}}}$. Similarly, the source projection of U is $U^*U = \mathrm{proj}_{\overline{|T|\mathcal{H}}}$.

Corollary 3.13. *Let $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$. Then $|T^*| = U|T|U^*$ and $T^* = U^*|T^*|$.*

Proof. Observe that $U|T|U^*$ is positive, and since $U^*U = \mathrm{proj}_{\overline{|T|\mathcal{H}}}$ and $T^* = (U|T|)^* = |T|U^*$,

$$(U|T|U^*)(U|T|U^*) = U|T|^2U^* = TT^*.$$

By the uniqueness of the square root, we have $U|T|U^* = (TT^*)^{1/2} = |T^*|$. From this we further deduce

$$U^*|T^*| = U^*U|T|U^* = |T|U^* = (U|T|)^* = T^*.$$ □

Exercise 3.14. Where possible, give geometric as well as algebraic explanations for the following statements about the polar decomposition:

(1) $U^*U|T| = |T|$,
(2) $U^*T = |T|$, and
(3) $UU^*T = T$.

Exercise 3.15. Show that T is compact iff $|T|$ is compact.

In general, for $T \in B(\mathcal{H})$, the partial isometry U in the polar decomposition $T = U|T|$ is not in $\mathrm{C}^*(T)$. However, it turns out that if you take any continuous function $f \in C(\sigma(|T|)\backslash\{0\})$, the operator $Uf(|T|)$ is in $\mathrm{C}^*(T)$.

Proposition 3.16. *Let $T \in B(\mathcal{H})$ with polar decomposition $T = U|T|$, and $f \in C_0(\sigma(|T|)\backslash\{0\})$. Then $Uf(|T|) \in \mathrm{C}^*(T)$. Moreover, $U^*US = S$ for all $S \in \mathrm{C}^*(T)$.*

Proof. By Stone-Weierstraß, any $f \in C(\sigma(|T|))$ is the norm limit of polynomials. Moreover, if $f(0) = 0$, then we can assume the same for an approximating sequence of polynomials. (In other words $f \in C(\sigma(|T|)\backslash\{0\})$ can be approximated by polynomials in $C(\sigma(|T|)\backslash\{0\})$. Note that these are polynomials with zero constant term, i.e. $p(0) = 0$.) So, if the claim holds for all polynomials p with $p(0) = 0$, it holds for any $f \in C(\sigma(|T|)\backslash\{0\})$. Let $p(z) = \sum_{k=1}^n \lambda_k z^k$. Then

$$Up(|T|) = \sum_{k=1}^{n} \lambda_k U|T|^k = \sum_{k=1}^{n} \lambda_k T|T|^{k-1} \in \mathrm{C}^*(T).$$

□

Exercise 3.17. Describe the projections in $C_0(X)$ where X is

(1) $(0, 1]$,
(2) $[0, 1]$,
(3) $[0, 1/3] \cup [1/3, 1]$.

Exercise 3.18. Let $\pi : A \to B$ be a surjective $*$-homomorphism between C^*-algebras and $b \in B$ a positive element. Show that b lifts to a positive element $a \in A$ – that is, there is $a \in A$ with $\pi(a) = b$ – such that $\|a\| = \|b\|$.

4. Ideals, Approximate Units, and $*$-homomorphisms

Preview of Lecture: To help guide your reading, we indicate here which of the following material we will address in lecture and which we will assume familiarity with:

The lecture for this section will focus on Theorem 4.11. The techniques in the proofs of Lemma 4.6 and Theorem 4.9 do not translate well to lecture, but that does not detract from their importance. In fact, they showcase a powerful yet technical tool: an approximate unit (a.k.a. approximate identity). Many C*-algebraists (guilty!) are intimidated by these at first. But the first time you use them in your own research, you'll love them for life.

Definition 4.1. An *approximate identity* in a C^*-algebra A is an increasing net $(e_\lambda)_{\lambda\in\Lambda}$ of positive contractive elements (i.e. $0 \le e_\gamma \le e_\lambda$ and $\|e_\lambda\| \le 1$ for all $\lambda, \gamma \in \Lambda$ with $\lambda \ge \gamma$) such that

$$\lim_\lambda \|e_\lambda a - a\| = \lim_\lambda \|ae_\lambda - a\| = \lim_\lambda \|e_\lambda a e_\lambda - a\| = 0.$$

Theorem 4.2. *Every C^*-algebra has an approximate identity. Moreover, if the C^*-algebra is separable, the identity can be chosen to be countable.*

The proof relies heavily on the functional calculus. We will not give it here, though it is not too sophisticated. Instead, we point to the proofs given in [5, Theorem 1.4.8] or [7, Theorem 3.1.1].

Remark 4.3. A C*-algebra with a countable approximate identity is called *σ-unital*. Any separable C*-algebra is σ-unital, but there exist non-separable σ-unital C*-algebras. A silly example is $B(\ell^2)$ since it's actually unital; a non-silly example is $C_0(X)$ where X is a locally compact but not σ-compact Hausdorff space. Many results that hold in the separable setting can be generalized to the σ-unital setting.

There are a few interesting characterizations of a σ-unital C*-algebra, such as containing a *strictly positive element*, which is an element $h \in A$ such that $\phi(h) > 0$ for every nonzero positive $\phi \in A^*$. For more on this, see [11, Section 3.10].

Example 4.4. In $K(\ell^2)$, the projections

$$P_n : (\xi_n)_n \mapsto (\xi_1, ..., \xi_n, 0, 0, ...),$$

form an approximate identity.

For a general Hilbert space, we form the approximate identity for the compact operators by nets of projections with finite rank where the order is given by the natural order on the projections, i.e. $p \leq q$ iff $pq = qp = p$.

Exercise 4.5. Determine an approximate identity for $C_0((0, 1])$. (A sketch will do.)

Here is a quick application of approximate units.

Lemma 4.6. *Every closed two-sided ideal in a* C*-*algebra is self-adjoint.*

Proof. Let J be a closed two-sided ideal in A. Then $B = J \cap J^*$ is a C*-subalgebra of A such that $x^*x, xx^* \in B$ for all $x \in J$. Let (e_λ) be an approximate identity for B. Then for any $x \in J$, we have $x^*x - xx^*e_\lambda \in J$ and hence

$$\begin{aligned} \lim_\lambda \|x^* - x^*e_\lambda\|^2 &= \lim_\lambda \|(x - e_\lambda x)(x^* - x^*e_\lambda)\| \\ &= \lim_\lambda \|(xx^* - xx^*e_\lambda) - e_\lambda(xx^* - xx^*e_\lambda)\| = 0. \end{aligned}$$

Since $x^*e_\lambda \in J$, it follows that $x^* \in J$ and so $J = J^*$. $\square$

This means that every ideal in a C*-algebra is a C*-subalgebra, which means that each ideal has an approximate unit. In fact, more is true. We say a net (a_λ) in a C*-algebra A is *quasi-central* if $\lim_\lambda \|a_\lambda b - ba_\lambda\| = 0$ for every $b \in A$. We have the following extension of the above theorem ([5, Theorem I.9.16]).

Theorem 4.7. *Every ideal of a* C*-*algebra has a quasi-central approximate unit.*

Exercise 4.8. Suppose A is a C*-algebra with closed two-sided ideal $J \triangleleft A$ and C*-subalgebra $I \subset A$ such that $I \triangleleft J$. Show that $I \triangleleft A$.

An approximate identity will also enable us to prove that the quotient of any C*-algebra by a closed two-sided ideal is again a C*-algebra.

Theorem 4.9. *Let A be a* C*-*algebra and $J \triangleleft A$. Then A/J is a* C*-*algebra.*

Proof. Since $J \subset A$ is a Banach subalgebra, a basic result from functional analysis (cf. [4, Theorems III.4.2 and VII.2.6]) implies that A/J is a Banach algebra under the norm $\|a + J\| = \inf_{x \in J} \|a + x\|$. (**Exercise:** Prove it!) Moreover, from the fact that $\|b\| = \|b^*\|$ for all $b \in A$, a two-line calculation shows that $\|a + J\| = \|a^* + J\|$ for all $a \in A$. So, we just check the C*-identity for $\|a + J\| = \inf_{x \in J} \|a + x\|$. Let $a \in A$ and (e_λ) an approximate identity for J. First, we claim that $\|a + J\| = \lim_\lambda \|a - ae_\lambda\|$. Since $ae_\lambda \in J$ for each λ, the $\leq$ inequality is clear. For the other direction, let $\varepsilon > 0$ and $x \in J$ such that $\|a + J\| + \varepsilon > \|a - x\|$. By possibly passing to $\tilde{A}$, we assume A is unital. Then by Exercise 3.11, $\|1 - e_\lambda\| \leq 1$, and

$$\begin{aligned} \lim_\lambda \|a - ae_\lambda\| &\leq \lim_\lambda \|(a - x)(1 - e_\lambda)\| + \|x - xe_\lambda\| \\ &\leq \lim_\lambda \|a - x\| + \|x - xe_\lambda\| \\ &\leq \|a - x\| < \|a + J\| + \varepsilon. \end{aligned}$$

Now, we can check the C*-norm:

$$\begin{aligned}\|(a+J)^*(a+J)\| = \|a^*a+J\| &= \lim_\lambda \|a^*a(1-e_\lambda)\| \geq \lim_\lambda \|1-e_\lambda\|\|a^*a(1-e_\lambda)\| \\ &\geq \lim_\lambda \|(1-e_\lambda)aa^*(1-e_\lambda)\| = \lim_\lambda \|a(1-e_\lambda)\|^2 = \|a+J\|^2 \\ &= \|a^*+J\|\|a+J\| \geq \|(a+J)^*(a+J)\|.\end{aligned}$$

□

Exercise 4.10. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras. Check that $\ker(\pi)$ is a closed two-sided ideal in A and the quotient map $q : A \to A/\ker(\pi)$ is a $*$-homomorphism.

Now, we are ready to build on Proposition 1.21 to get a very powerful theorem for $*$-homomorphisms.

Theorem 4.11. *An injective $*$-homomorphism between* C**-algebras is isometric. The image of any $*$-homomorphism between* C**-algebras is a* C**-algebra (in particular, the range of any $*$-homomorphism between* C**-algebras is closed.*

Proof. Recall from Proposition 1.21 that a $*$-homomorphism $\phi : A \to B$ between C*-algebras is contractive and for any $a \in A$, $\phi(\sigma(a)) \subset \sigma(a)$. We give the proof under the assumption that our C*-algebras and our maps are all unital and leave the adaption to the non-unital setting as an exercise.

Let $\phi : A \to B$ be an injective $*$-homomorphism. Note that for any $a \in A$, $\|a\|^2 = \|a^*a\|$ and $\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\|^2$, so by Theorem 3.10, it suffices to prove that $\|\phi(a)\| = \|a\|$ for $a \in A$ positive. Suppose $\|\phi(a)\| < \|a\|$ for some positive $a \in A$. Note that $\phi(a) \geq 0$ since $a = b^*b$ for some $b \in A$, and so $\phi(a) = \phi(b)^*\phi(b)$. So, the assumption that $\|\phi(a)\| < \|a\|$ is equivalent to the assumption that $r(a) := \alpha > \beta := r(\phi(a))$. Using the continuous functional calculus, we identify $C^*(a) = C_0(\sigma(a)\backslash\{0\}) \subset C_0((0,\alpha])$ and $C^*(\phi(a)) = C_0(\sigma(\phi(a))\backslash\{0\}) \subset C((0,\beta])$. Now, define $f \in C((0,\alpha])$ so that $f|_{(0,\beta]} = 0$, $f(\alpha) = 1$, and f is affine on $[\beta,\alpha]$.

Then

$$\|f(a)\| = \sup_{\lambda\in\sigma(a)} |f(\lambda)| = 1,$$

but

$$\|f(\phi(a))\| = \sup_{\lambda\in\sigma(\phi(a))} |f(\lambda)| = 0.$$

In particular, $f(a) \neq 0$ and $f(a) \in \ker\phi$, contradicting ϕ being injective.

Now, suppose $\pi : A \to B$ is a $*$-homomorphism with kernel $J = \ker(\pi)$. Then A/J is a C*-algebra by Theorem 4.9. Let $q : A \to A/J$ be the quotient map. Then q is a $*$-homomorphism and π factors through the quotient A/J, i.e. there exists a bijective $*$-homomorphism $\rho : A/J \to \pi(A)$ given by $\rho(q(a)) = \pi(a)$. (Indeed, this is just the first isomorphism theorem for algebras. The map ρ is $*$-preserving because q and π are: $\rho(q(a)^*) = \rho(q(a^*)) = \pi(a^*) = \pi(a)^* = \rho(q(a))^*$.)

So, it follows that $\rho : A/J \to B$ is an injective $*$-homomorphism between C*-algebras, which by the first part of this theorem, means that it is isometric. It follows from this that its image $\pi(A)$ is closed in B. □

Exercise 4.12. Extend this to the general case where the assumptions that A, B, and ϕ are not unital. Here's an idea of what to check. If A is not unital, then we can extend ϕ to $\tilde{A}$ as we did in Proposition 1.18 to map $1 \in \tilde{A}$ to $1 \in B$ or $1 \in \tilde{B}$ depending on whether or not B is unital. If A is unital, then check that $\phi(1)$ is the unit in the C*-subalgebra $C^*(\phi(A)) \subset B$, and we can just replace B with this C*-subalgebra in the proof.

Remark 4.13. There is a class of C*-subalgebras called *hereditary subalgebras*, which generalizes the notion of ideal. A C*-subalgebra $A \subset B$ is hereditary if for any positive elements $a \in A$ and $b \in B$, if $b \leq a$, then $b \in A$. It turns out that ideals are always hereditary ([5, Theorem 1.5.3]).

Definition 4.14. A *representation* of a C*-algebra A is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We say a representation π is *nondegenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

A paradigm example of a degenerate representation is where $\mathcal{H}$ decomposes as a nontrivial direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\pi(A)$ can be realized as a $*$-subalgebra of operators on $B(\mathcal{H}_1)$ identified with the operators whose kernels contain $\mathcal{H}_2$.

Remark 4.15. Non-degeneracy is a regular assumption, which avoids some obnoxious pitfalls. Many times theorems which are phrased for nondegenerate representations still hold without this assumption. The trick usually amounts to taking a degenerate representation $\pi : A \to B(\mathcal{H})$ and to define its restriction to the closure of $\pi(A)\mathcal{H}$. Though some delicacy may be required after this, depending on what statement you are trying to prove. We will point out an example later. (Theorem 11.17)

Exercise 4.16. We say a family of representations $\{\pi_i : A \to B(\mathcal{H}_i)\}_{i\in I}$ for a C*-algebra A is *separating* if for any $a, b \in A$, there exists $i \in I$ such that $\pi_i(a) \neq \pi_i(b)$. Define $\pi : A \to B(\oplus_i \mathcal{H}_i)$ by $\pi(a) = \oplus_i \pi_i(a)$. Show that π is a *faithful* representation, i.e. an isometric representation, if the family $\{\pi_i\}_{i\in I}$ is separating.

Now, suppose $\{a_j\}_{j\in J}$ is a dense subset of A. We cannot conclude from knowing that $\{\pi_i\}_{i\in I}$ is separating for $\{a_i\}_{i\in I}$ that π is faithful (why?). However, if we know that for each $j \in J$, there exists $i \in I$ such that $\|\pi_i(a_j)\| = \|a_j\|$, then we can conclude that π is faithful (why?).

Exercise 4.17. Suppose A is a commutative C*-algebra and X a locally compact Hausdorff space so that $A \simeq C_0(X)$. If A is separable, what does that say about X? (Yes, X is actually $\hat{A}$, but we call it X to remove all the extra distracting information.)

5. Group C*-algebras

Preview of Lecture: Today's C*-lectures will discuss 3 classes of examples of C*-algebras: group C*-algebras, AF algebras, and Cuntz–Krieger algebras. There's a quick preview at the beginning of each section.

For the group C*-algebras, in lecture, we'll discuss Proposition 5.7 and Example 5.9. We'll save Proposition 5.10 for Wednesday.

Most of the steps of the proof of Proposition 5.17 are relatively straightforward; the one which requires the most creativity is the fact that $h(\omega) \in \widehat{C_r^*(G)}$ for all $\omega \in \widehat{G}$ so we'll discuss that in lecture.

A useful source of examples and motivation for C*-theory are the *group C*-algebras*. Indeed, one can view a group C*-algebra as encoding the (infinite-dimensional) representations of the group. (See Exercise 5.12.) Understanding these representations better was a main motivation for a lot of the early work on C*-algebras, and group C*-algebras are still a fundamental source of examples and inspiration for research today.

Definition 5.1. Let G be a discrete group. The *complex group algebra* $\mathbb{C}G$ is the algebra generated by $\{u_g : g \in G\}$, where $u_g u_h = u_{gh}$.

By definition, then, $\mathbb{C}G$ consists of all finite products of finite linear combinations of $\{u_g : g \in G\}$.

Observe that $\mathbb{C}G$ is always unital (what's the unit?). Moreover, we have a natural involution on $\mathbb{C}G$:

$$(a_g u_g)^* := \overline{a_g} u_{g^{-1}}.$$

(Check for yourself that this formula indeed gives an involution.)

Given two finite linear combinations of generators $\sum_{g\in G} a_g u_g, \sum_{g\in G} b_g u_g \in \mathbb{C}G$, then the formula for the multiplication of the generators $\{u_g\}_{g\in G}$ implies that

$$\left(\sum_{g\in G} a_g u_g\right)\left(\sum_{g\in G} b_g u_g\right) = \sum_{h\in G}\left(\sum_{k\in G} a_k b_{k^{-1}h}\right) u_h.$$

This multiplication may look familiar if you've seen convolution multiplication or the Fourier transform before. For functions ϕ, ψ on a discrete group G, their *convolution product* is

$$\phi * \psi(g) := \sum_{h\in G} \phi(h)\psi(h^{-1}g).$$

That is, if we think of the coefficients $(a_g)_{g\in G}$ of an element $\sum_{g\in G} a_g u_g \in \mathbb{C}G$ as a function from G to $\mathbb{C}$, then the function associated to the product $(\sum_{g\in G} a_g u_g)(\sum_{g\in G} b_g u_g)$ is precisely the convolution product of the functions $(a_g)_{g\in G}$ and $(b_g)_{g\in G}$.

If we want to complete the $*$-algebra $\mathbb{C}G$ into a C*-algebra, we first need a norm. In our case this will come from a representation.

Definition 5.2. A *representation* of a $*$-algebra A is a $*$-preserving homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If A is unital, we will assume π is *unital* in that it takes the unit of A to the unit of $B(\mathcal{H})$. If π is injective we say that it is *faithful*.

Note that if π is a representation of $\mathbb{C}G$ and $a \in \mathbb{C}G$, then the fact that $B(\mathcal{H})$ is a C*-algebra implies that

$$\|\pi(a^*a)\| = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

In particular, the norm on A induced by π, $\|a\|_\pi := \|\pi(a)\|$, satisfies the C*-identity. Therefore,

$$C^*_\pi(G) := \overline{\pi(\mathbb{C}G)}$$

is a C*-algebra.

Exercise 5.3. If π is a representation of $\mathbb{C}G$, what sort of operator will $\pi(u_g)$ be? Can you say anything about $\|\pi(u_g)\|$?

There is a natural representation of $\mathbb{C}G$ on $\ell^2(G) = \overline{\operatorname{span}}\{\delta_g : g \in G\}$, called the *left regular representation* and often denoted by λ: On the generators, we define

$$\lambda(u_g)(\delta_h) = \delta_{gh},$$

and extend λ to $\mathbb{C}G$ by requiring it to be a linear multiplicative map.

Exercise 5.4. What is the adjoint of $\lambda(u_g)$? Is λ $*$-preserving?

Observe (check!) that λ is injective. So, we can think of $\mathbb{C}G$ as a subalgebra of $B(\ell^2(G))$. The *reduced group* C*-*algebra* $\mathrm{C}^*_r(G)$ is defined to be

$$\mathrm{C}^*_r(G) := \overline{\lambda(\mathbb{C}G)}.$$

So that we don't always have to choose a specific representation (and for abstract-nonsense reasons) we often want to work with the *universal group* C*-*algebra* $\mathrm{C}^*(G)$, which is defined to be the completion of $\mathbb{C}G$ in the *universal norm*

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G\}. \tag{5.1}$$

A reader who is familiar with set theory might notice that we have made no assertion about whether the collection of all representations of $\mathbb{C}G$ is a set. How, then, do we know that we can take the supremum in (5.1)? Recall that, for any $a \in \mathbb{C}G$ and any representation π of $\mathbb{C}G$, the quantity $\|\pi(a)\|$ is a real number, being the norm of an operator on some Hilbert space. So the collection in (5.1) is a subclass of the set of all real numbers, and basic results from set theory guarantee that a subclass of a set is still a set. It follows that the universal norm is well defined.

In fact, the universal norm is bounded above by the ℓ^1 norm:

Proposition 5.5. *If π is a representation of $\mathbb{C}G$, then for any $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$ we have $\|\pi(a)\| \leq \sum_{g\in F} |a_g|$.*

Proof. Since $\pi(u_g)$ is a unitary for all g, and hence has norm 1, the triangle inequality tells us that

$$\|\pi(a)\| \leq \sum_{g\in F} \|a_g u_g\| = \sum_{g\in F} |a_g|. \qquad \square$$

It follows that if a net in $\mathbb{C}G$ is Cauchy in the ℓ^1 norm, then that net is also Cauchy in $\mathrm{C}^*(G)$ (and $\mathrm{C}^*_r(G)$). In other words, we could alternatively think of $\mathrm{C}^*(G)$ and $\mathrm{C}^*_r(G)$ as completions in a C*-norm of $\ell^1(G)$. This will come in handy sometimes, for example in Section 5.1.

Proposition 5.6. *$\mathbb{C}G$ is dense in both $\mathrm{C}^*_r(G)$ and $\mathrm{C}^*(G)$.*

Proof. The fact that $\mathbb{C}G$ is dense in $\mathrm{C}^*_r(G)$ follows from the injectivity of λ. Similarly, to see that $\mathbb{C}G$ is dense in $\mathrm{C}^*(G)$, it will suffice to show that if $a \in \mathbb{C}G$ is nonzero, then $\|a\|_u \neq 0$. Since $\|a\|_u \geq \|\lambda(a)\|$ by the definition of the universal norm, it follows that $\|a\|_u = 0$ implies $a = 0$. $\square$

The reason we call $\mathrm{C}^*(G)$ the "universal group C*-algebra" is the following proposition. While the argument used in the proof is straightforward, it's a very powerful technique for constructing $*$-homomorphisms out of many examples of C*-algebras, not just group C*-algebras.

Proposition 5.7. *For any representation π of $\mathbb{C}G$, there is an associated surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(G) \to \mathrm{C}^*_\pi(G)$.*

Proof. We define $\hat{\pi}$ first for $a \in \mathbb{C}G \subseteq \mathrm{C}^*(G)$:

$$\hat{\pi}(a) := \pi(a) \in \mathrm{C}^*_\pi(G).$$

As π is a representation of $\mathbb{C}G$, in order to extend $\hat{\pi}$ to a $*$-homomorphism on all of $\mathrm{C}^*(G)$, I claim that it suffices to check that $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$. Why? Well, once we know that $\|\hat{\pi}(a)\| \leq \|a\|_u$ for all $a \in \mathbb{C}G$, then if $x \in \mathrm{C}^*(G)$ is a norm limit of elements in $\mathbb{C}G$, $x = \lim_i a_i$, then in particular, given any $\varepsilon > 0$, we can find I such that $\|a_i - a_j\|_u < \varepsilon$ whenever $i, j \geq I$. If $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$, then it follows that $(\hat{\pi}(a_i))_i$ is Cauchy in $\mathrm{C}^*_\pi(G)$. As $\mathrm{C}^*_\pi(G)$ is complete, $\lim_i(\hat{\pi}(a_i))_i$ has a limit, call it y. Defining $\hat{\pi}(x) := y$, one can check that $\hat{\pi}(x)$ is independent of the approximating Cauchy sequence $(a_i)_i \subseteq \mathbb{C}G \subseteq \mathrm{C}^*(G)$, and that this definition makes $\hat{\pi}$ into a $*$-homomorphism.

Thus, it (essentially) suffices to check that $\|\hat{\pi}(a)\| \leq \|a\|_u$ for all $a \in \mathbb{C}G \subseteq \mathrm{C}^*(G)$. However, the definition of the universal norm makes this immediate:

$$\|\hat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u. \qquad \square$$

Exercise 5.8. Fill in the gaps in the proof of Proposition 5.7. (This includes checking that $\hat{\pi}$ is surjective.)

Example 5.9. Let $G = \mathbb{Z}$ (under addition). Observe that if $u \in B(\mathcal{H})$ is a unitary, then we obtain a representation $\pi : \mathbb{C}\mathbb{Z} \to B(\mathcal{H})$ given by defining $\pi(u_1) = u$. (where u_1 corresponds to the cyclic generator of $\mathbb{Z}$). Conversely, any representation π of $\mathbb{C}\mathbb{Z}$ arises in this way.

It follows that, for any $u \in B(\mathcal{H})$, there is a surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(\mathbb{Z}) \to \mathrm{C}^*(\{u\})$. In other words, $\mathrm{C}^*(\mathbb{Z})$ is the *universal* C^**-algebra generated by a unitary.*

Now, consider $\mathrm{C}^*_r(\mathbb{Z})$. The Fourier transform $\mathcal{F}$ gives us a unitary isomorphism $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$,

$$\mathcal{F}(\xi)(z) = \sum_{n \in \mathbb{Z}} \xi_n z^n,$$

which takes convolution multiplication to pointwise multiplication. That is, if we define, for $f \in C(\mathbb{T})$, the operator $M_f \in B(L^2(\mathbb{T}))$ by

$$M_f \xi(z) = f(z)\xi(z),$$

then the Fourier transform implements an isomorphism

$$\mathrm{C}^*_r(\mathbb{Z}) \cong \{M_f : f \in C(\mathbb{T})\} \subseteq B(L^2(\mathbb{T})).$$

However, one easily checks that the $*$-algebra structure on $\{M_f : f \in C(\mathbb{T})\}$ agrees with the $*$-algebra structure on $C(\mathbb{T})$, and $\|M_f\| = \|f\|_\infty$, so $\{M_f : f \in C(\mathbb{T})\} \cong C(\mathbb{T})$ as C^*-algebras.

Finally, consider the C^*-algebra $C(\mathbb{T})$. The Stone-Weierstrass Theorem (cf. [4, Theorem I.5.6]) tells us that $C(\mathbb{T})$ is generated, as a C^*-algebra, by the function

$$f(z) = z.$$

It turns out that $C(\mathbb{T})$ can also be described as the universal C^*-algebra generated by a unitary. That is,

$$\mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}^*_r(\mathbb{Z}) \cong C(\mathbb{T}).$$

Proposition 5.10. *If $G \leq H$ then $\mathrm{C}^*(G)$ is a norm-closed subalgebra of $\mathrm{C}^*(H)$. The same is true for the reduced C^*-algebras.*

Proof. Let $\iota : \mathbb{C}G \to \mathbb{C}H$ denote the canonical inclusion. We first claim that if we view $\mathbb{C}G$ (respectively $\mathbb{C}H$) as a subalgebra of $\mathrm{C}^*(G)$ (resp. $\mathrm{C}^*(H)$), then ι is norm-decreasing. It then follows (using the same argument as in Proposition 5.7) that ι induces an $*$-homomorphism $\tilde{\iota} : \mathrm{C}^*(G) \to \mathrm{C}^*(H)$.

To see that ι is norm-decreasing, observe that every representation of $\mathbb{C}H$ restricts to a representation of $\mathbb{C}G$. Thus, the set used in (5.1) to compute the universal norm for G contains the set

$$\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G \text{ which extends to a representation of } \mathbb{C}H\}.$$

It follows that $\|\iota(a)\|_{u,H} \leq \|a\|_{u,G}$ for all $a \in \mathbb{C}G$.

The proof that $\tilde{\iota}$ is injective will be relatively straightforward once we've proved the Gelfand-Naimark-Segal Theorem, so we'll come back to it. $\square$

Here are two more structural results about $\mathrm{C}^*(G)$.

Proposition 5.11.

(1) $\mathrm{C}^*(G)$ is never simple unless $G = \{e\}$ is trivial.
(2) If $|G| = n$ and G is abelian, then $\mathrm{C}^*(G) \cong \mathbb{C}^n$.

Proof. (1) For any group G, there is a representation π of $\mathbb{C}G$ on $\mathbb{C}$, given by

$$\pi(u_g) = 1, \qquad \forall\, g \in G.$$

Observe that π is onto. If $G \neq \{e\}$, then we can choose $g \neq h \in G$, and

$$u_g - u_h \in \ker \pi.$$

Thus, $\ker \pi$ is a nontrivial ideal in $\mathrm{C}^*(G)$.

(2) As a vector space, $\mathbb{C}G = \mathbb{C}^{|G|}$, which is already complete, so $\mathbb{C}G \cong \mathrm{C}^*(G)$ is a finite dimensional vector space. Notice also (Exercise 5.14) that if G is abelian, so is $\mathbb{C}G$ and hence $\mathrm{C}^*(G)$. Since every finite dimensional C*-algebra is a direct sum of matrix algebras by Proposition 6.1 and any nontrivial matrix algebra is nonabelian, the result follows. □

Exercise 5.12. Recall that the set $U(\mathcal{H})$ of unitaries in $B(\mathcal{H})$ is a group under multiplication. A *unitary representation* of a group G is a group homomorphism $\rho : G \to U(\mathcal{H})$. Show that representations of $\mathbb{C}G$ are in bijection with unitary representations of G.

Remark 5.13. In this section we've focused on discrete groups and their C*-algebras. However, one can also define the group C*-algebra for any group G which has a locally compact Hausdorff topology with respect to which multiplication and inversion are continuous (for short, these are called *locally compact* groups). While a lot of the theory of (discrete) group C*-algebras goes through smoothly in the locally compact setting, Proposition 5.10 is a major exception: it is not true for locally compact groups. For example, consider $\mathbb{R}$ under addition. It turns out that $\mathrm{C}^*(\mathbb{R}) = C_0(\mathbb{R})$, and $\mathbb{Z}$ is a subgroup of $\mathbb{R}$, but $\mathrm{C}^*(\mathbb{Z}) \cong C(\mathbb{T})$ is not a subalgebra of $C_0(\mathbb{R})$. This example highlights the other major exception: Proposition 5.6. Notice that $C_0(\mathbb{R})$ is not unital. In particular, it contains no units, let alone a copy of $\mathbb{R}$– that's right, $\mathrm{C}^*(\mathbb{R})$ does not contain $\mathbb{R}$.

5.1. Abelian group C*-algebras.

If G is abelian, then $u_g u_h = u_h u_g$ for all $g, h \in G$, and so $\mathbb{C}G$ is also abelian.

Exercise 5.14. Show that any C*-completion of $\mathbb{C}G$ is an abelian C*-algebra.

By Exercise 5.14 and the Gelfand-Naimark Theorem (Theorem 2.11), it follows that $\mathrm{C}^*_r(G) = C_0(\widehat{G})$ for some locally compact Hausdorff space $\widehat{G}$. In fact, $\widehat{G}$ must be compact since $\mathbb{C}G$ (hence $\mathrm{C}^*_r(G)$) is unital. So what is this space $\widehat{G}$ exactly?

From the Gelfand-Naimark Theorem, we know we have $\widehat{G} = \widehat{\mathrm{C}^*_r(G)}$, the spectrum of $\mathrm{C}^*_r(G)$. However, I've used the new symbol $\widehat{G}$ deliberately.

Definition 5.15. For an abelian group G, $\widehat{G}$ denotes the *Pontryagin dual* of G:

$$\widehat{G} = \{\omega : G \to \mathbb{T} \text{ group homomorphism}\}. \tag{5.2}$$

Exercise 5.16. Show that $\widehat{G}$ is also a group, under pointwise multiplication. Do you need to assume G is abelian?

Our next main goal is to prove Proposition 5.17, which shows that $\widehat{G}$ and $\widehat{\mathrm{C}^*_r(G)}$ are homeomorphic. In order to do that, we need to identify the topology on $\widehat{G}$.

The topology on $\widehat{G}$ (when G is discrete) is the *point-norm topology*: a net $(\omega_i)_{i\in\Lambda} \subseteq \widehat{G}$ is Cauchy iff, for all $g \in G$, the nets $(\omega_i(g))_{i\in\Lambda} \subseteq \mathbb{T}$ are Cauchy.[3] Equivalently, a basis for the topology on $\widehat{G}$ consists of the sets

$$B_{\varepsilon,F}(\omega) := \{\eta \in \widehat{G} : |\eta(g) - \omega(g)| < \varepsilon \ \forall\ g \in F \text{ finite}\}.$$

Proposition 5.17. *The map $h : \widehat{G} \to \widehat{\mathrm{C}^*_r(G)}$ given by, for $\omega \in \widehat{G}$ and $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$,*

$$h(\omega)(a) = \sum_{g\in G} a_g \omega(g), \tag{5.3}$$

is a homeomorphism of topological spaces.

Proof. We first need to show that the formula for $h(\omega)$ given in Equation (5.3) does indeed define an element of $\widehat{\mathrm{C}^*_r(G)}$. We begin by showing that $h(\omega)$ is a $*$-algebra homomorphism. If $b = \sum_{g\in G} b_g u_g$ is another element of $\mathbb{C}G$,

$$h(\omega)(ab) = \sum_{g\in G} \left(\sum_{h\in G} a_h b_{h^{-1}g} \right) \omega(g),$$

[3] If G abelian but not discrete, its Pontryagin dual still exists, but the topology is that of uniform convergence on compact sets. For discrete groups, these are the same.

whereas the fact that ω is a group homomorphism implies that

$$h(\omega)(a)\cdot h(\omega)(b) = \left(\sum_{g\in G} a_g\omega(g)\right)\left(\sum_{h\in G} b_h\omega(h)\right) = \sum_{k\in G}\left(\sum_{h\in G} a_{kh^{-1}}b_h\right)\omega(k).$$

Making the change of variable $h\mapsto h^{-1}k$, we see that $h(\omega)(ab) = h(\omega)(a)\cdot h(\omega)(b)$ as claimed. Similarly, since $\omega(g^{-1}) = \omega(g)^{-1} = \overline{\omega(g)}$,

$$h(\omega)(a^*) = \sum_{g\in G}\overline{a_g}\omega(g^{-1}) = \overline{\sum_{g\in G} a_g\omega(g)} = (h(\omega)(a))^*.$$

To see that our formula for $h(\omega)$ extends to a bounded linear functional on $\mathrm{C}^*_r(G)$, we need to show that $|h(\omega)a| \le \|a\|_r$ for all $a\in\mathbb{C}G$. To that end, we first observe that for any $\chi\in\widehat{\mathrm{C}^*_r(G)}$, if we define

$$\tilde{a} = \sum_{g\in G} a_g\omega(g)\overline{\chi(u_g)}u_g,$$

then $h(\omega)(a) = \chi(\tilde{a})$. Since the Gelfand transform is isometric, it follows that

$$\|\tilde{a}\|_r = \sup\{|\eta(\tilde{a})| : \eta\in\widehat{\mathrm{C}^*_r(G)}\} \ge |\chi(\tilde{a})| = |h(\omega)(a)|.$$

We will therefore show that $\|\tilde{a}\|_r = \|a\|_r$. To that end, given $\xi\in\ell^2(G)$, define $\tilde{\xi}$ by

$$\tilde{\xi}_h = \chi(u_h^{-1})\overline{\omega(h)}\xi_h.$$

Since u_h is a unitary for each $h\in G$, and χ is a $*$-homomorphism, it follows that $\|\tilde{\xi}\|_2^2 = \|\xi\|_2^2$. Moreover,

$$\lambda(\tilde{a})\tilde{\xi}(g) = \sum_{k\in G} a_k\omega(k)\overline{\chi(u_k)}\tilde{\xi}_{k^{-1}g} = \sum_k a_k\omega(k)\overline{\chi(u_k)}\chi(u_{g^{-1}k})\overline{\omega(k^{-1}g)}\xi_{k^{-1}g},$$

and since both χ and ω are multiplicative, we see that

$$\lambda(\tilde{a})\tilde{\xi}(g) = \omega(g)\chi(u_g^{-1})\sum_k a_k\xi_{k^{-1}g} = \omega(g)\chi(u_g^{-1})(\lambda(a)\xi)(g).$$

As $|\omega(g)| = |\chi(u_g^{-1})| = 1$, we have $\|\lambda(\tilde{a})\tilde{\xi}\|_2^2 = \|\lambda(a)\xi\|_2^2$. It follows that

$$\|\tilde{a}\|_r \le \sup\{\|\lambda(\tilde{a})\tilde{\xi}\|_2 : \|\xi\|_2 = 1\} = \sup\{\|\lambda(a)\xi\|_2 : \|\xi\|_2 = 1\} = \|a\|_r.$$

(A symmetric argument shows the other inequality, so that $\|\tilde{a}\|_r = \|a\|_r$.) In other words,

$$|h(\omega)a| \le \|\tilde{a}\|_r = \|a\|_r,$$

so our formula for $h(\omega)$ determines an element of $\widehat{\mathrm{C}^*_r(G)}$ as claimed.

The fact that h is continuous is a fairly straightforward argument using the definition of the weak-$*$ topology. Suppose $(\omega_i)_{i\in\Lambda}\subseteq\widehat{G}$ is Cauchy. We need to see that $(h(\omega_i))_{i\in\Lambda}$ is Cauchy, i.e. we need to show that for any $a\in\mathrm{C}^*(G)$ the net $(h(\omega_i)(a))_{i\in\Lambda}\subseteq\mathbb{C}$ is Cauchy. If $a\in\mathbb{C}G$, so that $a = \sum_{g\in G} a_g u_g$ and $a_g = 0$ for all but finitely many g, choose

$$\varepsilon < \frac{1}{|\{g : a_g\neq 0\}|}\min\{\frac{1}{|a_g|} : a_g\neq 0\}.$$

Since $(\omega_i)_{i\in\Lambda}$ is Cauchy, and $a_g\neq 0$ for only finitely many g, we can choose I such that if $i,j\ge I$ then

$$|\omega_i(g) - \omega_j(g)| < \varepsilon \text{ whenever} a_g\neq 0.$$

For $i,j\ge I$, we have $|h(\omega_i)(a) - h(\omega_j)(a)| < \varepsilon$.

If $a\in\mathrm{C}^*(G)$ is the limit of a sequence $(a_n)_{n\in\mathbb{N}}\subseteq\mathbb{C}G$, then an $\varepsilon/3$ argument and the fact that each $h(\omega_i)$ is norm-decreasing will tell us that again, $(h(\omega_i)(a))_{i\in\Lambda}$ is Cauchy. It follows that $(h(\omega_i))_{i\in\Lambda}$ is Cauchy, as desired.

Checking that h is bijective is also straightforward. Given $\phi\in\widehat{\mathrm{C}^*_r(G)}$, define $\omega_\phi : G\to\mathbb{C}$ by

$$\omega_\phi(g) := \phi(u_g).$$

Observe first that since ϕ is a $*$-homomorphism, $\phi(u_g)\in\mathbb{T}$ for all g, so in order to show that $\omega\in\widehat{G}$ we only need to show that ω is multiplicative. But this follows immediately from the fact that ϕ is a $*$-homomorphism:

$$\omega_\phi(g)\omega_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_gu_h) = \phi(u_{gh}) = \omega_\phi(gh).$$

It is similarly immediate to check that for a fixed $\omega \in \widehat{G}$, $\omega_{h(\omega)} = \omega$, and that $h(\omega_\phi) = \phi$. It follows that $\omega \mapsto h(\omega)$ is a bijection.

Finally, we conclude the proof by showing that the inverse function $h^{-1} : \widehat{C^*_r(G)} \to \widehat{G}$, given by $h^{-1}(\phi) = \omega_\phi$, is continuous. Suppose that $(\phi_i)_i \subseteq \widehat{C^*_r(G)}$ is Cauchy – that is, for any $a \in C^*_r(G)$ the net $(\phi_i(a))_i \subseteq \mathbb{C}$ is Cauchy. In particular, the net

$$(\phi_i(u_g))_i = (\omega_{\phi_i}(g))_i \subseteq \mathbb{T}$$

is Cauchy for each $g \in G$. By definition, then, h^{-1} is continuous. □

6. AF algebras

Preview of Lecture: By definition, AF algebras are inductive limits. So, before reading this section, it would probably be a very good idea to review the section about inductive limits from the Prerequisite Notes.

The first page of this section will be touched on very lightly in lecture – which is to say, you should work through this material for yourself, and ask questions in office hours or lecture about any points where you get stuck.

We will talk about Bratteli diagrams in lecture, probably via Example 6.9.

The last three paragraphs of this section are meant to provide inspiration for future reading or research; no need to read them now (unless you're bored) and we won't discuss them in lecture.

Proposition 6.1. *If A is a C*-algebra which is finite dimensional as a vector space, then*

$$A \cong \bigoplus_{s=1}^{j} M_{n(s)}(\mathbb{C})$$

is a finite direct sum of matrix algebras.

This proof is surprisingly intricate, and relies on the Gelfand-Naimark-Segal Theorem, which we'll see on Wednesday. So we'll postpone the proof for now.

Definition 6.2. A C*-algebra A is an *AF algebra* or *approximately finite dimensional C*-algebra* if A is the inductive limit of a sequence of finite-dimensional C*-algebras.

The following Proposition was mentioned in the Prerequisite Notes, but not proved there.

Proposition 6.3. *If $A = \overline{\bigcup_n A_n}$ is the norm closure of an increasing union of subalgebras $A_n \subseteq A_{n+1} \subseteq \cdots \subseteq A$, then A is the inductive limit of the directed system (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map.*

Proof. It suffices to check that A satisfies the universal property of the inductive limit. So, suppose that B is a C*-algebra and that we have $*$-homomorphisms $\psi_n : A_n \to B$ such that $\psi_m \circ \iota_{mn} = \psi_n$ whenever $n \leq m$. Given $a \in A$, write $a = \lim_{n\to\infty} a_n$ where $a_n \in A_n$. The fact that our connecting maps are inclusions means that if $m \geq n$, $a_n = \iota_{mn}(a_n) \in A_m$. Thus, if N is large enough that $\|a_m - a_n\| < \varepsilon$ if $m \geq n \geq N$, then

$$\|\psi_m(\iota_{mn} a_n) - \psi_m(a_m)\| = \|\psi_m(a_n - a_m)\| < \varepsilon.$$

As $\psi_m \circ \iota_{mn} = \psi_n$, it follows that $(\psi_n(a_n))_n$ is Cauchy in B. We define $\psi : A \to B$ by $\psi(a) = \lim_n \psi_n(a_n)$ if $a = \lim_n a_n$ with $a_n \in A_n$. □

Exercise 6.4. Complete the proof of Proposition 6.3 by showing that ψ is well-defined (independent of the choice of sequence $(a_n)_n$); $*$-preserving; and multiplicative.

Example 6.5 (cf. Example 6.2 from the Prerequisite Notes)**.** $K(\ell^2)$ is an AF algebra. To see this, write P_n for the projection onto $\text{span}\{e_1, \ldots, e_n\}$ and observe that $M_n \cong P_n K(\ell^2) P_n$. Since $\overline{\bigcup_n P_n K(\ell^2) P_n} = \overline{FR(\ell^2)} = K(\ell^2)$, the result follows by applying the previous Proposition.

Remark 6.6. In the above example, we were discussing the compact operators on a fixed $\mathcal{H} = \ell^2$. However, (cf. Exercise 7.54 from Day 1) if two Hilbert spaces $\mathcal{H}, \mathcal{K}$ have the same dimension, with orthonormal bases

$\{\xi_n\}_n, \{\eta_n\}_n$ respectively, then the map $U : \mathcal{H} \to \mathcal{K}$ given by $U(\xi_n) = \eta_n$ is a unitary. In particular (this is another **exercise**) the map $\mathrm{Ad}(U) : B(\mathcal{H}) \to B(\mathcal{K})$ given by

$$\mathrm{Ad}(U)(T) = UTU^*$$

is a C*-algebra isomorphism. In particular, it takes $FR(\mathcal{H})$ to $FR(\mathcal{K})$ and $K(\mathcal{H})$ to $K(\mathcal{K})$.

So, if $\mathcal{H}$ is *any* Hilbert space with a countable orthonormal basis, then $K(\mathcal{H})$ is isomorphic to $K(\ell^2)$ (and in particular is an AF algebra). Because of this, and the fact that algebras of compact operators are (as we'll see) both ubiquitous and indispensable, we often talk about "the compact operators" as shorthand for $K(\ell^2)$, or $K(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$. In the literature, the Hilbert space is often dropped altogether, and the compact operators are denoted $\mathcal{K}$ (not to be confused with the Hilbert space $\mathcal{K}$ that we have occasionally used in these notes).

By construction, Example 6.3 of the Prerequisite Notes describes an AF algebra. Here it is again.

Example. Let $A_n = M_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Letting $\phi_{n,m} := \phi_{m,m-1} \circ \cdots \circ \phi_{n,n+1}$ whenever $m > n$, we see that by construction this forms a directed system. Since these are inclusions, one can identify the inductive limit with $\bigcup_{n\in\mathbb{N}} A_n$. ■

This is a particularly important one, known as M_{2^∞} or the CAR algebra. In fact, it's an example of a UHF algebra.

Definition 6.7. An AF algebra A is a *UHF* or *uniformly hyperfinite* algebra if A is the inductive limit of a sequence of full matrix algebras, where the connecting maps are unital embeddings.

Exercise 6.8. Is $K(\ell^2)$ a UHF algebra?

Example 6.9. [5, Example III.3.7] One can obtain quite different C*-algebras from the same sequence of finite-dimensional C*-algebras (A_n), if one uses different connecting maps.

For example, let $A_n = \mathbb{C}^{2^n}$. On the one hand, let X denote the standard middle-third Cantor set, so that $X = \bigcap_n C_n$, where $C_n \subseteq [0, 1]$ is the collection of 2^n intervals that remain after step n in the construction of X. We can construct $C(X)$ as an inductive limit of the algebras A_n, by identifying A_n with the set of functions on C_n that are locally constant.

In this case, since $C_n \supseteq C_{n+1}$, the connecting maps $\iota_n : A_n \to A_{n+1}$, and the structure maps $\phi^n : A_n \to C(X)$, are given by restriction. It follows that the connecting maps are injective, so $\varinjlim(A_n, \iota_n) = \overline{\bigcup_n A_n}$ by Proposition 6.3. And a straightforward $\varepsilon - \delta$ proof will show you that the set of functions which are constant on some C_n is dense in $C(X)$ – that is, $C(X) = \overline{\bigcup_n A_n} = \varinjlim(A_n, \iota_n)$.

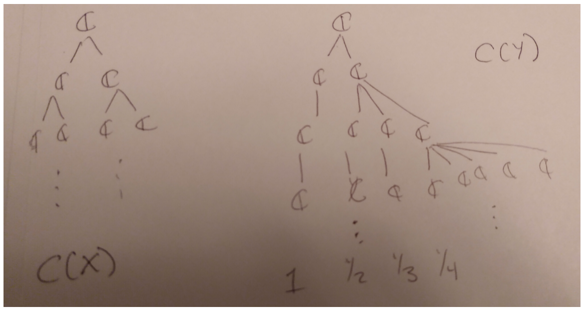
On the other hand, consider the space $Y = \{0\} \cup \{1/n : n \in \mathbb{Z}_{>0}\}$. Write $B_n \subseteq C(Y)$ for the set of functions which are constant on $[0, 2^{-n}]$. Then $B_n \cong C(\{1/k : 1 \leq k \leq 2^n\}) \cong \mathbb{C}^{2^n} \cong A_n$. Again, the connecting maps $\jmath_n : B_n \to B_{n+1}$ are given by inclusion, and $\bigcup_n B_n$ is dense in $C(Y)$, so $C(Y) = \varinjlim(B_n, \jmath_n)$. But clearly $C(Y) \not\cong C(X)$.

What do the connecting maps $\iota_n, \jmath_n$ look like when we identify both A_n and B_n with $\mathbb{C}^{2^n}$? We have

$$\iota_n(f)(z_1, \ldots, z_{2^{n+1}}) = f(z_1, z_3, \ldots, z_{2^{n+1}-1}), \qquad \text{and} \qquad \jmath_n(f)(z_1, \ldots, z_{2^{n+1}}) = f(z_1, \ldots, z_{2^n}).$$

In other words, $\iota_n(z_1, z_2, \ldots, z_{2^n}) = (z_1, z_1, z_2, z_2, \ldots, z_{2^n}, z_{2^n})$ and $\jmath_n(z_1, \ldots, z_{2^n}) = (z_1, z_2, \ldots, z_{2^n}, z_{2^n}, \ldots, z_{2^n})$.

One sees the difference even more clearly via the *Bratteli diagram* of the AF algebras. If $A = \varinjlim(A_n, \phi_n)$, with $A_n = \bigoplus_{j=1}^{k(n)} M_{r(j)}$, and the connecting maps $\phi_n : A_n \to A_{n+1}$ are inclusions, the Bratteli diagram consists of $\mathbb{N}$ levels, with $k(n)$ nodes at each level, and an edge from a node v at level n to a node w at level $n + 1$ if ϕ_n maps the vth matrix algebra into the wth matrix algebra. For example, below are the Bratteli diagrams for $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, \jmath_n)$.



Exercise 6.10. Show that any AF algebra has an approximate identity which consists of an increasing sequence of projections.

Exercise 6.11. Show that any AF algebra is isomorphic to a direct limit of finite-dimensional C*-algebras with *injective* connecting maps.

Exercise 6.12. Show that if $A = \varinjlim(A_n, \phi_n)_n$ and we set $B_n := A_{n+k}$ for a fixed $k \geq 0$, then we also have $A = \varinjlim(B_n, \phi_{n+k})_n$.

Because AF algebras are quite tractable, it's natural to ask which C*-algebras are subalgebras of AF algebras. That is, given a C*-algebra A, when can we find an injective $*$-homomorphism $\phi : A \to B$ for some AF algebra B? This simple-seeming question was only answered recently [Schafhauser 2018], under mild assumptions on A.

Exercise 6.13.

(1) Prove that $C([0,1])$ is not an AF algebra.
(2) If X is the Cantor set, show that $C(X)$ is AF.
(3) Show that a subalgebra of an AF algebra needn't be AF, by constructing an embedding of $C([0,1])$ into $C(X)$.

However, despite the intricacy of the structure of the subalgebras of AF algebras, the lattice of ideals of an AF algebra is easy to describe: [5, Theorem III.4.2] the ideals of an AF algebra are in bijection with directed hereditary subsets of its Bratteli diagram.

One can have two different directed systems that give rise to the same C*-algebra. An example is the UHF algebra $M_{2^\infty 3^\infty} = \varinjlim(A_n, \iota_n) = \varinjlim(B_n, \iota_n)$, where

$$A_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n+1)/2}3^{(n-1)/2}}, & n \text{ odd}; \end{cases} \qquad B_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n-1)/2}3^{(n+1)/2}}, & n \text{ odd}. \end{cases}$$

The nodes at odd levels in the Bratteli diagrams of $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, \iota_n)$ are not isomorphic, nor is the number of edges between levels.

Fortunately, there is a complete invariant for AF algebras – a way to tell whether or not two AF algebras are isomorphic. G. Elliott proved in 1978 that the ordered K-theory $(K_0(A), K_0(A)_+, [1])$ of an AF algebra is a classifying invariant for A, in that given two AF algebras A, B, their K-theory groups are order isomorphic – $(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B])$ – if and only if $A \cong B$. You'll hear about K-theory from Mark Tomforde next week, and [5, Chapter IV] has a proof of Elliott's classification theorem for AF algebras.

7. Cuntz–Krieger Algebras

Preview of Lecture: This section is a quick introduction to a class of (I think) fascinating C*-algebras. Unfortunately, a lot of what makes them so fascinating is beyond the scope of GOALS, but if you want to learn more, I'd recommend picking up Raeburn's book [12] on graph algebras.

For today, try to get a feel for the algebraic consequences of the relations (7.1) defining a Cuntz–Krieger algebra; you may want to pick a (small-ish) matrix B and think about what the associated C*-algebra might look like. Infinite and purely infinite C*-algebras show up in a lot of places, so it's also a good idea to build an understanding of these by playing with some examples and non-examples (cf. Exercise 7.4.)

Again, the last five paragraphs of this section are in there to inspire you to dig deeper into these Cuntz–Krieger algebras in the future;[4] don't worry too much about them now.

In this section, B will denote an $n \times n$ matrix with entries from $\{0,1\}$. The *Cuntz–Krieger algebra* $\mathcal{O}_B$ [Cuntz-Krieger 1981] associated to B is the universal C*-algebra generated by n partial isometries $s_1, \ldots, s_n$ such that, for each $1 \leq i \leq n$, we have

$$s_i^* s_i = \sum_{j=1}^{n} B_{ij} s_j s_j^* \qquad \text{and} \qquad s_i^* s_j = 0 \text{ if } i \neq j. \tag{7.1}$$

What do I mean by the "universal C*-algebra"? As with the group C*-algebra, $\mathcal{O}_B$ is the "largest" C*-algebra generated by n partial isometries which satisfy (7.1). That is, if $S_1, \ldots, S_n$ are partial isometries in a C*-algebra A which satisfy Equation (7.1), then there is a surjective $*$-homomorphism $\pi_S : \mathcal{O}_B \to C^*(\{S_1, \ldots, S_n\})$ such that $\pi_S(s_i) = S_i$. One can prove (cf. [12, Proposition 1.21] or [2, II.8.3]) that this universal object exists.

Proposition 7.1. *If B is a finite matrix, the Cuntz–Krieger algebra $\mathcal{O}_B$ is unital.*

Proof. Let $S = \sum_{i=1}^n s_i s_i^*$. Observe that, for any i,

$$Ss_i = s_i + \sum_{j \neq i} s_j s_j^* s_i = s_i, \qquad s_i S = s_i s_i^* s_i S = s_i \left(\sum_{j=1}^{n} B_{ij} s_j s_j^* \right) \left(\sum_{k=1}^{n} s_k s_k^* \right) = s_i \left(\sum_{j=1}^{n} B_{ij} s_j s_j^* \right) = s_i.$$

The fact that S is a projection (**Exercise:** check this!) implies that we consequently have, for any word w in the generators s_i and their adjoints, $Sw = wS$. In other words, S is the unit of $\mathcal{O}_B$. ☐

One can define a Cuntz–Krieger algebra for an infinite matrix, too, as long as the matrix is *row-finite* – for each i, the entries in row i of B have a finite sum. We need B to be row-finite because otherwise the first equation in (7.1) would involve an infinite sum of projections, which are mutually orthogonal by the second condition of (7.1). But an infinite sum of mutually orthogonal projections cannot converge in norm, yet the first equation in (7.1) requires that.

Example 7.2. If B is the $n \times n$ matrix of all 1s, then $s_i^* s_i = S$ for all i. That is, each s_i is an isometry, not merely a partial isometry, and $\sum_{i=1}^n s_i s_i^* = 1$. In this case, $\mathcal{O}_B$ is the *Cuntz algebra* $\mathcal{O}_n$.

The Cuntz algebras were introduced by J. Cuntz in 1977 as the first explicit examples of separable simple infinite C*-algebras.

Definition 7.3. A unital C*-algebra A is *infinite* if there exists $a \in A$ with $a^*a = 1$ but $aa^* \neq 1$.

Exercise 7.4.

(1) Is $B(\ell^2)$ infinite? What about $\mathcal{K}(\ell^2)$?
(2) If a unital C*-algebra A is infinite, when can it have a trace?

Cuntz showed that, moreover, the algebras $\mathcal{O}_n$ are all *purely infinite*: for any nonzero $x \in \mathcal{O}_n$, there exist $a, b \in \mathcal{O}_n$ with $axb = 1$. (Observe that any unital purely infinite C*-algebra is *a fortiori* simple.)

In addition to being separable and purely infinite, the algebras $\mathcal{O}_n$ have a lot of other intriguing properties that you'll learn about in the coming weeks (or in your future classes on C*-algebras): they're nuclear, they can be realized as a crossed product of a UHF algebra, they're not inductive limits of type I C*-algebras. $\mathcal{O}_2$ and $\mathcal{O}_\infty$ (defined to be the universal C*-algebra generated by infinitely many isometries $s_i, i \in \mathbb{N}$, such that for any n we have $\sum_{i=1}^n s_i s_i^* \leq 1$) behave particularly nicely with respect to tensor products.

Some of above properties are shared by general Cuntz–Krieger algebras $\mathcal{O}_B$. They are again nuclear, for example – the proof of this is based on the description of $\mathcal{O}_B$ as a groupoid C*-algebra. (You'll see more about groupoid C*-algebras in Robin Deeley's expository talk next week.) The groupoid picture of $\mathcal{O}_B$ arises from a certain type of dynamical system, called a *shift of finite type*, associated to B, and it turns out [Cuntz-Krieger 1981; Franks 1984; Rørdam 1995] that the K-theory of $\mathcal{O}_B$ is a classifying invariant for

[4] A word of warning, though: in the literature, Cuntz–Krieger algebras are usually denoted $\mathcal{O}_A$. I broke with tradition in these notes because we wanted to continue to reserve the letter A for C*-algebras.

these shifts of finite type. That is, the shifts of finite type associated to matrices B_1, B_2 are flow equivalent iff $K_0(\mathcal{O}_{B_1}) \cong K_0(\mathcal{O}_{B_2})$.

Another useful perspective on $\mathcal{O}_B$ is as a *graph* C*-*algebra.* One can think of B as being the adjacency matrix of a directed graph E_B on n vertices: in E_B, there is an edge from vertex i to vertex j iff $B_{ij} \neq 0$. The graph C*-algebra (cf. [12]) $C^*(E_B)$ is isomorphic to $\mathcal{O}_B$.

It turns out [12, Theorem 4.9], as for AF algebras, the ideals in a Cuntz–Krieger algebra $\mathcal{O}_B$ are in bijection with hereditary saturated subsets of the vertices of E_B.

Cuntz–Krieger algebras, graph C*-algebras, and generalizations such as higher-rank graph algebras and groupoid C*-algebras, are very active areas of current research.

8. The Gelfand-Naimark-Segal (GNS) Theorem

Preview of Lecture: In lecture, we won't discuss the proofs of the technical results we'll need about states for this lecture (eg Lemmas 8.8 and 8.11). However, these are important both for von Neumann algebraic applications and for C*-algebras, so you should read the proofs carefully and ask questions in office hours if you're confused.

We will prove Theorem 8.9 in lecture as well as Theorem 8.1. We'll discuss irreducible representations but, depending on time, perhaps not the proof of Proposition 6.1. We will, however, discuss the proof of Proposition 5.10.

There are a lot of exercises in this section! If there's time, we'll discuss a few in lecture (so please let us know if there are any that you'd particularly like to see).

The main goal of this section is the following theorem:

Theorem 8.1 (Gelfand-Naimark)**.** *Every* C**-algebra A admits a faithful nondegenerate representation* $\pi : A \to B(\mathcal{H})$*. If A is separable, π can be chosen to be separable.*

As an immediate corollary, every C*-algebra A is isomorphic to a norm-closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. (It can be useful to take this as the definition of a C*-algebra, which justified our using the term "C*-algebra" for abstract (not concretely represented) C*-algebras.)

Throughout this section, we generally assume A is unital, for simplicity; the arguments can all be made in general, by taking some care with approximate identities. See Exercise 8.15 below (and Remark 8.16 for why we can't just unitize our way out of this one).

Our first step on the road to proving Theorem 8.1 has to do with *states*.

Definition 8.2. A *state* on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ which is *positive* in that $\phi(a) \geq 0$ whenever $a \geq 0$, and such that

$$\|\phi\| := \sup\{|\phi(a)| : \|a\| = 1\} = 1.$$

The closed convex (**exercise**) subset $\mathcal{S}(A) \subset A^*_{\leq 1}$ consisting of states is called the state space.

Example 8.3. If π is a representation of A on $\mathcal{H}$, and $h \in \mathcal{H}$ has norm 1, the function

$$\phi(a) := \langle \pi(a)h, h\rangle$$

is a state on A.

Exercise 8.4. If $\pi : A \to \mathbb{C}$ is a character, then it is a representation (by 2.15). What is the state corresponding to a character π?

Exercise 8.5. Show that any positive linear functional $\phi : A \to \mathbb{C}$ is $*$-preserving, i.e. $\phi(a^*) = \overline{\phi(a)}$ for all $a \in A$.

Exercise 8.6. Show that $\mathcal{S}(A)$ is a closed convex subset of $A^*_{\leq 1}$. It follows from Alaoglu's theorem that it is weak*-compact. What does the Krein-Milman theorem say about $\mathcal{S}(A)$?

Given a state[5] ϕ on A, if we define $[a, b]_\phi := \phi(b^*a)$, then this form on A is sesquilinear (linear in the first variable, conjugate linear in the second variable) and satisfies the Cauchy-Schwarz inequality:

Exercise 8.7. Show that $|[a, b]_\phi| \leq [a, a]_\phi \, [b, b]_\phi = \phi(a^*a)\phi(b^*b)$.

Here are a few facts about states that we will need later.

Lemma 8.8. *Let A be a unital* C**-algebra.*

(1) If ϕ is a state on A, then $\phi(1) = 1$.
(2) If ϕ is a bounded linear functional on A which satisfies $1 = \|\phi\| = \phi(1)$, then ϕ is a state.

---

[5] Actually, all you need is a positive linear functional for the following assertions and exercise.

Proof. If ϕ is a state, then $|\phi(1)| = \phi(1) \leq \|\phi\| = 1$. For the other inequality, Exercise 8.7 tells us that
$$|\phi(a)|^2 \leq \phi(1)\phi(a^*a).$$
Moreover, since $\|a^*a\| \geq a^*a$ by the functional calculus, we have $\phi(a^*a) \leq \|a^*a\|\phi(1)$. It follows that for any $a \in A$ with $\|a\| = 1$,
$$|\phi(a)|^2 \leq \phi(1)^2,$$
and hence $\|\phi\| \leq \phi(1)$. We conclude that $1 = \|\phi\| = \phi(1)$.

If $1 = \phi(1) = \|\phi\|$, then once we know that ϕ is positive, ϕ must be a state. Pick $a \in A_+$ and write $\phi(a) = \alpha + i\beta$. If necessary, replace a with $-a$ to ensure that $\beta \geq 0$. We begin by showing that $\beta = 0$. Fix $n \in \mathbb{N}$ and observe that, since $a^* = a$,
$$\|n - ia\|^2 = \|(n + ia^*)(n - ia)\| = \|n^2 + in(a^* - a) + a^2\| \leq n^2 + \|a\|^2.$$
On the other hand, $|\phi(n - ia)|^2 = |n\phi(1) - i\alpha + \beta|^2 = (n+\beta)^2 + \alpha^2$. So,
$$(n^2 + \|a\|^2) = \|\phi\|^2(n^2 + \|a\|^2) \geq \|\phi\|^2 \, \|n - ia\|^2 \geq |\phi(n - ia)|^2 = n^2 + 2n\beta + \beta^2 + \alpha^2.$$
In order for this inequality to hold for all $n \in \mathbb{N}$ we must have $\beta = 0$, as claimed.

To complete the proof that ϕ is positive, fix a positive $a \in A_+$ with $\|a\| \leq 1$. Then Proposition 3.6 implies that $\|1 - a\| \leq 1$. Since $\|\phi\| = 1$ by hypothesis,
$$1 \geq \|1 - a\| \geq \phi(1 - a) = \phi(1) - \phi(a) = 1 - \phi(a).$$
It follows that $\phi(a) \geq 0$ for any positive a. □

The next theorem is the cornerstone of our proof of Theorem 8.1.

Theorem 8.9 (GNS construction)**.** *If ϕ is any state on a unital* C*-*algebra A, there is a nondegenerate representation $\pi_\phi : A \to B(\mathcal{H})$ and a unit vector $h \in \mathcal{H}$ such that $\phi(a) = \langle \pi_\phi(a)h, h\rangle$ for any $a \in A$.*

Such a vector h is called a *cyclic vector* for the representation π.

Proof. We will build $\mathcal{H}$ out of A itself. Let $N_\phi = \{a \in A : [a,a]_\phi = 0\}$. Observe (check!) that N_ϕ is a vector subspace of A, which is closed in norm. (The fact that N_ϕ is closed under addition follows from Exercise 8.7. Proving that N_ϕ is closed in norm is also a good exercise.)

In fact, Exercise 8.7 actually proves that N_ϕ is a left ideal in A: if $x \in N_\phi$ and $a \in A$ then
$$|[ax, ax]_\phi| = |\phi(x^*(a^*ax))| \leq \phi(x^*x)\phi(x^*(a^*a)^2x) = 0,$$
so $ax \in N_\phi$.

Therefore, let X be the vector space quotient $X = A/N_\phi$, and define an inner product on X by
$$\langle a + N_\phi, b + N_\phi\rangle_\phi := \phi(b^*a).$$
The fact that N_ϕ is a left ideal means that $\langle \cdot, \cdot \rangle_\phi$ is well defined.

Take $\mathcal{H}_\phi$ to be the completion of X with respect to the norm induced by $\langle \cdot, \cdot\rangle_\phi$. Then our representation $\pi_\phi : A \to B(\mathcal{H}_\phi)$ is given by left multiplication: $\pi_\phi(a)(b + N_\phi) = ab + N_\phi$.

To see that π_ϕ is actually a representation, we need to check that $\pi_\phi(a)$ is a bounded linear operator for all a, and also check that π_ϕ is linear, multiplicative and $*$-preserving. In checking that π_ϕ is $*$-preserving, you will see why we defined $\langle \cdot, \cdot\rangle_\phi$ as we did.

We use the functional calculus to show that $\pi_\phi(a)$ is a bounded operator. Since $a^*a \in A$ is positive, we have $\|a^*a\|1 - a^*a \geq 0$ is a positive element of A. Thus, for any $x \in A$, Exercise 3.11 tells us that
$$0 \leq x^*(\|a^*a\|1 - a^*a)x = \|a^*a\|x^*x - x^*a^*ax,$$
so $(ax)^*ax \leq \|a^*a\|x^*x$. In particular,
$$\begin{aligned}\|\pi_\phi(a)\|^2 &= \sup\{\langle \pi_\phi(a)(x + N_\phi), \pi_\phi(a)(x + N_\phi)\rangle_\phi : \phi(x^*x) = 1\}\\ &= \sup\{\phi((ax)^*(ax)) : \phi(x^*x) = 1\}\\ &\leq \|a^*a\| = \|a\|^2.\end{aligned}$$

So, we conclude that π_ϕ is a representation of A on a Hilbert space $\mathcal{H}_\phi$. To see that π_ϕ is nondegenerate, suppose that $\pi_\phi(a)(x + N_\phi) = 0$ for all a. In particular, taking $a = 1$,
$$0 = \|\pi_\phi(1)(x + N_\phi)\|^2 = \langle x + N_\phi, x + N_\phi\rangle_\phi = \phi(x^*x),$$

so we must have $x \in N_\phi$.

Finally, note that the unit vector h such that $\phi(a) = \langle \pi(a)h, h\rangle$ is $h = 1 + N_\phi$. □

Remark 8.10. If you ever see in some proof in the literature a representation unceremoniously associated to some state (in particular for a trace that is moreover a state), it's assumed to be the GNS representation constructed as above.

To prove Theorem 8.1, we will take the direct sum of a lot of the representations whose existence we have just established.

Lemma 8.11. *Let A be a* C*-*algebra and a a nonzero normal element of A. Then there is a state τ on A such that $|\tau(a)| = \|a\|$.*

Proof. Let $B = C^*(\{a, 1\}) \subseteq \widetilde{A}$. Fix $\lambda \in \sigma(a)$ with $|\lambda| = r(a)$ maximal, and let $g_\lambda : C(\sigma(a)) \to \mathbb{C}$ be given by evaluation at λ. Observe that g_λ is a positive linear functional, and $g_\lambda(1) = 1$, so g_λ is a state on $C(\sigma(a))$.

Because g_λ can be viewed as a linear functional on the closed subspace B of $\widetilde{A}$, the Hahn-Banach Theorem tells us there is a norm one linear functional τ on A which extends g_λ. As $\tau(1) = g_\lambda(1) = 1$, Lemma 8.8 tells us that τ is also a state. Furthermore, as the Gelfand transform $\Gamma : B \overset{\cong}{\to} C(\sigma(a))$ takes a to the function $f(z) = z$, it follows that $|\tau(a)| = |\lambda| = r(a)$, which equals $\|a\|$ by the fact that the Gelfand transform is isometric. □

Corollary 8.12. *If $F \subseteq \mathcal{S}(A)$ is a subset of the states of A which is dense in the weak-$*$ topology, then for any $a \in A$,*

$$\sup\{|\phi(a)| : \phi \in F\} = \|a\|.$$

We are finally ready to prove our main theorem.

Proof of Theorem 8.1. Choose a subset F of $\mathcal{S}(A)$ which is dense in the weak-$*$ topology on $\mathcal{S}(A) \subseteq A^*$. Define $\pi := \bigoplus_{\phi \in F} \pi_\phi$, where π_ϕ is the representation arising from the state ϕ as in the previous Theorem.

Fix $a \in A$. Since $\phi(1) = 1$,

$$\|\pi(a)\|^2 = \sup_{\phi \in F} \|\pi_\phi(a)\|^2 = \sup\{\langle \pi_\phi(a^*a)\xi, \xi\rangle : \phi \in F, \xi \in \mathcal{H}_\phi\} \geq \sup_{\phi \in F} \langle \pi_\phi(a^*a)1, 1\rangle = \sup_{\phi \in F} \phi(a^*a) = \|a\|^2.$$

As π is a $*$-homomorphism and therefore norm-decreasing, it follows that $\|\pi(a)\| = \|a\|$ for all $a \in A$. The fact that π is nondegenerate follows from the fact that each π_ϕ is nondegenerate, which in turn follows from our construction of $\mathcal{H}_\phi$ as a completion of (a quotient of) A.

If A is separable, then [4, Theorem V.5.1] implies that $A^* \supseteq \mathcal{S}(A)$ is too, so we can take the set F to be countable. The separability of A implies the separability of $\mathcal{H}_\phi$ for each ϕ,[6] so $\mathcal{H}$ is separable. □

Definition 8.13. The representation

$$\pi_u := \bigoplus_{\phi \in \mathcal{S}(A)} \pi_\phi : A \to B\left(\bigoplus_{\phi \in \mathcal{S}(A)} \mathcal{H}_\phi\right) =: B(\mathcal{H}_u)$$

is called the *universal representation* of A.

For the sake of a faithful representation, we could instead form the direct sum over a weak*-dense subset of $\mathcal{S}(A)$.

Remark 8.14. This representation has a special extra property in that the associated von Neumann algebra $\pi_u(A)''$ is isometrically isomorphic to A^{**} (ask Roy). Both are often called the *enveloping von Neumann algebra* of A.

Exercise 8.15. Generalize the results in this section to non-unital C*-algebras. (In particular, you will have to show that an approximate unit $(e_\lambda)_\lambda$ becomes Cauchy in A/N_ϕ for any state ϕ, and hence gives rise to a cyclic vector in any GNS representation π_ϕ.)

[6] There is something to check here, since the norm on $\mathcal{H}_\phi$ is not the same as the norm on A. **Exercise:** How do they relate?

Remark 8.16. Why can't we just unitize in Exercise 8.15? Well, as easy as it was to always guarantee a unique extension of a $*$-homomorphism to the unitization, it is no longer true in general for positive linear maps. (We'll return to this in Proposition 9.21.) In fact, the non-unital version of Theorem 8.9 is required to prove this for states because it allows us to borrow from this fact for representations. The proof of this fact takes us a little off course, so we will state it here with reference:

[5, Corollary 1.9.7] Every state on a nonunital C*-algebra A extends uniquely to a state on $\tilde{A}$.

What does Theorem 8.1 say about Abelian C*-algebras? In this case, the Riesz-Markov-Kakutani representation theorem tells us that states on $C_0(X)$ are in bijection with probability measures on X, so that $\phi(f) = \int_X f\, d\mu_\phi$. Note that N_ϕ consists of the set of C_0 functions on X which are 0 off a μ_ϕ-null set. Thus, $\mathcal{H}_\phi = \overline{C_0(X)/N_\phi} \cong L^2(X, \mu_\phi)$, and π_ϕ represents $C_0(X)$ on $L^2(X, \mu_\phi)$ as multiplication operators:

$$\pi_\phi(f)\xi = x \mapsto f(x)\xi(x).$$

To me at least, this is reminiscent of the link between the continuous and the Borel functional calculus.

Exercise 8.17. What does the universal representation of an Abelian C*-algebra look like?

Exercise 8.18. Let A be a C*-algebra.

(1) Show that for any $b \in A$, there exists a representation $\pi : A \to B(\mathcal{H})$ and unit vector $h \in \mathcal{H}$ so that $\|\pi(b)x\| = \|b\|$. (Hint: Apply Lemma 8.11 to $a = b^*b$.)
(2) Use Exercise 4.16 to give a different argument for the last claim in Theorem 8.1, i.e. that any separable C*-algebra has a faithful separable representation.

8.1. **Applications.** We've already seen the GNS theorem invoked several times, for structural results about C*-algebras. Here are some of those delayed proofs.

Exercise 8.19. Show that if $0 \le a \le b$, then $\|a\| \le \|b\|$, without assuming a and b commute.

Exercise 8.20. Show that if the C*-algebra A is finite dimensional as a vector space, then we may take the Hilbert space $\mathcal{H}$ of Theorem 8.1 to be finite dimensional. *Hint:* Show that you only need finitely many states $\phi \in F$, and that H_ϕ is finite dimensional for all ϕ.

Exercise 8.21. Use the GNS theorem to give a very quick proof of Theorem 3.10.

Exercise 8.22. For a commutative C*-algebra A, what would a weak$*$-dense subspace of $\mathcal{S}(A)$ look like?

The following should rightfully be called a Definition/Theorem. The proof uses results that take us a little far afield, so we give it as a definition and refer you to [5, Lemma 1.9.1-Theorem 1.9.4] for a proof.

Definition 8.23. A representation $\pi : A \to B(\mathcal{H})$ of a C*-algebra A is *irreducible* if one of the following equivalent conditions hold:

(1) π has no proper invariant subspaces, i.e. no subspace $V \subsetneq \mathcal{H}$ so that $\pi(a)V \subset V$ for all $a \in A$.
(2) π has no proper invariant manifolds (i.e. subspaces which may or may not be closed).
(3) $\pi(A)' = \mathbb{C}1_{\mathcal{H}}$.

Under the additional assumption that π has a cyclic unit vector $h \in \mathcal{H}$, these are also equivalent to

(4) The state $a \mapsto \langle \pi(a)h, h\rangle$ is *pure*, i.e. it is an extreme point in the state space $\mathcal{S}(A)$.

Remark 8.24. We have a couple remarks on irreducible representations:

(1) First, it's sometimes helpful to see a non-example: Let $\pi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$ be two nondegenerate representations of A. Then $\pi_1 \oplus \pi_2 : A \to B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is not irreducible. (Evidently we don't bother with calling things "reducible".)
(2) Notice that a character on a C*-algebra is a pure state. (Indeed, for any states ϕ_1, ϕ_2 and $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$, the map $\alpha_1\phi_1 + \alpha_2\phi_2$ will not be multiplicative.) It turns out ([5, Lemma 1.9.10]) that you can use a Krein-Milman argument to strengthen parts (1) and (2) of Exercise 8.18 to hold for pure states/irreducible representations. Then an argument like part (3) will allow you to prove the conclusion of Corollary 8.12 where F consists of all pure states of A.
(3) Not every C*-algebra has a faithful irreducible representation. Such C*-algebras are called *primitive*.

Exercise 8.25. If $\pi : A \to B(\mathcal{H})$ is irreducible, what does that say about the von Neumann algebra $\pi(A)''$? (Looking for a one word answer.)

Proof of Proposition 6.1. Suppose that A is a finite dimensional C*-algebra. By GNS, view A as a subalgebra of $B(\mathcal{H})$, where $\mathcal{H}$ is finite dimensional. Thus, A is an algebra of *compact* operators.

It turns out [5, Corollary I.10.6] that every irreducible representation of $K(\mathcal{H})$ is unitarily equivalent to the identity representation. Thus, decompose the (identity) representation of A on $\mathcal{H}$ into a direct sum of irreducible representations $\pi_i : A \to B(\mathcal{H}_i)$ where $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Then each $\mathcal{H}_i$ must be finite dimensional, and we must have only finitely many terms in this direct sum decomposition, since $\mathcal{H}$ is finite dimensional. In other words, if $\mathcal{H}_i \cong \mathbb{C}^{n_i}$, then $\pi_i(A) \cong M_{n_i}$. Thus,

$$A = \bigoplus_i \pi_i(A) \cong \bigoplus_i M_{n_i}$$

as desired. ☐

The proof of Proposition 5.10 relies on *positive definite functions* on groups, and their connection with states on $\mathrm{C}^*(G)$.

Definition 8.26. Let G be a discrete group. A function $\psi : G \to \mathbb{C}$ is *positive definite* if, for any finite subset $F \subseteq G$, the matrix M^ψ in $M_F(\mathbb{C})$ given by

$$M^\psi_{s,t} = \psi(s^{-1}t)$$

is positive.

Proposition 8.27. *If ϕ is a state on $\mathrm{C}^*(G)$, then the function $\psi^\phi(g) = \langle \pi_\phi(u_g)1, 1\rangle_\phi$ is positive definite. Conversely, every positive definite function defines a state on $\mathrm{C}^*(G)$.*

Proof. If ϕ is a state on $\mathrm{C}^*(G)$, we compute that

$$M^{\psi^\phi}_{s,t} = \langle \pi_\phi(u_{s^{-1}t})1, 1\rangle = \langle \pi_\phi(u_t)1, \pi_\phi(u_s)1\rangle = \phi(u_t)\overline{\phi(u_s)}.$$

In other words, if T is the matrix with entries indexed by elements of G, such that the first column consists of the entry $\phi(u_s)$ in the sth row, and T is zero in all other columns, then $M^{\psi^\phi} = T^*T$ is positive. So ψ^ϕ is positive definite, as claimed.

For the converse, given a positive definite function ψ, define $\phi_\psi(\sum_g a_g u_g) := \frac{1}{\psi(e)} \sum_g a_g \psi(g)$. By construction, ϕ_ψ is a linear functional on $\mathbb{C}G$. Considering the set $F = \{e\}$ tells us that $\psi(e) > 0$, so ϕ_ψ is well defined, and moreover that

$$\phi_\psi(u_e) = \frac{\psi(e)}{\psi(e)} = 1. \tag{8.1}$$

Moreover, ϕ_ψ is bounded with respect to $\|\cdot\|_u$, because

$$|\phi_\psi(\sum_g a_g u_g)| = |\langle \pi_{\phi_\psi}(\sum_g a_g u_g)1, 1\rangle| \le \|\pi_{\pi_\psi}(\sum_g a_g u_g)\| \le \|\sum_g a_g u_g\|_u. \tag{8.2}$$

It now follows that $\|\phi_\psi\| = 1$: equation (8.1) implies that $\|\phi_\psi\| = \sup\{|\phi_\psi(f)| : \|f\|_u = 1\} \ge |\phi_\psi(u_e)| = 1$, and equation (8.2) implies that $\|\phi_\psi\| \le 1$. Thus, Lemma 8.8(2) tells us that ϕ_ψ extends to a state on $\mathrm{C}^*(G)$. ☐

Proof of Proposition 5.10. We first address the case of the reduced C*-algebras. Suppose $G \le H$ are discrete groups, and decompose $\ell^2(H) = \bigoplus_h \ell^2(Gh)$ via the right cosets of G. Notice that the left regular representation of $\mathbb{C}G \subseteq \mathbb{C}H$ on $\ell^2(H)$ preserves this decomposition, and $\ell^2(Gh) \cong \ell^2(G)$ (via a canonical isomorphism) for any $h \in H$. As the operator norm of a direct sum satisfies

$$\|f \oplus g\| = \max\{\|f\|, \|g\|\},$$

it follows that the norm induced on $\mathbb{C}G$ by the left regular representation λ^H is the same as the norm induced by λ^G. In other words, the inclusion $\mathbb{C}G \subseteq \mathbb{C}H$ is isometric with respect to the reduced norm, so $\mathrm{C}^*_r(G) \subseteq \mathrm{C}^*_r(H)$.

Now, we show that if $G \le H$ (and G is countable) then $\mathrm{C}^*(G) \le \mathrm{C}^*(H)$. The fact that G countable implies that $\mathrm{C}^*(G)$ is separable. In this case,$\mathrm{C}^*(G)^*$ is also separable, so there exists a faithful state ϕ on $\mathrm{C}^*(G)$: namely, for a weak-$*$ dense subset $\{\omega_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}(A)$, take $\phi = \sum_n 2^{-n}\omega_n$. It is straightforward to check that, thanks to the density of $\{\omega_n\}_n$, $\phi(a) = 0$ implies $a = 0$, so ϕ is indeed faithful.

Consider the positive definite function ψ^ϕ on G which Proposition 8.27 associates to ϕ. Extend it to ψ on H by setting $\psi(h) = 0$ whenever $h \notin G$. To see that ψ is positive definite, note first that if $s, t \in H$ and $sG \neq tG$, then $s^{-1}t \notin G$ and therefore $M^\psi_{s,t} = 0$. In other words, for any finite set F, M^ψ is block diagonal, where each block is indexed by $F \cap sG$ for a single *left* coset sG of G. Block diagonal matrices are positive precisely when each block is positive, so to see that ψ is positive definite it suffices to consider the matrices M^ψ associated to finite sets $F \subseteq sG$ which are contained in a single coset. For any $g, h \in G$,

$$M^\psi_{sg,sh} = \psi(g^{-1}h) = \psi^\phi(g^{-1}h) = M^{\psi^\phi}_{g,h},$$

so the fact that ψ^ϕ is positive definite implies that ψ is as well.

Now, consider the GNS representation π_ψ associated to ϕ_ψ. As ϕ_ψ and ϕ agree on $\mathrm{C}^*(G)$, it follows that for any $f \in \mathrm{C}^*(G)$,

$$\|\tilde\iota(f)\|_{u,H} \geq \|\pi_\psi(f)\| = \|\pi_\phi(f)\|.$$

The fact that ϕ is faithful means that π_ϕ is injective and therefore isometric, by Theorem 4.11: if $\pi_\phi(f) = 0$ then

$$\begin{aligned} 0 = \|\pi_\phi(f)\|^2 &= \sup\{\|\pi_\phi(f)[a]\|^2 : [a] \in \mathcal{H}_\phi = \overline{\mathrm{C}^*(G)}^{\|\cdot\|_\phi}, \|[a]\| = 1\} \\ &= \sup\{\phi(a^*f^*fa) : \phi(a^*a) = 1\} \geq |\phi(f^*f)| = |\phi(f)|^2 \end{aligned}$$

by Exercise 8.5. The fact that ϕ is faithful then implies that $f = 0$. In other words, $\|\pi_\phi(f)\| = \|f\|_{u,G}$ for any $f \in \mathrm{C}^*(G)$. So $\|\tilde\iota(f)\|_{u,H} \geq \|f\|_{u,G}$. As we saw in Monday's notes that $\tilde\iota : \mathrm{C}^*(G) \to \mathrm{C}^*(H)$ is norm-decreasing, it now follows that $\|\tilde\iota(f)\|_{u,H} = \|f\|_{u,G}$. Consequently, $\tilde\iota$ must be injective: if $\tilde\iota(f) = 0$ then $f = 0$. □

Remark 8.28. For those that are wondering whether all of this rigamarole about positive definite functions is really necessary: If you try to extend a state from $\mathrm{C}^*(G)$ to $\mathrm{C}^*(H)$ by just making it zero on all elements not coming from $\mathrm{C}^*(G)$, it's hard to prove directly that this extension is still a state.

Now that we have faithful representations, we are ready to give our first proof of a powerful and useful tool in C*-algebras. Aloud we usually reference it by saying something like, "contractions lift to contractions" (with the assumption that we can scale to get the full result).

Proposition 8.29. *Let $\pi : A \to B$ be a $*$-homomorphism between* C**-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.*

Proof. First, by possibly identifying B with its image inside $\tilde B$, we assume B is unital. Moreover, it suffices to show the claim for $\|b\| = 1$ (why?).

We by choosing any $a \in A$ with $\pi(a) = b$. Then $1 = \|\pi(a)\| \leq \|a\|$. If we have equality, then there is nothing to do. So, we assume $\|a\| > 1$, and hence also that its positive part $|a|$ has norm strictly greater than 1. By taking a faithful representation, we assume $A \subset B(\mathcal{H})$ and let $a = u|a|$ be the polar decomposition of a in $B(\mathcal{H})$.[7] Define a function $f \in C[0, \|a\|]$ by

$$f(t) = \begin{cases} t & ; \quad t \in [0,1] \\ 1 & ; \quad t \in (1, \|a\|] \end{cases}$$

Note that $f(|a|) \in \mathrm{C}^*(a) \subset \mathrm{C}^*(a, 1_\mathcal{H}) \simeq \tilde A$. Now, we define $g \in C[0, \|a\|]$ by

$$g(t) = \begin{cases} 1 & ; \quad t \in [0,1] \\ t^{-1} & ; \quad t \in (1, \|a\|] \end{cases}$$

If A is unital, then $g(|a|) \in A$, and if not, it's in $\mathrm{C}^*(A, 1_\mathcal{H})$. But notice that $tg(t) = f(t)$ for all $t \in [0, \|a\|]$ with $f(|a|) \in \mathrm{C}^*(|a|)$, and so[8]

$$uf(|a|) = u|a|g(|a|) = ag(|a|) \in \mathrm{C}^*(a) \subset A.$$

Moreover,

$$\|uf(|a|)\| \leq \|u\|\|f(|a|)\| = \|f(|a|)\| \leq 1.$$

[7] Yes, we are deviating from the "operators are capitalized" notation. Don't tell Brent and Rolando.

[8] Alternatively, we know automatically from Proposition 3.16 that $uf(|a|) \in A$.

(**Exercise** Why is $\|u\| = 1$?) (We could give a better estimate for the norm here, but this is all we need.) We claim $uf(|a|)$ is the desired lift of b. Indeed, let $\tilde{\pi}$ denote the unital extension of π to $\mathrm{C}^*(A, 1_{\mathcal{H}})$ (where we take $\tilde{\pi} = \pi$ if $\mathrm{C}^*(A, 1_{\mathcal{H}}) = A$, i.e. if A is unital). Then, by Exercise 2.20,

$$\pi(uf(|a|)) = \tilde{\pi}(uf(|a|)) = \tilde{\pi}(ag(|a|)) = \tilde{\pi}(a)g(|\tilde{\pi}(a)|) = bg(|b|).$$

But as a continuous function in $C(\sigma(|b|))$, $g = 1$. Hence $g(|b|) = 1$, and so $bg(|b|) = b$. So, $\pi(uf(|a|)) = b$ and moreover,

$$\|b\| \geq \|uf(|a|)\| \geq \|\pi(uf(|a|))\| = \|b\|.$$

$\square$

9. Completely Positive Maps

This section gives a very quick introduction to completely positive maps for C*-algebraists. If this is your first time seeing such maps defined, we recommend ignoring the non-unital generalities for this go around. Once you have a grasp of the unital setting, you'll understand what's going on, and you will know where to look if you ever need the non-unital generalizations in the future. With the exception of a few examples, we will stick with the unital assumption in lecture.

The lecture will focus mostly on understanding key examples of completely positive maps (Examples 9.5, 9.8, 9.9 and Exercise 9.10), the characterization of completely positive maps afforded by Stinespring's Dilation theorem (Theorem 9.22), and an understanding Arveson's Extension theorem (9.28) for completely positive maps into $B(\mathcal{H})$.

With time, we will give an overview of the proof of Stinespring's Dilation Theorem, which is a direct generalization of the GNS construction. In which case, it will be beneficial to have the GNS construction proof handy. This proof goes through some algebraic tensor products for vector spaces. If it feels too confusing, try revisiting it after we've had a treatment of tensor products next week.

Section 9.1 establishes some preliminary results and delves into dilation techniques. We encourage you to read through the various dilation tricks and try the corresponding exercises in Section 9.1. These are valuable tools, which we will not address in lecture.

This section concerns maps that preserve positivity even after matrix amplification. We will have to forego several important facts and results on (completely) positive maps. For a full treatment, we highly recommend Vern Paulsen's book: [9, Chapters 2,3,6,7].

We begin with what we mean by matrix amplification. Ignoring the norm for a moment, given a $*$-algebra A and some $1 \leq n < \infty$, we define $\mathrm{M}_n(A)$ to be the $n \times n$ matrices with entries in A (just as we would in more general ring theory).

$$\mathrm{M}_n(A) := \{[a_{ij}]_{1\leq i,j\leq n} \ : \ a_{ij} \in A, 1 \leq i,j \leq n\} \tag{9.1}$$

We will usually suppress the usual subscripts on the matrices, i.e. we write $[a_{ij}]$ for $[a_{ij}]_{1\leq i,j\leq n}$ (sometimes also $[a_{ij}]_{ij}$).

This also comes with a natural involution where $[a_{i,j}]^* = [a_{j,i}^*]$ for all $[a_{i,j}] \in \mathrm{M}_n(A)$.

Definition 9.1. For a linear map $\phi : A \to B$ between $*$-algebras we define, for each $n \geq 1$, the linear map

$$\phi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B), \quad \phi^{(n)}([a_{ij}]) = [\phi(a_{ij})].$$

The map $\phi^{(n)}$ is often called a *matrix amplification* of ϕ.

When A is a C*-algebra, there is a natural C*-norm on $\mathrm{M}_n(A)$, which is inherited from the norm on A in the following sense:

Recall from Exercise 7.50 from Day 1 Lectures that $\mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$ for any Hilbert space $\mathcal{H}$. Now (using Theorem 8.1), we faithfully represent A on some Hilbert space $\mathcal{H}$ with an injective $*$-homomorphism $\pi : A \to B(\mathcal{H})$. This induces a $*$-homomorphism $\pi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, which is also injective (check). Then we can define a norm on $\mathrm{M}_n(A)$ by $\|[a_{ij}]\| := \|\pi^{(n)}([a_{ij}])\|$ (injectivity implies this is a norm and not just a semi-norm), which will satisfy the C*-identity (because $(\pi^{(n)})^{-1} : \pi^{(n)}(\mathrm{M}_n(A)) \to \mathrm{M}_n(A)$ is a $*$-homomorphism).

The following inequality is a useful exercise, but we already have plenty of exercises. The argument is outlined in [13, Exercise 1.13].

Proposition 9.2. *For any* C*-*algebra A, $n \geq 1$, and $[a_{ij}] \in \mathrm{M}_n(A)$, we have*

$$\max_{i,j}\{\|a_{ij}\|\} \leq \|[a_{ij}]\| \leq \sum_{ij} \|a_{ij}\|.$$

9.1. Preliminary results on cp maps.

Unlike with the Gelfand-Naimark Theorem for commutative C*-algebras, we will not start from scratch here. However, results in this section are developed nicely in [9, Chapter 2]. The proofs therein are well-written and easy to follow, but we are after bigger fish and therefore will just take these as means to an end.

Definition 9.3. We say a linear map $\phi : A \to B$ between C*-algebras is *positive* if it maps positive elements to positive elements. We say it is *n-positive* if $\phi^{(n)}$ is positive, and we say that it is *completely positive* (c.p. or cp) if it is n-positive for all $n \geq 1$. A completely positive map that is unital is abbreviated *ucp*.

Remark 9.4. For notation and terminology: often the word "linear" is dropped when discussing cp maps, and $\phi^{(n)}$ is sometimes denoted by ϕ_n.

One important class of examples that we have already seen is positive linear functionals (such as the states used in the GNS representation theorem).

Example 9.5. For a unital C*-algebra A, a positive linear functional $\phi \in A^*$ is completely positive. Indeed, (for the unital case) note that $\phi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(\mathbb{C})$, so we check positivity by checking for positive-definiteness. To that end, let $\xi \in \mathbb{C}^n$ and $[a_{ij}] \in \mathrm{M}_n(A)$ positive. Then by Exercise 3.11,

$$\begin{bmatrix} \bar{\xi}_1 & \dots & \bar{\xi}_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \xi_n & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \tag{9.2}$$

is positive in $\mathrm{M}_n(A)$. Then $\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}$ is positive in A,[9] which means its image under ϕ is positive by assumption. Then we compute

$$\begin{aligned} \langle \phi^{(n)}([a_{ij}])\xi, \xi\rangle = \langle [\phi(a_{ij})]\xi, \xi\rangle &= \left\langle \begin{bmatrix} \sum_{j=1}^n \phi(a_{1j})\xi_j \\ \vdots \\ \sum_{j=1}^n \phi(a_{n,j})\xi_j \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle \\ &= \sum_{i,j=1}^n \overline{\xi_i}\xi_j \phi(a_{ij}) = \phi(\sum_{i,j=1}^n \overline{\xi_i}\xi_j a_{ij}) \geq 0 \end{aligned}$$

Exercise 9.6. Show that the composition of completely positive maps is completely positive.

Exercise 9.7. Let $\phi : A \to B$ be a positive map between C*-algebras. Show that ϕ is $*$-preserving, i.e. $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Exercise 9.8. Show that the matrix amplification of any $*$-homomorphism between C*-algebras is again a $*$-homomorphism. Conclude that any $*$-homomorphism is completely positive.

Example 9.9. To get more examples of completely positive maps we build them out of known examples.

The idea is to conjugate another cp map: Let $\psi : A \to B$ be a cp map between C*-algebras and $b \in B$. Then the map $\phi := b^*\psi(\cdot)b : A \to B$ is linear and positive by Exercise 3.11. It is moreover completely positive. Indeed, for each $n \geq 1$ and positive element $[a_{ij}] \in \mathrm{M}_n(A)$,

$$\phi^{(n)}([a_{ij}]) = \begin{bmatrix} b^*\phi(a_{11})b & \dots & b^*\phi(a_{1n})b \\ \vdots & \ddots & \vdots \\ b^*\phi(a_{n1})b & \dots & b^*\phi(a_{nn})b \end{bmatrix} = \begin{bmatrix} b^* & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b^* \end{bmatrix} \begin{bmatrix} \phi(a_{11}) & \dots & \phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(a_{n1}) & \dots & \phi(a_{nn}) \end{bmatrix} \begin{bmatrix} b & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b \end{bmatrix}.$$

Observe (**exercise**) that when $\|b\| \leq 1$, ϕ is cp. Moreover, ϕ is *contractive:* $\|\phi(a)\| \leq \|a\|$ for all a.

Exercise 9.10. Now consider a more concrete setting of $B(\ell^2)$, and consider the rank n projection P defined on the basis vectors by $Pe_i = 1$ if $i \leq n$ and $Pe_i = 0$ if $i > n$. If we write an operator $A \in B(\ell^2)$ as a matrix array, what does its image under the completely positive map $A \mapsto PAP$ look like? (This is where the word "compression" comes from.)

Now, we identify $PB(\ell^2)P \simeq B(P\ell^2) \simeq \mathrm{M}_n(\mathbb{C})$ (like in Example 6.5). These are $*$-isomorphisms, which means their composition with the above compression by P gives a completely positive map $B(\ell^2) \to \mathrm{M}_n(\mathbb{C})$.

[9] Perhaps there is a quicker argument, but here is one through tensor products. We'll go ahead and record it so you can come back after we've covered them. Exercise 11.11 tells us that $\mathrm{M}_n \odot A = \mathrm{M}_n(A)$. So, the positive matrix in (9.2) is of the form $x = p \otimes b \in \mathrm{M}_n \otimes A$, where p is the projection onto the first coordinate. Then $x = |x| = \sqrt{p \otimes b^*b}$, which must also equal $|b| \otimes p$ by uniqueness of positive square roots. Then $p \otimes b - p \otimes |b| = 0$ implies $b = |b| \geq 0$.

Example 9.11. One important class of completely positive maps are conditional expectations, which feature more prominently in von Neumann algebras. Recall from the von Neumann lecture notes that a conditional expectation is a contractive linear projection $E : A \to B$ from a C*-algebra onto a C*-subalgebra $B \subset A$ such that $Eb = b$ for all $b \in B$. By a theorem of Tomiyama, any conditional expectation is automatically completely positive and contractive. In this exercise, we consider a class of these that we will use a few times in these notes.

Recall that a finite dimensional C*-algebra has the form $F = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$. We define a conditional expectation $\mathrm{M}_L(\mathbb{C}) \to F$ as follows: for each j, let P_j denote the projection onto the jth component of F, and define $\rho_j : \mathrm{M}_L \to \mathrm{M}_{l_j}$ as the compression $E_j(\cdot) = P_j \cdot P_j$ (where we identify $\mathrm{M}_{l_j}(\mathbb{C})$ with its copy in $\mathrm{M}_L(\mathbb{C})$). Then $E : \mathrm{M}_L(\mathbb{C}) \to F$, given by $\sum_j E_j$, is a ucp map (**exercise** check). (Why do we automatically know F is unital?)

Theorem 9.12 (Russo-Dye)**.** *Let A and B be unital* C*-*algebras and $\phi : A \to B$ a positive map. Then $\|\phi\| = \|\phi(1)\|$.*

This is [9, Corollary 2.9], where it appears as a Corollary to von Neumann's inequality [9, Corollary 2.7], which we will not treat here.

In the subsection on nonunital C*-algebras in [9, Chapter 2], Paulsen gives this non-unital extension of the Russo-Dye theorem.

Proposition 9.13. *Any positive map between* C*-*algebras is bounded.*

Finally, we record the following examples for future use. The proof is short, but we leave it for [9, Theorem 3.9].

Proposition 9.14. *For any unital* C*-*algebra A and any compact Hausdorff space X, any unital positive map $\phi : A \to C(X)$ is ucp.*

Remark 9.15. The converse holds too. This is a theorem of Stinespring (not to be confused with his dilation theorem in the next section). ([9, Theorem 3.11])

Dilation Tricks:

Though our goals are Theorems 9.22 and 9.28, we would be doing a disservice to come this close to dilation tricks and not give you a feel for the techniques. Also, we'll want some of these facts later.

Lemma 9.16. *Let A be a unital* C*-*algebra and $a, b \in A$. Then $\|a\| \le 1$ iff $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$ is positive in $M_2(A)$.*

Proof. We assume A is faithfully (and unitally) represented on a Hilbert space $B(\mathcal{H})$, whence we check for positive-definiteness. For $a \in A$, if $\|a\| \le 1$, then for any $\xi, \eta \in \mathcal{H}$, we have

$$\left\langle \begin{pmatrix} 1_{\mathcal{H}} & a \\ a^* & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle = \langle \xi, \xi \rangle + \langle a\eta, \xi \rangle + \langle \xi, a\eta \rangle + \langle \eta, \eta \rangle$$
$$\ge \|\xi\|^2 - 2\|a\|\|\eta\|\|\xi\| + \|\eta\|^2 \ge 0.$$

On the other hand, if $\|a\| > 1$, then there exist unit vectors $\xi, \eta \in A$ such that $\langle a\eta, \xi \rangle < -1$, which would make the inner product above negative.

□

Definition 9.17. We say a linear map $\phi : A \to B$ between C*-algebras is *completely bounded* if

$$\sup_n \|\phi^{(n)}\| < \infty.$$

Corollary 9.18. *Any completely positive map is completely bounded. Moreover, if A and B are unital* C*-*algebras and $\phi : A \to B$ is a completely positive map, then*

$$\|\phi(1)\| = \|\phi\| = \sup_n \|\phi^{(n)}\|.$$

We prove the case where ϕ is unital, i.e. $\phi(1) = 1$, which also means $\phi^{(n)}(1) = 1_{\mathrm{M}_n(A)}$ for all $n \ge 1$. The more general case needs one additional fact and is addressed in [9, Proposition 3.6], but the main idea is already in the unital case.

Proof. We know already that $\|\phi(1)\| \leq \|\phi\| \leq \sup_n \|\phi^{(n)}\|$. Moreover, we know $\|\phi^{(n)}(1)\| = \|1_{\mathrm{M}_n(B)}\| = 1$ for all $n \geq 1$. So, we want to prove that $\sup_n \|\phi^{(n)}\| \leq 1$. So, let $a = [a_{ij}] \in \mathrm{M}_n(A)$ with $\|a\| \leq 1$. Then by Lemma 9.16,

$$\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix} \in \mathrm{M}_{2n}(A)$$

is positive. Since ϕ is completely positive, $\phi^{(n)}$ is 2-positive, and so

$$\phi^{(2n)}\left(\begin{pmatrix} 1_{\mathrm{M}_n(A)} & a \\ a^* & 1_{\mathrm{M}_n(A)} \end{pmatrix}\right) = \begin{pmatrix} 1_{\mathrm{M}_n(B)} & \phi^{(n)}(a) \\ \phi^{(n)}(a)^* & 1_{\mathrm{M}_n(B)} \end{pmatrix}$$

is positive. By Lemma 9.16, this implies $\|\phi^{(n)}(a)\| \leq 1$, as desired. $\square$

More abbreviations:
Corollary 9.18 says that any ucp map is completely positive and completely contractive, abbreviated by cpc (or some permutation of those letters).

Exercise 9.19. Let A be a unital C*-algebra and $a \in A$ such that $\|a\| \leq 1$. Show that the following is a unitary in $\mathrm{M}_2(A)$:

$$\begin{pmatrix} a & \sqrt{1 - aa^*} \\ \sqrt{1 - a^*a} & -a^* \end{pmatrix}.$$

This is sometimes referred to Halmos' Dilation.

Now that we've tried a few dilation tricks, we (you) are ready to give another proof of Proposition 8.29.

Exercise 9.20. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.

(1) Consider the element $x = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \in \mathrm{M}_2(B)$. Show that $\|x^*x\| = \|b^*b\|$.
(2) Apply Exercise 2.21 to x and $\pi^{(2)}$ to get some lift $y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \mathrm{M}_2(A)$ (i.e. $\pi^{(2)}(y) = x$) with y self-adjoint and $\|y\| = \|x\| = \|b\|$.
(3) Show that y_{12} is a lift of b.
(4) Now use Proposition 9.2 to finish the argument. (Don't forget to mention why $\|y_{12}\| \leq b$.)

We close with one important fact that holds for cpc maps that does not hold in general is that any cpc maps between C*-algebras extends to a ucp map between their unitizations. The proof is short but digs into some surprisingly technical aspects of double duals of C*-algebras, so we leave it to [3, Proposition 2.2.1].

Proposition 9.21. *Let A and B be C*-algebras with A non-unital and B unital, and let $\phi : A \to B$ be a cpc map. Then ϕ extends to a ucp map $\tilde{\phi} : \tilde{A} \to B$, which is given by*

$$\tilde{\phi}(a + \lambda 1_{\tilde{A}}) = \phi(a) + \lambda 1_B.$$

9.2. **Stinespring's Dilation Theorem.** We saw in the previous section that compressing a $*$-homomorphism gives a completely positive map. What Stinespring's Dilation Theorem tells us is that that's basically how *every* completely positive map arises! That's right, when we are working with completely positive maps, we are really just looking at "compressed" $*$-homomorphisms.[10] That's what makes Stinespring's theorem so powerful: cp (ucp) maps are more abundant than $*$-homomorphisms, but when you have a cp map, you can draw a lot of conclusions pertaining to its structure by appealing to its "Stinespring Dilation" $*$-homomorphism.

Enough prelude. Here's the theorem.

Theorem 9.22 (Stinespring's Dilation Theorem)**.** *Let A be a unital C*-algebra and $\phi : A \to B(\mathcal{H})$ a cp map. Then there exists a Hilbert space $\mathcal{H}'$, a unital representation $\pi : A \to B(\mathcal{H}')$ and a linear map $V : \mathcal{H} \to \mathcal{H}'$ such that*

$$\phi(a) = V^*\pi(a)V$$

for every $a \in A$. In particular, $\|\phi\| = \|V\|^2 = \|V^*V\| = \|\phi(1)\| = \sup_n \|\phi^{(n)}\|$.

[10] "Compressed" is in quotations because in the non-unital setting it will be conjugation but not necessarily by a projection as in Definition 4.2.1 in the von Neumann notes.

Moreover, if ϕ is unital, then V is an isometry and $V^* = P_{V\mathcal{H}}$ is the projection onto $V\mathcal{H} \subset \mathcal{H}'$. In this case we identify $\mathcal{H}$ with a subspace $V\mathcal{H} \subset \mathcal{H}'$ and have

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

Remark 9.23. We have a few remarks on this.

(1) When ϕ is unital, we think of $\pi(a)$ as

$$\pi(a) = \begin{bmatrix} \phi(a) & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{12} : \mathcal{H}^\perp \to \mathcal{H}$, $T_{21} : \mathcal{H} \to \mathcal{H}^\perp$ and $T_{22} : \mathcal{H}^\perp \to \mathcal{H}^\perp$ are some bounded linear maps.
Notice how the unital case generalizes Example 9.10 (with $\pi = id$).

(2) There is a non-unital version. Follow [3, Remark 1.5.4].

(3) One usually hears the term "minimal Stinespring dilation." Consider a Stinespring representation $(\pi, \mathcal{H}', V)$ for $\phi : A \to B(\mathcal{H})$. Let $\mathcal{H}_0 \subset \mathcal{H}$ be the closed linear span of $\pi(A)V\mathcal{H}$, which is reducing for $\pi(A)$ (as in vN notes) and hence the co-restriction $\pi : A \to B(\mathcal{H}_0)$ is a representation. Whenever $\pi(A)V\mathcal{H}$ is dense in $\mathcal{H}'$, (i.e. its closure is $\mathcal{H}_0$), then the Stinespring dilation is unique up to unitary equivalence. (See [9, Proposition 4.2].)

The proof is exactly a generalization of the GNS construction of a representation corresponding to a state. The technique in general is sometimes called "separation and completion": first you define a semi-norm (or semi-inner-product in this case), then you mod out by the null set– hence making it a genuine norm (or inner product), then complete the quotient space with respect to your new norm. Since we have already seen the technical side of the GNS proof, let's see the overarching idea this time around in order to better understand how to potentially use this technique in other settings. (For a proof that checks all the details, see [9, Theorem 4.1].)

Proof of Stinespring's Dilation Theorem. Let $\phi : A \to B(\mathcal{H})$ be a cp map, and consider the algebraic tensor product

$$A \odot \mathcal{H} := \operatorname{span}\{a \odot \xi \ : \ a \in A, \xi \in \mathcal{H}\}.$$

We define a symmetric bilinear function $\langle \cdot, \cdot \rangle$ by

$$\langle a \odot \xi, b \odot \eta \rangle = \langle \phi(b^* a)\xi, \eta \rangle_{\mathcal{H}},$$

for $a, b \in A$ and $\xi, \eta \in \mathcal{H}$ (extending linearly to $A \odot \mathcal{H}$). One then checks that this is positive semidefinite (i.e. $\langle x, x \rangle \geq 0$), which means it's an inner product modulo the fact that we could potentially have $\langle x, x \rangle = 0$ for non-zero $x \in A \odot \mathcal{H}$. No worries. It turns out the space consisting of such elements $\mathcal{N} = \{x \in A \odot \mathcal{H} : \langle x, x \rangle = 0\}$ is a subspace of $A \odot \mathcal{H}$, which means we can take the quotient $(A \odot \mathcal{H})/\mathcal{N}$. The symmetric bilinear function $\langle \cdot, \cdot \rangle$ from before now induces a genuine inner product on $(A \odot \mathcal{H})/\mathcal{N}$ given by

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle := \langle x, y \rangle.$$

So, when we complete $(A \odot \mathcal{H})/\mathcal{N}$ with respect to this inner product, we get a Hilbert space. Let's suggestively call it $\mathcal{H}'$.

For $a \in A$, we define the linear map $\pi(a) : A \odot \mathcal{H} \to A \odot \mathcal{H}$ by left multiplication, i.e. for $a \in A$ and $b \odot \xi \in A \odot \mathcal{H}$, we have

$$\pi(a)(b \odot \xi) = ab \odot \xi,$$

and we extend linearly. A computation shows that $\pi(a) : \mathcal{N} \to \mathcal{N}$, and so it induces a linear map on the quotient $(A \odot \mathcal{H})/\mathcal{N}$, which we still denote by $\pi(a)$. Moreover, it turns out that $\|\pi(a)(x + \mathcal{N})\| \leq \|a\|\|x + \mathcal{N}\|$ for all $x + \mathcal{N} \in (A \odot \mathcal{H})/\mathcal{N}$ (where $\|x + \mathcal{N}\|^2 = \langle x + \mathcal{N}, x + \mathcal{N} \rangle$), which means we can extend $\pi(a)$ to a bounded linear operator on all of $\mathcal{H}'$. One then checks that this is indeed a unital $*$-homomorphism.

Define $V : \mathcal{H} \to \mathcal{H}'$ by $V(\xi) = (1_A \odot \xi) + \mathcal{N}$. Then we compute for each unit vector $\xi \in \mathcal{H}$, using Exercise 7.48 from the Day 1 Lecture Notes,

$$\|V\xi\|^2 = \langle 1_A \odot \xi, 1_A \odot \xi \rangle = \langle \phi(1_A^* 1_A)\xi, \xi \rangle \leq \|\phi(1)\|\|\xi\|^2 = \|\phi(1)\|.$$

It follows that $\|V\| = \|\phi(1)\|$ and moreover that V is an isometry when ϕ is unital.

Finally, since V is an isometry, we conclude that for all $\xi, \eta \in \mathcal{H}$,

$$\langle V^* \pi(a)V\xi, \eta \rangle_{\mathcal{H}} = \langle \pi(a)V\xi, V\eta \rangle = \langle \pi(a)((1 \odot \xi) + \mathcal{N}), (1 \odot \eta) + \mathcal{N} \rangle = \langle \phi(a)\xi, \eta \rangle_{\mathcal{H}},$$

and hence $V^*\pi(a)V = \phi(a)$. □

Exercise 9.24. Describe in words how (the proof of) Stinespring's Dilation Theorem generalizes the Gelfand Naimark Segal Theorem. In particular, when ϕ is a state, what is $\mathcal{H}$? $A \odot \mathcal{H}$?

Yes, there is a generalization of Stinespring's Dilation Theorem called the Kasparov-Stinespring Dilation Theorem. This is phrased in either the language of Hilbert C*-modules (see [6] for a nice introduction) or multiplier algebras (in Kasparov's original paper).

Frankly, Stinespring's theorem admits several generalizations. For instance, there is one for maps that are just considered completely bounded, i.e. linear maps with $\sup_n \|\phi^{(n)}\| < \infty$.

For the sake of seeing Stinespring's Dilation Theorem in action, we introduce another useful concept for ucp maps: multiplicative domains. Here's how we define a multiplicative domain.

Definition 9.25. Let A and B be unital C*-algebras and $\phi : A \to B$ ucp. Then the set

$$\{a \in A : \phi(a)\phi(b) = \phi(ab) \text{ and } \phi(b)\phi(a) = \phi(ba)\ \forall\ b \in A\}$$

is a C*-subalgebra of A called the *multiplicative domain of* ϕ.

Notice that ϕ is a $*$-homomorphism when restricted to this set. In fact, this is the largest C*-subalgebra on which the ucp map acts as a $*$-homomorphism, though the fact that it is a C*-algebra requires proof. To prove this, we use Stinespring's Dilation theorem to prove the following alternative description.

Proposition 9.26. *Let A and B be unital* C**-algebras and $\phi : A \to B$ ucp. Then the multiplicative domain of ϕ is equal to the set*

$$\{a \in A : \phi(a)^*\phi(a) = \phi(a^*a) \text{ and } \phi(a)\phi(a)^* = \phi(aa^*)\}.$$

Proof of Proposition 9.26. Let A be a unital C*-algebra and $\phi : A \to B$ a ucp map. One inclusion is immediate. We will work through the other inclusion.

Since B can be faithfully represented on some $B(\mathcal{H})$ (and the composition of that representation with ϕ is still cp), we assume $B \subset B(\mathcal{H})$ and view ϕ as a map into $B(\mathcal{H})$. Let $(\pi, V, \mathcal{H}')$ be a Stinespring Dilation for $\phi : A \to B(\mathcal{H})$, i.e. $\pi : A \to B(\mathcal{H}')$ is a representation of A and $V : \mathcal{H} \hookrightarrow \mathcal{H}'$ an isometric embedding so that $\phi(a) = V^*\pi(a)V$ for all $a \in A$. Then for any $a \in A$, we have

$$\begin{aligned}
\phi(a^*a) - \phi(a)^*\phi(a) &= V^*\pi(a^*a)V - V^*\pi(a)^*VV^*\pi(a)V \\
&= V^*\pi(a)^*1_{\mathcal{H}'}\pi(a)V - V^*\pi(a)^*VV^*\pi(a)V \\
&= V^*\pi(a)^*(1_{\mathcal{H}'} - VV^*)\pi(a)V
\end{aligned}$$

Now, suppose $a \in A$ so that $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$. Since V is an isometry, VV^* is a positive contraction, and so by Exercise 3.11, $1_{\mathcal{H}'} - VV^*$ is a positive contraction, which has a unique positive square root. With that observation, we compute

$$\begin{aligned}
0 = \phi(a^*a) - \phi(a)^*\phi(a) &= V^*\pi(a)^*(1_{\mathcal{H}'} - VV^*)\pi(a)V \\
&= V^*\pi(a)^*(\sqrt{1_{\mathcal{H}'} - VV^*})^2\pi(a)V \\
&= [\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V]^*[\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V].
\end{aligned}$$

It follows (from say the C*-identity) that $\sqrt{1_{\mathcal{H}'} - VV^*}\pi(a)V = 0$.

With that, we let $b \in A$ and compute

$$\phi(ba) - \phi(b)\phi(a) = V^*\pi(b)(1_{\mathcal{H}'} - VV^*)\pi(a)V = 0.$$

A symmetric argument shows that $\phi(ab) = \phi(b)\phi(a)$ for all $b \in A$, which completes the argument. □

Exercise 9.27. Conclude that the multiplicative domain of a cpc map $\phi : A \to B$ from a unital C*-algebra is a C*-subalgebra.

9.3. **Arveson's Extension Theorem.** The other major theorem for completely positive maps (as far as C*-algebraists are usually concerned) is Arveson's Extension Theorem. Just as Stinespring's Dilation Theorem was a generalization of GNS, which was a generalization of GN, Arveson's Extension Theorem is a generalization of Krein's Theorem, which is a strengthening of the Hahn-Banach Theorem for C*-algebras. On the other hand, where Stinespring's proof was a generalization of the proofs that came before, Arveson's proof builds on the proofs that came before.

Theorem 9.28 (Arveson's Extension Theorem)**.** *Let B be a unital* C*-*algebra, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to B(\mathcal{H})$ a cp map. Then there exists a cp map $\tilde{\phi} : B \to B(\mathcal{H})$ extending ϕ, i.e. $\tilde{\phi}|_A = \phi$.*

Remark 9.29. In an abuse of categorical language, $B(\mathcal{H})$ is often called *injective* in the category of C*-algebras with morphisms given by cpc maps. (It's an abuse of language because we always assume an embedding $A \subset B$ is a $*$-homomorphism embedding.)

This theorem plays a big role in the next section when we see a characterization of nuclear C*-algebras in terms of completely positive maps. For now, we just give an idea of the proof via the results it generalizes.

Theorem 9.30 (Krein)**.** *Let B be a unital* C*-*algebra, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to \mathbb{C}$ a positive linear map. Then ϕ extends to a positive map on B.*

This intermediate result is [9, Theorem 6.2].

Proposition 9.31. *Let B be a unital* C*-*algebra, $n \geq 1$, $A \subset B$ a unital* C*-*subalgebra, and $\phi : A \to \mathrm{M}_n(\mathbb{C})$ completely positive. Then ϕ extends to a completely positive map $B \to \mathrm{M}_n(\mathbb{C})$.*

From this to Arveson's theorem, we take a completely positive map $\phi : A \to B(\mathcal{H})$ and an increasing net of finite rank projections $P_i \in B(\mathcal{H})$. Then each compression $\phi_i : A \to P_i B(\mathcal{H}) P_i \simeq \mathrm{M}_{\mathrm{rank} P_i}(\mathbb{C})$, given by $P_i \phi(\cdot) P_i$, is a completely positive map with completely positive extension. From here you take a point-ultraweak cluster point of the ϕ_i's (ask Brent and Rolando), and that's your cp extension of ϕ!

Exercise 9.32. Suppose $C \subset B(\mathcal{H})$ is a unital C*-subalgebra of $B(\mathcal{H})$ (meaning its unit is $1_{\mathcal{H}}$) and $E : B(\mathcal{H}) \to C$ is a conditional expectation (which we recall from Exercise 9.11 is completely positive by Tomiyama's theorem). Show that Arveson's Extension Theorem holds for C as well, i.e. for any unital C*-algebras $A \subset B$ and cp map $\phi : A \to C$, there exists a cp map $\tilde{\phi} : B \to C$ extending ϕ, i.e. $\tilde{\phi}|_A = \phi$.

Using Example 9.11, conclude that Arveson's Extension Theorem holds for all finite dimensional C*-algebras.

Remark 9.33. If you've peeked at some of the reference texts, you'll notice that many of the theorems from this section are given for operator systems. What are those? You'll learn more about them in Sam Kim's expository lecture next week, but for now, here's an idea.

Notice that completely positive maps completely preserve the structure of positive elements in a C*-algebra. So, there is a lot to be gained from considering self-adjoint unital subspaces of C*-algebras.

An *operator system* is a unital self-adjoint subspace of a C*-algebra. (Not necessarily closed.) Arveson's extension theorem is actually stated where we assume that $A \subset B$ is not a C*-algebra but an operator system inside B.

10. Completely Positive Approximations

This section introduces what is historically known as the "completely positive approximation property," which, in the hindsight provided by a major theorem of Choi-Effros and also Kirchberg (which we give next week), is now called nuclearity. In essence, a C*-algebra has the completely positive approximation property when it can be well approximated by cpc maps that factor through finite dimensional C*-algebras. This is, at its heart, a property of maps, which is where we start in section 10.1.

However, in the lecture, we will focus on nuclearity of C*-algebras (10.2) and hence will take the material in Section 10.1 for granted. We will go through the discussion on $K(\ell^2)$ at the beginning of this section in the context of the definition of nuclearity. We will prove Proposition 10.15 and Proposition 10.9 in the separable setting. Arveson's Extension theorem will feature prominently.

Though we will not be able to treat it in lecture, we highly recommend reading the argument that commutative C*-algebras are nuclear (Proposition 10.10) and working out the hands-on example in Exercise 10.12.

Many of the C*-algebras we can get our hands on have some reasonable connection to finite-dimensional C*-algebras. AF algebras in particular were built out of finite-dimensional subalgebras. More generally, they can be approximated by their finite dimensional subalgebras in a way that can be generalized to a much larger class of C*-algebras. To get a better feeling for what we mean, let us start with a motivating example.

We know (Example 6.5) that $K(\ell^2)$ is built as a union of finite-dimensional algebras as follows:

$$K(\ell^2) = \overline{\bigcup_n P_n K(\ell^2) P_n}$$

where P_n is the projection onto $\operatorname{span}\{e_1, \dots, e_n\}$. Since the projections $(P_n)_n$ form an approximate unit for $K(\ell^2)$, we have for each $T \in K(\ell^2)$,

$$\|T - P_n T P_n\| \to 0.$$

We saw in the previous section that the map $T \mapsto P_n T P_n$ is a completely positive comtractive map. Compose that with the $*$-isomorphism $P_n K(\ell^2) P_n \simeq \mathrm{M}_n(\mathbb{C})$, and we have a cpc map $\psi_n : K(\ell^2) \to \mathrm{M}_n(\mathbb{C})$. Moreover, when we compose that with the $*$-homomorphism embedding $\phi_n : \mathrm{M}_n \to P_n K(\ell^2) P_n \subset K(\ell^2)$, we can write

$$\|T - \phi_n \psi_n(T)\| \to 0.$$

This is called a *completely positive approximation* of $K(\ell^2)$, and the existence of such an approximation is what it means (in modern terms) to be nuclear.

For the sake of simplicity, many results here are not stated in their full generality. If you find this section interesting, we suggest [3, Chapter 2], which covers this material quite well, save a dearth of hands-on examples.

10.1. Nuclear Maps.

We start with the definition of a nuclear map between C*-algebras.

Definition 10.1. A cpc map $\theta : A \to B$ between C*-algebras is called *nuclear* if there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$, for $i \in I$, so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology, i.e. for each $a \in A$,

$$\lim_{i \in I} \|\phi_i(\psi_i(a)) - \theta(a)\| = 0.$$

Remark 10.2. There's lots to say here. This idea is thoroughly researched and nuanced, and there are so many variations on the above definition. We'll keep these remarks brief.

- If A is separable, then it can be written as a countable union of finite subsets. Then we can choose the net I in Definition 10.1 to be a sequence.
- The requirements placed on the maps ψ_i and ϕ_i can vary. It turns out we could equivalently relax the contractive requirement. On the other hand, we could equivalently keep the requirement that they are cpc and demand moreover that they have certain (approximate) orthogonality preserving properties (known as order zero). There's a fair bit of research in this direction by Winter, Zacharias, Kirchberg, Hirchberg, Brown, and Carrion to name a few. (FYI: Nate Brown will be one of our expository speakers, and José Carrion will speak at our conference.)

- The convergence in Definition 10.1 could have be given with respect the point-ultraweak (aka σ-weak) topology (in which case the map would be called *weakly nuclear*). This is the first step on the bridge between nuclearity for C*-algebras and semidiscreteness/ hyperfiniteness for von Neumann algebras (ask Brent and Rolando what those terms mean), but we are getting ahead of ourselves.

This is really a local property, as the following proposition shows.

Proposition 10.3. *A cpc map $\theta : A \to B$ is nuclear iff for any $\varepsilon > 0$ and finite set $F \subset A$, there exists $n \geq 0$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that*

$$\|\phi(\psi(a)) - \theta(a)\| < \varepsilon$$

for each $a \in F$.

Proof. Suppose there exist cpc maps $\psi_i : A \to \mathrm{M}_{k(i)}(\mathbb{C})$ and $\phi_i : \mathrm{M}_{k(i)}(\mathbb{C}) \to B$ for $i \in I$ so that $\phi_i \circ \psi_i \to \theta$ in the point norm topology. Then for any $\varepsilon > 0$ and $F \subset A$ finite, we choose $i \in I$ so that $\|\phi_i(\psi_i(a)) - \theta(a)\| < \varepsilon$ for each $a \in F$.

Now, we assume the localized version. As in Examples 5.3 and 5.8 in the Prerequisite materials, we form a directed set

$$\{(\varepsilon, F) \ : \ \varepsilon > 0, F \subset A \text{ finite}\}.$$

For each (ε, F), let $\phi_{(\varepsilon,F)}$ be a cpc map so that $\|\phi_{(\varepsilon,F)}(\psi_{(\varepsilon,F)}(a)) - \theta(a)\| < \varepsilon$ for each $a \in F$. Then for each $a \in A$, we have the desired convergence. $\square$

Exercise 10.4. Show that a map $\theta : A \to B$ is nuclear if there exist finite dimensional C*-algebras F_i and cpc maps $\psi_i : A \to F_i$ and $\phi_i : F_i \to B$ so that $\phi_i \circ \psi_i$ converges pointwise in norm to θ.
Hint: Recall that a finite dimensional C*-algebra has the form $F = \oplus_{j=1}^m \mathrm{M}_{l_j}(\mathbb{C}) \subset \mathrm{M}_L(\mathbb{C})$ where $L = \sum l_j$, and use Example 9.11.

Exercise 10.5. Let A and B be C*-algebras and $C \subset B$ a C*-subalgebra. Show that if $\theta : A \to C$ is a nuclear map, then so is θ when viewed as a map from A to B. Suppose we have a map $\rho : A \to C$ that is nuclear as a map from A to B. What could prevent ρ from being a nuclear map as a map from A to C?

10.2. **Completely Positive Approximation Property.**

Definition 10.6. A C*-algebra is *nuclear* if the identity map $\mathrm{id}_A : A \to A$ is nuclear, i.e. there exist cpc maps $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$ such that for each $a \in A$,

$$\|a - \phi_i(\psi_i(a))\| \to 0.$$

In the separable setting, the usual image one presents is something like the following approximately commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_2"] & & \cdots \\
& \mathrm{M}_{k(0)}(\mathbb{C}) \arrow[ur, "\phi_0"] & & \mathrm{M}_{k(1)}(\mathbb{C}) \arrow[ur, "\phi_1"] & & \mathrm{M}_{k(2)}(\mathbb{C}) & \cdots
\end{tikzcd}$$

Remark 10.7. Sometimes these C*-algebras are called amenable. Sometimes for mathematical reasons—sometimes because the word "nuclear" in a grant application means one must fill out many many more forms.

A C*-algebra satisfying Definition 10.6 is also said to satisfy the *completely positive approximation property* (CPAP).

Example 10.8. It follows from Exercise 10.4 that finite dimensional C*-algebras are nuclear.

Proposition 10.9. *Ideals of nuclear* C*-*algebras are nuclear.*

Proof. Suppose A is nuclear with completely positive approximation $A \xrightarrow{\psi_i} \mathrm{M}_{k(i)}(\mathbb{C}) \xrightarrow{\phi_i} A$ for $i \in I$. Let $J \triangleleft A$ be an ideal and $\{e_\lambda\}_\Lambda$ an approximate unit of J (with $0 \leq e_\lambda \leq e_\gamma \leq 1$ when $\lambda \leq \gamma$). Let $\iota : J \to A$ denote the inclusion of J into A (i.e. $\iota(a) = a$ for all $a \in J$). For each λ, define $\rho_\lambda : A \to J$ by $\rho_\lambda(a) = e_\lambda a e_\lambda$. Since each e_λ is self-adjoint and contractive, the maps ρ_λ are cpc by Exercise 9.9. Since the compositions of cpc maps are cpc (Exercise 9.6), for each i, λ, the maps $\psi'_{i,\lambda} := \psi_i \circ \iota : J \to \mathrm{M}_{k(i)}$ and $\phi'_{i,\lambda} := \rho_\lambda \circ \phi_i : \mathrm{M}_{k(i)} \to J$

are cpc. (Yes, the λ is a superflous index on ψ_i'.) Moreover, $\{(i,\lambda)\}_{I\times\Lambda}$ is a directed set with $(i,\lambda)\leq(j,\gamma)$ when $i\leq j$ and $\lambda\leq\gamma$.

Let $a\in J$ and $\varepsilon>0$, and choose $(i_0,\lambda_0)\in I\times\lambda$ so that $\|a-\phi_i\circ\psi_i(a)\|<\varepsilon/2$ and $\|a-\rho_\lambda(a)\|<\varepsilon/2$ for each $i\geq i_0$ and $\lambda\geq\lambda_0$. Then for each $(i,\lambda)\geq(i_0,\lambda_0)$,

$$\begin{aligned}\|a-\phi_{i,\lambda}'\circ\psi_{i,\lambda}'(a)\| &= \|a-e_\lambda(\phi_i\circ\psi_i(a))e_\lambda\|\\ &\leq \|a-e_\lambda a e_\lambda\|+\|e_\lambda a e_\lambda - e_\lambda(\phi_i\circ\psi_i(a))e_\lambda\|\\ &\leq \|a-e_\lambda a e_\lambda\|+\|e_\lambda\|\|a-\phi_i\circ\psi_i(a)\|\|e_\lambda\|\\ &\leq \|a-e_\lambda a e_\lambda\|+\|a-\phi_i\circ\psi_i(a)\|\\ &< \varepsilon.\end{aligned}$$

□

In approximately commutative diagrams, the picture from the above proof looks like this.

$$\begin{tikzcd}
J \arrow[rrrr, "id_J"] \arrow[dr, "\iota"] & & & & J \\
& A \arrow[rr, "id_A"] \arrow[dr, "\psi_i"] & & A \arrow[ur, "\rho_\lambda"] & \\
& & \mathrm{M}_{k(i)}(\mathbb{C}) \arrow[ur, "\phi_i"] & &
\end{tikzcd}$$

It's not a proof, but it's a good intuition to guide the proof.

Proposition 10.10. *Abelian* C^**-algebras are nuclear.*

The proof uses what is known as a "partition of unity argument." Generalizing the idea of a partition of unity has proved very fruitful in certain areas of research in recent years, so we give this proof as an example.

We take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1,...,U_n$, there exist continuous functions $h_1,...,h_n:X\to[0,1]$ so that $\mathrm{supp}(h_j)\subset U_j$ and $\sum_j h_j=1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* (in fact a rather nice one).

Proof. Let A be an abelian C^*-algebra. If A is not unital, then $A\triangleleft\tilde{A}$, and by Proposition 10.9, it suffices to show that $\tilde{A}$ is nuclear. So, we assume A is unital and moreover that $A=C(X)$ for some compact Hausdorff space X. Combining Proposition 10.3 and Exercise 10.4, we conclude that it suffices to show that for any $F\subset C(X)$ finite and $\varepsilon>0$, there exists a finite dimensional C^*-algebra C (in our case, it will be $\mathbb{C}^n=\oplus_1^n\mathrm{M}_1(\mathbb{C})$) and cpc maps $C(X)\xrightarrow{\psi}C\xrightarrow{\phi}C(X)$ so that $\|f-\phi\circ\psi(f)\|<\varepsilon$ for every $f\in F$.

Let $F\subset C(X)$ be a finite subset and $\varepsilon>0$. For each $x\in X$, let

$$U_x:=\bigcap_{f\in F}f^{-1}(B_{\varepsilon/2}(f(x))).$$

Then $U_x\subset X$ is an open neighborhood of x such that for each $y\in U_x$ and $f\in F$, we have $|f(y)-f(x)|<\varepsilon/2$. Since X is compact, we choose $x_1,...,x_n$ so that a finite subcover $U_{x_1},...,U_{x_n}$ covers X, and moreover for each $f\in F$ and $y\in U_i$,

$$|f(y)-f(x_i)|<\varepsilon.$$

Then we choose a partition of unity $h_1,...,h_n:X\to[0,1]$ so that $\mathrm{supp}(h_j)\subset U_{x_j}$ and $\sum_j h_j=1$.

Define $\psi:C(X)\to\mathbb{C}^n$ by $\psi(g)=(g(x_1),...,g(x_n))=\oplus_{j=1}^n ev_{x_j}$, where ev_{x_j} denotes the point evaluation $g\mapsto g(x_j)$. Then ψ is a unital $*$-homomorphism. Define $\phi:\mathbb{C}^n\to C(X)$ by

$$(\lambda_1,...,\lambda_n)\mapsto\sum\lambda_i h_i.$$

Then ϕ is a positive map, which is moreover unital since $\phi(1)=\sum h_i=1$. Hence by Proposition 9.14, it is ucp, and, in particular, cpc by Corollary 9.18.

So, we estimate for $f \in F$,

$$\begin{aligned}\|f - \phi \circ \psi(f)\| &= \| \left(\sum h_i\right) f - \sum f(x_i)h_i\| = \| \sum fh_i - f(x_i)h_i\| \\ &= \sup_{y \in X} |\sum (f(y) - f(x_i))h_i(y)| \leq \sup_{y\in X} \sum |f(y) - f(x_i)|h_i(y) \\ &\leq \sum \varepsilon h_i(y) = \varepsilon.\end{aligned}$$

□

Remark 10.11. There has been a significant push in the classification program for nuclear C*-algebras (that satisfy a nice list of adjectives) to come up with a non-commutative version of this partition of unity argument. With it comes certain non-commutative dimension theories (see for example Winter and Zacharias's paper on Nuclear Dimension). Some of this may be featured in José Carrion's conference talk.

Exercise 10.12. Partitions of unity are nicer when you have a concrete example. For each $n \geq 2$, cover $[0, 1]$ by $2^n - 1$ open intervals of equal length. (What are they? Also, we could start with $n = 1$, but it's too simple to pick up on a pattern.) Call this cover $\mathcal{U}_n$. Define (sketch) a partition of unity for $\mathcal{U}_n$. (Hint: think zig-zags.)

Now, construct a sequence of completely positive maps $C([0, 1]) \xrightarrow{\psi_n} \mathbb{C}^{k_n} \xrightarrow{\phi_n} C([0, 1])$, (what is k_n?) that give a completely positive approximation of $C([0, 1])$.

Proposition 10.13. *Suppose for each finite subset $F \subset A$ and $\varepsilon > 0$, there exists a nuclear* C*-*subalgebra $B \subset A$ such that for each $a \in F$, there exists $b \in B$ such that $\|a - b\| < \varepsilon$. Then A is nuclear.*

Proof. By Proposition 10.3, it suffices to show that for any $\varepsilon > 0$ and finite set $F \subset A$, there exists $n \geq 0$ and cpc maps $\psi : A \to \mathrm{M}_n(\mathbb{C})$ and $\phi : \mathrm{M}_n(\mathbb{C}) \to B$ such that

$$\|\phi(\psi(a)) - \theta(a)\| < \varepsilon$$

for each $a \in F$. Let $\{a_1, ..., a_m\} \subset A$ be a finite subset $\varepsilon > 0$ and let $B \subset A$ nuclear so that for each a_j, there exists a $b_j \in B$ such that $\|a_j - b_j\| < \varepsilon/3$. Let $n \geq 0$ and $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ and $\phi_B : B \to \mathrm{M}_n(\mathbb{C})$ be cpc maps so that $\|b_j - \phi_B\psi_B(b_j)\| < \varepsilon/3$ for each $1 \leq j \leq m$.

But how do we get a map ψ defined on all of A? Easy, since $\mathrm{M}_n(\mathbb{C}) = B(\mathbb{C}^n)$, the cpc map $\psi_B : B \to \mathrm{M}_n(\mathbb{C})$ extends to a cpc map $\psi : A \to \mathrm{M}_n(\mathbb{C})$ by Arveson's Extension Theorem.[11] Since $\phi_B : \mathrm{M}_n(\mathbb{C}) \to B \subset A$, we don't need to change it, so we choose $\phi = \phi_B$.

Now, all that's left is to compute for each $1 \leq j \leq m$:

$$\begin{aligned}\|a_j - \phi\psi(a_j)\| &\leq \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|\phi\psi(b_j) - \phi\psi(a_j)\| \\ &\leq \|a_j - b_j\| + \|b_j - \phi\psi(b_j)\| + \|b_j - a_j\| \\ &< \varepsilon\end{aligned}$$

□

Exercise 10.14. Using the above proposition, show that nuclearity is closed under taking direct limits. Conclude that AF algebras are nuclear.

The above proof is perhaps a little abstract. Here's a version that's a little more tangible. First, we recall once more the construction of the CAR algebra:

Let $\mathrm{M}_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The inductive limit is denoted $\mathrm{M}_{2^\infty} = \overline{\bigcup_n \mathrm{M}_{2^n}(\mathbb{C})}$. Note that by construction, for each $n \geq 0$, the copy of M_{2^n} in M_{2^∞} is unital.

Proposition 10.15. *The CAR algebra is nuclear.*

[11]Acutally, it's overkill here– one of the preliminary results leading up to Arveson's would work in finite dimensions.

Proof. For each $n \geq 0$, define $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$ be the inclusion where we identify M_{2^n} with its copy inside M_{2^∞}. The restriction of this map to its image is a $*$-isomorphism, so we call its inverse $\phi_n^{-1} : \phi_n(\mathrm{M}_{2^n}(\mathbb{C})) \to \mathrm{M}_{2^n}(\mathbb{C})$. This is a unital $*$-homomorphism from a C*-subalgebra of M_{2^∞} to $\mathrm{M}_{2^n}(\mathbb{C}) = B(\mathbb{C}^{2^n})$. So Arveson's Extension theorem says ϕ_n^{-1} has a ucp extension $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$. So, we have ucp maps $\psi_n : \mathrm{M}_{2^\infty} \to \mathrm{M}_{2^n}(\mathbb{C})$ and $\phi_n : \mathrm{M}_{2^n}(\mathbb{C}) \to \mathrm{M}_{2^\infty}$. Moreover, for each $a \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$, there exists an $N \geq 0$ so that $a \in \mathrm{M}_{2^n}(\mathbb{C})$ for all $n \geq N$, which means $\phi_n \circ \psi_n(a) = \phi_n \circ \phi_n^{-1}(a) = a$ for all $n \geq N$.

Now, suppose $a \in \mathrm{M}_{2^\infty}$, and $a_0 \in \bigcup_n \mathrm{M}_{2^n}(\mathbb{C})$ so that $\|a - a_0\| < \varepsilon/2$. Choose $N \geq 0$ so that $\phi_n \circ \psi_n(a_0) = a_0$ for all $n \geq N$. Then for all $n \geq N$,

$$\|a - \phi_n \circ \psi_n(a)\| \leq \|a - a_0\| + \|a_0 - \phi_n \circ \psi_n(a_0)\| + \|\phi_n \circ \psi_n(a_0 - a)\| < \varepsilon$$

$\square$

Exercise 10.16. Generalize the proof of Proposition 10.15 to get another proof that all separable AF algebras are nuclear.
Hint: Consider a inductive (aka directed) system of finite dimensional C*-algebras (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map, and let A be the direct (inductive) limit of this system. Then use Exercise 9.32.

Chapter 2 in [3] does an excellent job of introducing the operations that do and do not preserve nuclearity. Since we do not wish to re-write their book, we will just collect them here. These range from easy exercises to deep theorems.

(1) Nuclearity passes to direct limits and direct sums ($\bigoplus_i A_i$) (but not direct products $\prod_i A_i$).
(2) Nuclearity passes to quotients.
There are essentially two proofs for this. The first is a consequence of Connes' Fields Medal work involving showing hyperfinite $\Leftrightarrow$ injective– ask Brent and Rolando. Otherwise, it follows from the fact that exactness (defined soon) passes to quotients. The proof of this (due to Kirchberg) is one of the most difficult proofs in C*-algebras, resting some of the deepest and most difficult theorems in von Neumann algebra theory. – See [3, Chapter 9] for a (not self-contained) outline.
(3) Nuclearity does not necessarily pass to subalgebras.
The easiest examples come from crossed products, which we'll see next week. (See [3, Remark 4.4.4].) For a more sophisticated appeal, we have Kirchberg's $\mathcal{O}_2$ embeddability theorem, which implies that the non-nuclear C*-algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ embeds into the nuclear C*-algebra $\mathcal{O}_2$. (We will see next week why $\mathrm{C}^*_r(\mathbb{F}_2)$ is not nuclear. We take for granted that the Cuntz-Krieger algebras are nuclear.)
(4) Nuclearity passes to ideals (Proposition 10.9) (even hereditary subalgebras) and C*-subalgebras to which there exists a conditional expectation.
(5) Nuclearity passes to extensions, i.e. if $0 \to J \to A \to B \to 0$ is short exact and both J and B are nuclear, then so is A. (This one is easier with next week's characterization.)

We wrap up this section with a slight weakening of nuclearity that is still a very powerful property.

As we saw in Exercise 10.5, the range of a cpc map has a lot of bearing on whether or not it is nuclear. It may be that a C*-algebra fails to be nuclear but still has a faithful nuclear representation. These are still a nice class of C*-algebras.

Definition 10.17. A C*-algebra A is *exact* if there exists a faithful nuclear representation $\pi : A \to B(\mathcal{H})$.

Every nuclear C*-algebra is exact– moreover for nuclear C*-algebras, the map $\pi : A \to \pi(A)$ is nuclear. A non-nuclear example of an exact C*-algebra is $\mathrm{C}^*(\mathbb{F}_2)$ (due to Wasserman).

Exercise 10.18. Show that exactness *does* pass to C*-subalgebras. What does that tell you about every C*-subalgebra of a nuclear C*-algebra?

The name "exact" is hardly justified here. We will see it again later in the tensor product section, where it will make more sense.

11. Tensor Products of C*-algebras

Overall, sections 11.1 and 11.2 will be treated as preliminary material in the lecture, which will focus more on sections 11.3 and 11.7. Section 11.4 goes into much more difficult problems concerning injectivity and exactness for tensor products. The point there is to just give a feel for the the questions and obstacles in both settings. With time, we will touch on the topics in lecture. Section 11.5 gives a tensor product characterization of nuclearity (Theorem 11.50) and highlights some important examples (Remark 11.53). We will mention these in lecture but without much discussion. Section 11.6 establishes an important class of examples (Theorem 11.57), which we will be sure to mention in lecture, but without much word on the proof.

Section 11.3 defines the two primarily studied C*-norms on tensor products. These are quite analogous to the universal and reduced norms for discrete groups, and we will explore several tensor product analogies to results we saw for groups, e.g. Corollary 11.29, Proposition 11.34, and Proposition 11.33. Section 11.7 justifies our use of *completely* positive maps. We will cover Example 11.60 and mention how Stinespring's Dilation theorem is used in the proof of Theorem 11.61.

The way you read these notes will depend on your background and comfort level. If algebraic tensor products are new to you, spend more time in section 11.1. Regardless of your comfort level with algebraic tensors, be sure you've digested Exercise 11.11, which is quite foundational to the later sections. If you are still shaky on Hilbert space operators, linger in section 11.2. If you feel comfortable with (assuming) the material in these sections, but still want some more fundamental examples and arguments under your belt, check out sections 11.6 and 11.5.

One of the most important constructions in C*-algebras is the tensor product. Given two C*-algebras A and B, we form a C*-tensor product $A \otimes_\alpha B$ by taking the $*$-algebraic tensor product $A \odot B$ and completing with some C*-norm. In this section, we consider the two most prominent ones. This section is taken heavily from the first half of [3, Chapter 3].

One word on notation. Because there is so much significance to the norm on a given tensor product, we will denote algebraic tensor products by $\odot$ and tensor products that are also complete with respect to a norm by $\otimes$ (possibly with decoration to denote which norm). Sometimes $\otimes$ is used in the literature to denote an algebraic tensor product, and sometimes it is used to indicate the normed tensor product space with the spatial tensor product norm Definition 11.22. Usually authors are good about warning you of this.

11.1. **Facts about algebraic tensor products.** In this section we list some relevant facts about algebraic tensor products that we will take for granted in the lecture. Many of these are proved in [3, Section 3.1-3.2].

We give a non-constructive definition since it highlights the key properties: Let A and B be $\mathbb{C}$-vector spaces. Their tensor product is the vector space $A \odot B$, together with a bilinear map $\odot : A \times B \to A \odot B$, such that $A \odot B$ is universal in the following sense:
For any $\mathbb{C}$-vector space C and any bilinear map $\phi : A \times B \to C$, there exists a unique bilinear map $\tilde{\phi} : A \odot B \to C$ so that $\tilde{\phi}(a \odot b) = \phi(a, b)$ for all $a \in A$ and $b \in B$. The bilinearity of the map $\odot : A \times B \to A \odot B$ means that we have the following algebraic relations in $A \odot B$:

(1) $(a_1 + a_2) \odot b = (a_1 \odot b) + (a_2 \odot b)$ and
$a \odot (b_1 + b_2) = (a \odot b_1) + (a \odot b_2)$ for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$; and
(2) $\lambda(a \odot b) = (\lambda a) \odot b = a \odot (\lambda b)$ for all $a \in A$, $b \in B$, and $\lambda \in \mathbb{C}$.

Elements of the form $a \odot b$ for $a \in A$ and $b \in B$ are called *simple tensors*. Note that if $a = 0$ or $b = 0$, then $a \odot b = 0$.

Remark 11.1. $A \odot B$ is spanned by its simple tensors, but consists of many more elements. For example, in general the element $(a_1 \odot b_1) + (a_2 \odot b_2)$ cannot be written as a simple tensor $a \odot b$.

As a vector space, the notion of linear independence in an algebraic tensor product is a little technical but also technically very useful. We lay out the following propositions for later use.

As far as linear independence goes, the following propositions can be useful:

Proposition 11.2. *Suppose $\{a_1, ..., a_n\} \subset A$ are linearly independent and $\{b_1, ..., b_n\} \subset B$. Then*

$$\sum_1^n a_i \odot b_i = 0 \Rightarrow b_i = 0, \text{ for } 1 \le i \le n.$$

Proposition 11.3. *If $\{e_i\}_{i\in I}$ is a basis for A and $\{e'_j\}_{j\in J}$ is a basis for B, then $\{e_i \odot e'_j\}_{(i,j)\in I\times J}$ is a basis for $A \odot B$.*

Proposition 11.4. *If $\{e_i\}_{i\in I}$ is a basis for A and $x \in A \odot B$, then there exists a unique finite set $I_0 \subset I$ and $\{b_i\}_{i\in I_0}$ so that $x = \sum_{i\in I_0} e_i \odot b_i$.*

Just as we take tensor products of linear spaces, we can take tensor products of linear maps.[12] The following is more of a proposition/ definition; existence and uniqueness of these maps come from the above universal property.

Definition 11.5. Suppose $A_1 A_2, B_1, B_2$ are $\mathbb{C}$-vector spaces and $\phi_i : A_i \to B_i$, $i = 1, 2$ are linear maps. Then there is a unique linear map

$$\phi_1 \odot \phi_2 : A_1 \odot B_1 \to A_2 \odot B_2$$

so that $\phi_1 \odot \phi_2(a \odot b) = \phi_1(a) \odot \phi_2(b)$ for all $a \in A_1$, $b \in A_2$. This is called the *tensor product* of the maps ϕ_1 and ϕ_2.

The tensor product of linear maps preserves both injectivity and exact sequences:

Proposition 11.6. *Suppose A_1, A_2, B_1, B_2 are $\mathbb{C}$-vector spaces and $\phi_i : A_i \to B_i$, $i = 1, 2$ are injective linear maps. Then $\phi_1 \odot \phi_2$ is also injective.*

Proposition 11.7. *Suppose J, A, B, C are $\mathbb{C}$-vector spaces. If $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ is a short exact sequence (i.e. ι is injective, π is surjective, and $\ker(\pi) = \iota(J)$), then so is*

$$0 \to J \odot C \xrightarrow{\iota\odot id_C} A \odot C \xrightarrow{\pi\odot id_C} B \odot C \to 0.$$

We highlight a special case of this tensor product map when $B_1 = B_2$ is an algebra.

Definition 11.8. Suppose A_1, A_2 are $\mathbb{C}$-vector spaces, B a $\mathbb{C}$-algebra, and $\psi_i : A_i \to B$ are linear maps. Then there exists a unique linear map

$$\psi_1 \times \psi_2 : A_1 \odot A_2 \to B$$

so that $\psi_1 \times \psi_2(a \odot b) = \psi_1(a)\psi_2(b)$ for all $a \in A_1$, $b \in A_2$. This is called the *product* of the maps ψ_1 and ψ_2.

Exercise 11.9. Explain what is meant by $\psi_1 \times \psi_2$ is a "special case" of a tensor product of maps. (Think of the universal property and the bilinear map $B \odot B \to B$ given on simple tensors by $b_1 \odot b_2 \mapsto b_1 b_2$.)

We are interested in particular in tensor products of C*-algebras. When A and B are C*-algebras, then the algebraic tensor product is a $*$-algebra with multiplication and involution defined on simple tensors as

$$(a \odot b)^* = a^* \odot b^* \quad \text{ and } \quad (a_1 \odot b_1)(a_2 \odot b_2) = a_1 a_2 \odot b_1 b_2,$$

and extended linearly to all of $A \odot B$.

When we take the product of two $*$-homomorphisms $\psi_1 : A_1 \to B$ and $\psi_2 : A_2 \to B$, we are forced to impose an extra condition to guarantee that the product $\psi_1 \times \psi_2$ is again a $*$-homomorphism: the images must commute, i.e. for each $a_1 \in A_1$ and $a_2 \in A_2$, $\psi_1(a_1)\psi_2(a_2) = \psi_2(a_2)\psi_1(a_1)$.

Exercise 11.10. Justify the claim above, i.e. the product $\psi_1 \times \psi_2$ of two $*$-homomorphisms $\psi_1 : A_1 \to B$ and $\psi_2 : A_2 \to B$ is a $*$-homomorphism provided that the ranges $\psi_1(A_1)$ and $\psi_2(A_2)$ commute in B.

Recall from Section 9 where we defined a natural C*-norm on

$$\mathrm{M}_n(A) := \{[a_{ij}] \;:\; a_{i,j} \in A, 1 \le i, j \le n\}. \tag{11.1}$$

Exercise 11.11. Let A be any C*-algebra, $1 \le n < \infty$, and let $E_{i,j}$ denote the matrix units on $\mathrm{M}_n(\mathbb{C})$ (i.e. the matrices with 1 in the i, j coordinate and 0 elsewhere). Define a map $\pi : \mathrm{M}_n(A) \to \mathrm{M}_n \odot A$ by $\pi([a_{i,j}]) = \sum_{i,j=1}^n E_{i,j} \odot a_{ij}$. Show that this is an algebraic $*$-isomorphism.

[12] For those categorically inclined, tensors play well with linear categories and act like "multiplication" for objects/ morphisms. Ask Corey Jones after his expository talk.

11.2. **Tensor Products of Hilbert Space Operators.** We saw in the prereqs how to define a tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. (Recall that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of the algebraic tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the norm coming from the inner product which is given on simple tensors by $\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle$.)

Given operators $T_i \in B(\mathcal{H}_i)$ for $i = 1, 2$, we have a natural algebraic tensor product mapping $T_1 \odot T_2 : \mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ given on simple tensors by

$$(T_1 \odot T_2)(\xi \odot \eta) = T_1 \xi \odot T_2 \eta.$$

This extends linearly to a linear map $\mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ defined on sums of simple tensors by

$$T_1 \odot T_2 \sum_1^n c_j (\xi_j \otimes \eta_j) = \sum_1^n c_j (T_1 \xi_j \otimes T_2 \eta_j).$$

This map extends to an operator $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the following proposition.

Proposition 11.12. *Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and operators $T_i \in B(\mathcal{H}_i)$, $i = 1, 2$, there is a unique linear operator $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that*

$$T_1 \otimes T_2(\xi_1 \otimes \xi_2) = T_1 \xi_1 \otimes T_2 \xi_2$$

for all $\xi_i \in \mathcal{H}_i$, $i = 1, 2$, and moreover $\|T_1 \otimes T_2\| = \|T_1\| \|T_2\|$.

Proof. First, we want to show that the operator $T_1 \odot T_2$ is bounded on $\mathcal{H}_1 \odot \mathcal{H}_2$, which means we can extend it to a bounded operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Assume for now that $T_2 = 1_{\mathcal{H}_2}$, and write $T = T_1$. Let $\sum_1^n c_j (\xi_j \odot \eta_j) \in \mathcal{H}_1 \odot \mathcal{H}_2$. Using a Gram-Schmidt process, we may assume η_j are orthonormal (check). Then we compute

$$\begin{aligned}
\left\| T \odot 1_{\mathcal{H}_2} (\sum_{j=1}^n c_j (\xi_j \odot \eta_j)) \right\|^2 &= \left\| \sum_{j=1}^n c_j T \xi_j \odot \eta_j \right\|^2 = \left| \langle \sum_{i=1}^n c_i T \xi_i \odot \eta_i, \sum_{j=1}^n c_j T \xi_j \odot \eta_j \rangle \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \langle T \xi_i, T \xi_j \rangle \langle \eta_i, \eta_j \rangle \right| = \sum_{j=1}^n |c_j|^2 \|T \xi_j\|^2 \leq \|T\|^2 \sum_{j=1}^n |c_j|^2 \|\xi_j\|^2 \\
&= \|T\|^2 \left| \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \langle \xi_i, \xi_j \rangle \langle \eta_i, \eta_j \rangle \right| = \|T\|^2 \left\| \sum_{j=1}^n c_j (\xi_j \odot \eta_j) \right\|^2 .
\end{aligned}$$

Then $\|T \odot 1_{\mathcal{H}_2}\| \leq \|T\|$ on $\mathcal{H}_1 \odot \mathcal{H}_2$, meaning it extends to an operator in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, denoted by $T \otimes 1_{\mathcal{H}_2}$, with $\|T \otimes 1_{\mathcal{H}_2}\| \leq \|T\|$. Similarly, one shows that for any $T_2 \in B(\mathcal{H}_2)$, we have $1_{\mathcal{H}_1} \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Now, for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, we compose $(1_{\mathcal{H}_1} \otimes T_2)(T_1 \otimes 1_{\mathcal{H}_2})$ to get $T_1 \otimes T_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\|T_1 \otimes T_2\| \leq \|T_1\| \|T_2\|$ and

$$T_1 \otimes T_2(\xi_1 \otimes \xi_2) = T_1 \xi_2 \otimes T_2 \xi_2$$

for all $\xi_i \in \mathcal{H}_i$. To show that this norm inequality is an equality, we find, for any $\varepsilon > 0$, unit vectors $\xi_i \in \mathcal{H}_i$ with $|\|T_i \xi_i\| - \|T_i\|| < \varepsilon (2 \max_i \|T_i\|)^{-1}$ for $i = 1, 2$. Then, using Exercise 7.49 from Day 1, we have

$$\|(T_1 \otimes T_2)(\xi_1 \otimes \xi_2)\| = \|T_1 \xi_1 \otimes T_2 \xi_2\| = \|T_1 \xi_1\| \|T_2 \xi_2\| \sim_\varepsilon \|T_1\| \|T_2\|.$$

(That's shorthand for $\|T_1 \xi_1\| \|T_2 \xi_2\|$ is within epsilon of $\|T_1\| \|T_2\|$.) $\square$

We will take for granted that taking tensor products of operators is well-behaved with respect to addition, (scalar) multiplication, and adjoints.

Exercise 11.13. For $A = [a_{ij}] \in M_2(\mathbb{C}) = B(\mathbb{C}^2)$ and $B = [b_{i,j}] \in M_3(\mathbb{C}) = B(\mathbb{C}^3)$, write a matrix array for $A \otimes B \in B(\mathbb{C}^2 \otimes \mathbb{C}^3)$. (This is called a Kronecker product.)

In infinite dimensions, we do not have $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (the former is no longer automatically closed).

Proposition 11.14. *For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we define $*$-homomorphisms $\iota_i : B(\mathcal{H}_i) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by identifying $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C} 1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C} 1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$. These induce a product $*$-homomorphism $\iota_1 \times \iota_2 : B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, which is injective.*

Proof. Since $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ (**Exercise:** check) and $B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $\mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ commute in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (**Exercise:** check), we have from Section 11.1 the product $*$-homomorphism

$$B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

given by

$$\sum_{j=1}^n S_j \odot T_j \mapsto \sum_{j=1}^n (S_j \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes T_i) = \sum_{j=1}^n S_j \otimes T_j.$$

We just need to show that this map is injective, i.e. if the operator $\sum_{j=1}^n S_j \otimes T_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is zero, then the sum of elementary tensors $\sum_{j=1}^n S_j \odot T_j \in B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is also zero. By possibly re-writing the coefficients of the S_j, we may assume that the operators $\{S_j\}$ are linearly independent. If $0 = \sum_{j=1}^n S_j \otimes T_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then for all vectors $\xi_1, \eta_1 \in \mathcal{H}_1$ and $\xi_2, \eta_2 \in \mathcal{H}_2$, we have

$$\begin{aligned} 0 = \langle (\sum_{j=1}^n S_j \otimes T_j)(\xi_1 \otimes \xi_2), (\eta_1 \otimes \eta_2) \rangle &= \sum_{j=1}^n \langle S_j \xi_1 \otimes T_j \xi_2, \eta_1 \otimes \eta_2 \rangle \\ = \sum_{j=1}^n \langle S_j \xi_1, \eta_1 \rangle \langle T_j, \xi_2, \eta_2 \rangle &= \sum_{j=1}^n \langle \big(\langle T_j \xi_2, \eta_2 \rangle \big) S_j \xi_1, \eta_1 \rangle. \end{aligned}$$

Since this holds for all $\xi_1, \eta_1 \in \mathcal{H}_1$ the operator $\sum_{j=1}^n \langle T_j \xi_2, \eta_2 \rangle S_j \in B(\mathcal{H}_1)$ is zero (by Exercise 7.45 from Day 1 Lectures). Since we assumed the $\{S_j\}$ are linearly independent, it follows from Proposition 11.2 that the coefficients $\langle T_j \xi_2, \eta_2 \rangle$ must all be 0. Again, since this holds for all $\xi_2, \eta_2 \in \mathcal{H}_2$, it follows that each $T_j = 0 \in B(\mathcal{H}_2)$, which finishes the proof. $\square$

Corollary 11.15. *Given two representations $\pi_i : A_i \to B(\mathcal{H}_i)$, $i = 1, 2$, there is an induced representation*

$$\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

such that $\pi_1 \odot \pi_2(a_1 \odot a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_i \in A_i$, $i = 1, 2$.

We have discussed extending pairs of linear maps to tensor products, but what about restricting maps on tensor products to the tensor factors? Given a $*$-homomorphism on an algebraic tensor product of C*-algebras $\phi : A \odot B \to C$, when can we define restrictions $\phi|_A : A \to C$ and $\phi|_B : B \to C$? In general this is not so easy. In the unital setting, there is a natural way to do this.

Exercise 11.16. Suppose A, B, and C are C*-algebras with A and B unital and $\phi : A \odot B \to C$ a $*$-homomorphism. Then there exist $*$-homomorphisms $\phi_A : A \to C$ and $\phi_B : B \to C$ with commuting ranges such that $\phi = \phi_A \times \phi_B$.

A little harder to prove is the following (without the assumption that A and B are unital). See [3, Theorem 3.6.2].

Theorem 11.17. *Let A and B be* C**-algebras and $\pi : A \odot B \to B(\mathcal{H})$ a nondegenerate $*$-homomorphism. Then there exist nondegenerate representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ so that $\pi = \pi_A \times \pi_B$.*

Exercise 11.18. Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, show that the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images.

11.3. **C*-norms on tensor products.** For C*-algebras A and B, $A \odot B$ is a $*$-algebra. In order to turn it into a C*-algebra, we need to be able to define a C*-norm $\|\cdot\|$ on $A \odot B$. With this, $(A \odot B, \|\cdot\|)$ will be a *pre-*C**-algebra*, i.e. its completion is a C*-algebra. Much like the situation with groups, we are guaranteed the following:

- C*-norms on algebraic tensor products of C*-algebras always exist;
- there can be (very) many different C*-norms on a given algebraic tensor product of two C*-algebras;
- but we know how to describe the largest and smallest;[13] and
- it is extremely interesting to ask when the two coincide (and this is related to the notion of amenability for groups because math is beautiful).

[13]The second part of this statement is a deep theorem due to Takesaki.

Definition 11.19. For C*-algebras A and B, a *cross norm* on a $A\odot B$ is a norm $\|\cdot\|$ such that $\|a\otimes b\| = \|a\|\|b\|$ for every $a \in A$ and $b \in B$.

Example 11.20. We verified in the previous section that for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, the norm on $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ inherited from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a cross norm. In fact as a consequence of Takesaki's theorem[14] (which we will discuss more later in this section) every C*-norm on $A \odot B$ is a cross norm. We will take this as a fact as we proceed.

In Exercise 11.11, we saw that there is an algebraic $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \simeq \mathrm{M}_n(A)$, the latter being a C*-algebra with norm induced by the norm of A. Hence pulling back the norm along this $*$-isomorphism gives a C*-norm on $\mathrm{M}_n(\mathbb{C}) \odot A$ (i.e. $\|[\lambda_{ij}] \odot a\| = \|[\lambda_{ij}a]\|$). Moreover, $\mathrm{M}_n(\mathbb{C}) \odot A$ is already complete with respect to this norm, which means it is a C*-algebra. Hence any other C*-norm we define on $\mathrm{M}_n(A)$ agrees with this norm. (See remarks after Proposition 1.21.) That means we have proved the following proposition.

Proposition 11.21. *Let A be a C*-algebra and $1 \le n < \infty$. Then there is a unique C*-norm on the algebraic tensor product $\mathrm{M}_n(\mathbb{C}) \odot A$, which comes from the $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \simeq \mathrm{M}_n(A)$. Hence we write $\mathrm{M}_n(\mathbb{C}) \otimes A$.*

This identification also introduces very convenient notation, e.g. for the diagonal matrix in $\mathrm{M}_n(A)$ with $a \in A$ down the diagonal:

$$I_n \otimes a \;\leftrightarrow\; \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a \end{bmatrix}.$$

For general C*-algebras A and B, it should not be taken for granted that a C*-norm exists at all on $A \odot B$. However, it turns out the two most natural candidates both yield C*-norms.

The first is the spatial norm, i.e. the norm inherited as a subspace of bounded operators on a tensor product of Hilbert spaces. Recall that as a consequence of the GNS construction, every C*-algebra has at least one faithful representation on some Hilbert space.

Definition 11.22 (Spatial Norm)**.** Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. The *spatial* norm on $A_1 \odot A_2$ is

$$\left\|\sum a_i \odot b_i\right\|_{\min} = \left\|\sum \pi_1(a_i) \otimes \pi_2(b_i)\right\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

Remark 11.23. We will explain the $\|\cdot\|_{\min}$ notation later with Takesaki's theorem, which we keep mentioning.

Exercise 11.24. Check that $\|\cdot\|_{\min}$ is a semi-norm satisfying the C*-identity.

Proposition 11.25. *The semi-norm $\|\cdot\|_{\min}$ is a norm, i.e. for each $x \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then $x = 0$.*

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. Then the algebraic tensor product map $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is injective. By Proposition 11.14, we can view $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ as a $*$-subalgebra of $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and consequently have $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ injective. Then for any $x = \sum_{i=1}^n a_i \odot b_i \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then

$$0 = \|x\|_{\min} = \|\sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\| = \|(\pi_1 \odot \pi_2)(x)\|,$$

which by injectivity means $x = 0$. □

Hence $\|\cdot\|_{\min}$ is a norm, and we can define the C*-algebra

$$A \otimes B := \overline{A \odot B}^{\|\cdot\|_{\min}}.$$

It is sometimes denoted $A \otimes_{\min} B$, but we choose the undecorated notation to match the literature. In most cases this the unofficial "default" norm to take on a tensor product of C*-algebras.[15]

[14] Full disclosure, using this theorem is wayyyy overkill. A functional calculus argument could prove this, but this section is already long enough.

[15] For groups, it's the other way around and the maximal C*-completion of the group algebra is often the undecorated one.

For a sense of perspective, dropping the representation notation, we view $A_1 \subset B(\mathcal{H}_1)$ and $A_2 \subset B(\mathcal{H}_2)$. Then there is a natural way to stick them into a common C*-algebra, i.e. $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, from whence they can inherit the C*-norm, i.e. $A_1 \otimes A_2$ is the closure of the $*$-subalgebra $A_1 \odot A_2 \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

However, the norm was defined with an arbitrary choice of faithful representations. Fortunately, the value of the norm is independent of that choice.

Proposition 11.26. *Given faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$ and $\pi'_i : A_i \to B(\mathcal{H}'_i)$, then the minimal tensor norms $\|\cdot\|_{\min}$ and $\|\cdot\|'_{\min}$ defined by each pair of faithful representations agree.*

The proof is nice to see because it highlights two useful techniques. The first, yet again, is approximate identities. The second is the fact that there is only one C*-norm on $\mathrm{M}_n(B)$ for any C*-algebra B.

In our proof, we limit ourselves to the countable setting to avoid the extra notation involved with nets.

Proof. By symmetry, it suffices to prove the case where $\mathcal{H}_1 = \mathcal{H}'_1$ and $\pi_1 = \pi'_1$.

We first consider the case where $A_1 = \mathrm{M}_n(\mathbb{C})$ for some n. Since both $\|\cdot\|_{\min}$ and $\|\cdot\|'_{\min}$ are C*-norms, by Proposition 11.21, for every $x = \sum_{i=1}^m T_i \odot a_i \in \mathrm{M}_n(\mathbb{C}) \odot A_2$,

$$\left\|\sum_{i=1}^n \pi_1(T_i) \otimes \pi_2(a_i)\right\| = \|x\|_{\min} = \|x\|'_{\min} = \left\|\sum_{i=1}^n \pi_1(T_i) \otimes \pi'_2(a_i)\right\|. \tag{11.2}$$

Now, for the general separable case, take an increasing net of finite-rank projections $P_1 \leq P_2 \leq \ldots$ in $B(\mathcal{H}_1)$ where the rank of P_n is n and such that $\|P_n\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}_1$ (i.e. P_n converge in SOT to $1_{\mathcal{H}_1}$). Then for every $T \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})$ converges in $*$-SOT[16] to T, and so we have (check)

$$\|T\| = \sup_n \|(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})\|.$$

That means for any $x = \sum_{i=1}^m a_i \odot b_i \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup_n \left\|\sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2(b_i)\right\|$$

$$\|x\|'_{\min} = \sup_n \left\|\sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi'_2(b_i)\right\|.$$

For $n \geq 1$, define a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to P_nB(\mathcal{H})P_n$. Since ϕ is a faithful representation of $\mathrm{M}_n(\mathbb{C})$, by (11.2), we have

$$\begin{aligned}\left\|\sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi_2(b_i)\right\| &= \left\|\sum_{i=1}^m \phi(\phi^{-1}(P_n\pi(a_i)P_n)) \otimes \pi_2(b_i)\right\| \\ &= \left\|\sum_{i=1}^m \phi(\phi^{-1}(P_n\pi(a_i)P_n)) \otimes \pi'_2(b_i)\right\| \\ &= \left\|\sum_{i=1}^m P_n\pi(a_i)P_n \otimes \pi'_2(b_i)\right\|.\end{aligned}$$

It follows that $\|x\|_{\min} = \|x\|'_{\min}$. ∎

So, given C*-algebras A_1 and A_2 and faithful nondegenerate representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we complete $\pi_1 \odot \pi_2$ to a faithful representation

$$\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

There is another often useful description of the minimal tensor norm.

Proposition 11.27. *For C*-algebras A_1 and A_2, and $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$,*

$$\|x\|_{\min} = \sup\{\|\sum_{j=1}^n \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \text{ (nondegenerate) representations}\}.$$

[16] $S_n \to S$ in $*$-SOT if $S_n \to S$ in SOT and $S_n^* \to S^*$ in SOT.

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}_i')$ be faithful representations. Then by Exercise 4.16, $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}_i')$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}_i')$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2$... □

Exercise 11.28. Finish the proof of Proposition 11.27. This is an example of a technique where one can *dilate* a map to one with a desired property (e.g. faithfulness) and then *cut down* to the original map to draw the desired conclusion.

Corollary 11.29. *For a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism*

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2.$$

Proof. We are charged with showing that $\phi_1 \odot \phi_2$ is continuous with respect to the topologies on $A_1 \odot A_2$ and $B_1 \odot B_2$ induced by their respective $\|\cdot\|_{\min}$ norms. We know that there exist faithful representations $\pi_i^A : A_i \to B(\mathcal{H}_i^A)$ and faithful representations $\pi_i^B : B_i \to B(\mathcal{H}_i^B)$. So if $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$, the fact that $*$-homomorphisms are norm-decreasing means that

$$\|x\|_{A_1 \otimes_{\min} A_2} = \|\sum_{j=1}^n \pi_1^A(a_j) \otimes \pi_2^A(b_j)\| \geq \|\sum_{j=1}^n \pi_1^B(\phi_1(a_j)) \otimes \pi_2^B(\phi_2(b_j))\| = \|\phi_1 \odot \phi_2(x)\|_{B_1 \otimes_{\min} B_2}.$$

But each $\pi_i^B \phi_i : A_i \to B(\mathcal{H}_i^B)$ is a representation of A_i, so we complete the proof via an appeal to the preceding proposition. □

Just as with groups, there is another natural norm which comes from taking all possible representations.

Definition 11.30 (Maximal Norm)**.** Let A and B be C*-algebras. We define the maximal C*-tensor norm on $A \odot B$ by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \text{ a (nondegenerate) rep}\}$$

for all $x \in A \odot B$.

The first question is if this is even finite; it is by Theorem 11.17. Indeed, given $\pi : A \odot B \to B(\mathcal{H})$, with restrictions $\pi|_A$ and $\pi|_B$, then we have for all simple tensors $a \odot b \in A \odot B$,

$$\|\pi(a \odot b)\| = \|\pi|_A(a)\pi|_B(b)\| \leq \|\pi|_A(a)\|\|\pi|_B(b)\| \leq \|a\|\|b\|. < \infty.$$

Just as we argued for groups (Proposition 5.7), this with the triangle inequality guarantees that $\|x\|_{\max} < \infty$ for all $x \in A \odot B$.

Exercise 11.31. Check that $\|\cdot\|_{\max}$ is a semi-norm satisfying the C*-identity.

For any pair of faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we get a representation $\pi = \pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It follows that for any $x \in A_1 \odot A_2$,

$$\|x\|_{\min} = \|\pi(x)\| \leq \|x\|_{\max}.$$

So, for any $x \in A_1 \odot A_2$,

$$\|x\|_{\max} = 0 \Rightarrow \|x\|_{\min} = 0 \Rightarrow x = 0,$$

which means $\|\cdot\|_{\max}$ is a norm. Hence we define the C*-algebra

$$A_1 \otimes_{\max} A_2 := \overline{A_1 \odot A_2}^{\|\cdot\|_{\max}}.$$

Remark 11.32. Note that by definition, the $*$-algebra $A_1 \odot A_2$ is a dense subalgebra in $A_1 \otimes_{\max} A_2$ and $A_1 \otimes A_2$.

Just as with groups, the maximal tensor product enjoys the following universal property.

Proposition 11.33. *If $\phi : A_1 \odot A_2 \to C$ is a $*$-homomorphism, then there exists a unique $*$-homomorphism $A_1 \otimes_{\max} A_2 \to C$, which extends ϕ. In particular, any pair of $*$-homomorphisms $\phi_i : A_i \to C$ with commuting ranges induces a unique $*$-homomorphism*

$$\phi_1 \times \phi_2 : A \otimes_{\max} B \to C.$$

Note that this is really just a statement about norms, and it is a theme we've seen before (Proposition 5.7). Let's flesh out a more general idea that underlies both.

Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. (Notice that, unless $B_0 = B$, this means B_0 is *not* a C*-algebra.) The only obstruction to extending π to a $*$-homorphism on B is if π is not contractive on B_0, i.e. $\|\pi(b)\| > \|b\|$ for some $b \in B_0$. In other words, π extends to B iff π is contractive on B_0. The necessity is easy to see. Indeed, if π does extend to B, then the C*-norm on B forces π to be contractive on all of B, including B_0. On the other hand, if $\pi : B_0 \to C$ is a contractive $*$-homomorphism, then it is in particular bounded, which means it extends to a bounded homomorphism $\pi : B \to C$. Moreover, just as we saw in Proposition 5.7, for any $b \in B$ with $b_n \in B_0$ converging to b, we have $\pi(b_n) \to \pi(b)$ and hence $\pi(b_n)^* \to \pi(b)^*$. Then by uniqueness of limits, $\pi(b^*) = \pi(b)$ since

$$\|\pi(b_n)^* - \pi(b^*)\| = \|\pi(b_n^*) - \pi(b^*)\| \to 0.$$

For the sake of reference, we record this in a proposition:

Proposition 11.34. *Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. Then π extends to B iff π is contractive on B_0.*

With that digression, the proof of proposition 11.33 is quite immediate.

Proof of Proposition 11.33. Take a faithful nondegenerate representation $\pi : C \to B(\mathcal{H})$. Then $\pi \circ \phi : A_1 \odot A_2 \to B(\mathcal{H})$ is a contractive $*$-homomorphism (with respect to the $\|\cdot\|_{\max}$ norm) and hence extends to $A \otimes_{\max} A_2$. □

It follows from this that $\|\cdot\|_{\max}$ is the largest possible C*-norm on $A_1 \odot A_2$.

Corollary 11.35. *Given any C*-norm $\|\cdot\|$ on $A_1 \odot A_2$, there is a surjective $*$-homomorphism $A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ extending the identity map on $A_1 \odot A_2$.*

Proof. Suppose $\|\cdot\|$ is another C*-norm on $A_1 \odot A_2$. Then the identity map $A_1 \odot A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ is a $*$-homomorphism, which then extends to a $*$-homorphism

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}.$$

Since it is a $*$-homomorphism, its image is closed and contains the dense subset $A_1 \odot A_2$, and so it is a surjection. As a surjective $*$-homomorphism, it is contractive, and so $\|x\|_{\max} \geq \|x\|$ for all $x \in A_1 \odot A_2$. □

Remark 11.36. Very often in the literature, the closure of $A \odot B$ with respect to an arbitrary tensor norm is denoted by $A \otimes_\alpha B$ where the norm is denoted by $\|\cdot\|_\alpha$.

It turns out that the spatial norm $\|\cdot\|_{\min}$ *is* the minimal C*-norm on $A_1 \odot A_2$. This is an important theorem due to Takesaki whose proof involves some heavy work in extending states to tensor products. For the sake of time, we will have to take this for granted. The proof is worked out in [3, Section 3].

Theorem 11.37 (Takesaki)**.** *The spatial norm $\|\cdot\|_{\min}$ is the minimal C*-norm on $A_1 \odot A_2$. In other words, given any C*-norm $\|\cdot\|$ on $A_1 \odot A_2$, there are surjective $*$-homomorphisms*

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|} \to A_1 \otimes A_2$$

extending the identity map

$$A_1 \odot A_2 \to A_1 \odot A_2 \to A_1 \odot A_2.$$

It follows that if the *natural* surjection $A_1 \otimes_{\max} A_2 \to A_1 \otimes A_2$ is injective, then $A_1 \odot A_2$ has a unique tensor norm. This fact is often indicated by writing

$$A_1 \otimes_{\max} A_2 = A_1 \otimes A_2.$$

Remark 11.38. It is important here that it is this natural surjection that is also injective, i.e. the one that extends the identity map $A_1 \odot A_2$.

We have been avoiding the non-unital elephant in the room. We relegate the proof to [3, Corollary 3.3.12].

Proposition 11.39. *If A and B are* C*-*algebras with A non-unital, then any* C*-*norm on $A \odot B$ can be extended to a* C*-*norm on $\tilde{A} \odot B$ (meaning the norms agree on $A \odot B \subset \tilde{A} \odot B$). Similarly, when both A and B are non-unital, any* C*-*norm can be extended to $\tilde{A} \odot \tilde{B}$.*[17]

Exercise 11.40. For C*-algebras A and B, we have canonical[18] isomorphisms $A \otimes B \simeq B \otimes A$ and $A \otimes_{\max} B \simeq B \otimes_{\max} A$.

11.4. **Inclusions and Short Exact Sequences.** This section is dedicated to two properties that held automatically for algebraic tensor products but that can now fail for their C*-completions:

(1) They respect inclusions, i.e. if B and C are C*-algebras and $A \subset B$ a C*-subalgebra, then we have a natural inclusion
$$A \odot C \hookrightarrow B \odot C.$$
(2) They respect exact sequences, i.e. if B and C are C*-algebras and $J \triangleleft B$ an ideal, then the following sequence is exact.
$$0 \to J \odot C \to B \odot C \to B/J \odot C \to 0.$$

Proposition 11.41. *Let B and C be* C*-*algebras, $A \subset B$ a* C*-*subalgebra, and $J \triangleleft B$ an ideal. Then*

(1) We have a natural inclusion $A \otimes_{\min} C \subseteq B \otimes_{\min} C$.
(2) This can fail for the maximal tensor product.

Exercise 11.42. Check (1). (This is just a statement about norms on sums of simple tensors.)

For (2), that's where things get interesting. Questions about embeddability of maximal tensor products get hard quick. So, it's easiest to explain why it can go wrong. Recall that the maximal tensor product norm was defined as a supremum over all representations. A representation on $B \odot C$ restricts to one on $A \odot C$, but a representation on $A \odot C$ need not extend to $B \odot C$. So, in general the sup taken for the maximal norm on $A \odot C$ is taken over a larger set than the one for $B \odot C$.

Remark 11.43. One fact that will play a role promptly is that this *does* hold when A is an ideal in B. A representation from an ideal $J \triangleleft A$ in a C*-algebra does always extend to a representation on A (see [1, Section 1.3]). So when $J \triangleleft A$ is an ideal, then so is $J \odot C$ for any C*-algebra C, and we have $J \otimes_{\max} C \triangleleft A \otimes_{\max} C$.

Here are some examples of where this can go wrong. Unfortunately, we haven't built up sufficient terminology to explain why.

Example 11.44. Let $A \subset B(\mathcal{H})$ be a separable C*-algebra lacking Lance's Weak Expectation Property ([3, Exercise 2.3.14]), e.g. an exact C*-algebra that is non-nuclear (exactness due to Wasserman), such as $\mathrm{C}^*_r(\mathbb{F}_2)$. Then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

Using Kirchberg's $\mathcal{O}_2$ embedding theorem (a very difficult and sophisticated result in C*-theory) as well as his groundbreaking work enabling the recent solution to Connes' Embedding Problem (more on that later), we can give another example: $\mathrm{C}^*_r(\mathbb{F}_2)$ embeds into $\mathcal{O}_2$ (because it is exact and separable), but $\mathrm{C}^*_r(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$ does not embed into $\mathcal{O}_2 \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2)$.

Proposition 11.45. *Let B and C be* C*-*algebras and $J \triangleleft B$ an ideal. Then*

(1) The sequence
$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to B/J \otimes_{\max} C \to 0$$
is exact.
(2) This can fail for the minimal (i.e. spatial) tensor product.

For (1), the proof in full detail is provided in [3, Proposition 3.7.1]. We simply give an idea of what needs to be shown. In either case, $J \otimes_{\max} C \triangleleft B \otimes_{\max} C$ and $J \otimes C \triangleleft B \otimes C$. So we have exact sequences
$$0 \to J \otimes_{\max} C \to B \otimes_{\max} C \to (B \otimes_{\max} C)/(J \otimes_{\max} C) \to 0$$
and
$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0.$$

[17]In general (i.e. when we don't have $A = \tilde{A}$ or $B = \tilde{B}$, this is a larger algebra than $\widetilde{A \odot B}$.

[18]i.e. This is another way of saying "natural". In this setting, this means the maps extend the usual algebraic maps.

In both cases, from the algebraic identification $B/J \odot C \simeq (B \odot C)/(B \odot J)$ one argues that there is a C*-norm so that

$$(B \otimes_{\max} C)/(J \otimes_{\max} C) \simeq B/J \otimes_\alpha C \quad \textit{and} \quad (B \otimes_{\min} C)/(J \otimes_{\min} C) = B/J \otimes_\beta C.$$

It will follow from the maximality of $\|\cdot\|_{\max}$ that $\otimes_\alpha = \otimes_{\max}$. But for the other quotient, that won't always happen.

Definition 11.46. We say a C*-algebra C is *exact* if the sequence

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to (B \otimes_{\min} C)/(J \otimes_{\min} C) \to 0$$

is exact for any C*-algebra B and any ideal $J \triangleleft B$.

Though seemingly unrelated, the two definitions we have given for exactness are indeed equivalent, though the proof of this is not easy.

Theorem 11.47 (Kirchberg)**.** *A C*-algebra is exact in the sense of Definition 10.17 if and only if the functor $\otimes_{\min} A$ is exact, i.e. if the above definition holds.*

The question of when two C*-algebras have a unique C*-tensor norm is very difficult, and resolving this question for certain algebras is equivalent to resolving big open problems.

For example, thanks to deep and groundbreaking work of Kirchberg, we know that a famous recently-resolved problem, Connes' Embedding Problem, is equivalent to answering the question of whether or not $C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$. (Ask Brent and Rolando for the the original statement.) Further work (building on Kirchberg's results) connected this to what is known as Tsirelson's problem in quantum information theory, which was what was actually refuted earlier this year.

Another example is A. Thom's example of a hyperlinear group that is not residually finite. (Again, thanks to work of Kirchberg, this is equivalent to the full group C*-algebra of said group not having a unique tensor norm with $B(\mathcal{H})$.)

Another example is Junge and Pisier's proof that $B(\mathcal{H}) \odot B(\mathcal{H})$ does not have a unique C*-tensor norm when $\mathcal{H}$ is infinite dimensional, which was proven by Kirchberg to be equivalent to another collection of open problems.

Remark 11.48. You may have noticed that Kirchberg was very influential in a lot of results pertaining to tensor products of C*-algebras. Yeah.

Remark 11.49 (Remark on tensors and commutivity). Given C*-algebras A_1 and A_1, an example of a representation of $A_1 \odot A_2 \to B(\mathcal{H})$ is the tensor product of two representations,

$$\sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

But in general, there can be many representations that are not of this form, i.e. for some $x \in A_1 \odot A_2$, we could have

$$\begin{aligned}\|x\|_{\max} &= \sup\{\|\pi(x)\| \ : \pi : A_1 \odot A_2 \to B(\mathcal{H})\} \\ &> \sup\{\|\pi_1 \odot \pi_2(x)\| \ : \pi_i : A_i \to B(\mathcal{H}_i)\}.\end{aligned}$$

On an philosophical level, this is a question about commutivity. Given C*-algebras A_1 and A_2, is there any context (= C*-algebra they can be simultaneously embedded into) where A_1 and A_2 commute but *not* as tensors. Let's try to flesh this out a little.

Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images (Exercise 11.18). When $\pi = \sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have a much better idea of what the images are and why they commute. In this case the restrictions are given for $a_i \in A_i$ by

$$\pi_1(a_1) = \sigma_1(a_1) \otimes 1_{\mathcal{H}_1} \quad \text{and} \quad \pi_2(a_2) = 1_{\mathcal{H}_2} \otimes \sigma_2(a_2).$$

Then we have

$$\pi_1(a_1)\pi_2(a_2) = \big(\sigma_1(a_1)\otimes 1_{\mathcal{H}_1}\big)\big(1_{\mathcal{H}_2}\otimes\sigma_2(a_2)\big) = \sigma_1(a_1)\otimes\sigma_2(a_2) = \big(1_{\mathcal{H}_2}\otimes\sigma_2(a_2)\big)\big(\sigma_1(a_1)\otimes 1_{\mathcal{H}_1}\big) = \pi_2(a_2)\pi_1(a_1).$$

11.5. **Nuclearity.** On the other end of the spectrum are C*-algebras which always have unique tensor product norms. The term originally used for such C*-algebras was in fact "nuclear." But we've already used this term for C*-algebras satisfying the completely positive approximation property. That these two coincide is a remarkable theorem, independently proved by Choi-Effros and Kirchberg

Theorem 11.50 (Choi-Effros, Kirchberg)**.** *A* C**-algebra A satisfies the completely positive approximation property (Definition 10.6) if and only if $A \odot B$ has a unique* C**-tensor norm for any* C**-algebra B.*

The proof of this theorem would require us to build up a fair bit of theory first, so we simply point you to Chapters 2 and 3 in [3], where the argument and surrounding theory is laid out quite well.

In general, it's often easier to prove that a C*-algebra has the completely positive approximation property (an internal property) as opposed to always having a unique tensor product norm (an external property). However, it was not so hard to show the latter for one class of C*-algebras.

Example 11.51. From Proposition 11.21, we know that $\mathrm{M}_n(\mathbb{C})$ is nuclear for any $n \geq 1$. It turns out that any finite-dimensional C*-algebra is nuclear. (This mostly comes down to Proposition 6.1. See [7, Theorem 6.3.9] for more details.)

We have already seen that $K(\mathcal{H})$, as an AF algebra, is nuclear. Just for fun, here's an argument from the tensor product perspective.

Example 11.52. Let $\mathcal{K}$ denote the compact operators on some Hilbert space $\mathcal{H}$ and A any C*-algebra.

First we claim that $FR(\mathcal{H}) \odot A$ is a dense $*$-subalgebra of $\mathcal{K} \odot A$ with respect to any C*-norm on $\mathcal{K} \odot A$. We know from Day 1 lectures that $FR(\mathcal{H})$ is dense in $\mathcal{K}$. Now, suppose $S \odot a \in \mathcal{K} \odot A$ and $S_j \in FR(\mathcal{H})$ a sequence with $S_j \to S$. Recall that any C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$ is a cross norm, and so for any norm C*-norm $\|\cdot\|$ on $\mathcal{K} \odot A$, we have

$$\|(S \odot a) - (S_j \odot a)\| = \|(S - S_j) \odot a\| = \|S - S_j\|\|a\| \to 0.$$

Using the triangle inequality, we can extend this to show that any $x = \sum_{j=1}^m T_j \odot a_j \in \mathcal{K} \odot A$ can be approximated in any C*-norm by sums of simple tensors of finite rank operators.

So if we know $\|x\|_{\max} = \|x\|_{\min}$ for any $x \in FR(\mathcal{H}) \odot A$, then it follows that the natural surjection $\mathcal{K} \otimes_{\max} A \to \mathcal{K} \otimes A$ is isometric and $\mathcal{K}$ is nuclear. Fix $x = \sum_{j=1}^m T_j \odot a_j \in FR(\mathcal{H}) \odot A$, and let $\pi : \mathcal{K} \odot A \to B(\mathcal{H}')$ be a representation. Then there exists a projection $P \in B(\mathcal{H})$ of rank $n < \infty$ such that $T_j = PT_jP$ for each j, and $x = \sum_{j=1}^m PT_jP \odot a_j$. Hence $x \in PB(\mathcal{H})P \odot A$. From Exercise 7.41 from Day 1 Lectures, we have a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P$, and hence a representation $\pi' := \pi \circ (\phi \odot id_A) : \mathrm{M}_n(\mathbb{C}) \odot A \to B(\mathcal{H})$.

Since we know $\mathrm{M}_n(\mathbb{C}) \otimes_{\max} A = \mathrm{M}_n(\mathbb{C}) \otimes_{\min} A$, we know that for any faithful representations $\sigma_1 : \mathrm{M}_n(\mathbb{C}) \to B(\mathcal{H}_1)$ and $\sigma_2 : A \to B(\mathcal{H}_2)$,

$$\begin{aligned}
\|\sum_{j=1}^m \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)} &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\min} \\
&= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\max} \geq \|\pi'(\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j)\| \\
&= \|\pi(\sum_{j=1}^m PT_jP \odot a_j)\| = \|\pi(x)\|.
\end{aligned}$$

In particular, this holds for the faithful representations $\sigma_1 = \mathrm{id}_{\mathcal{K}} \circ \phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P \subset \mathcal{K} \hookrightarrow B(\mathcal{H})$ and any faithful representation σ_2 of A. But then we have

$$\begin{aligned}
\|x\|_{\min} &= \|\sum_{j=1}^m \mathrm{id}_{\mathcal{K}}(S_j) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\
&= \|\sum_{j=1}^m \sigma_1(\phi^{-1}(PS_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\
&\geq \|\pi(x)\|.
\end{aligned}$$

Since $\pi : \mathcal{K} \odot A \to B(\mathcal{H}')$ was arbitrary, it follows that

$$\|x\|_{\min} \geq \|x\|_{\max},$$

which finishes the proof.

Remark 11.53. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. It follows from Example 11.52 that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \odot \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). This will be a closed two-sided ideal in $B(\ell^2 \odot \ell^2)$, which means it must be the compact operators $\mathcal{K}(\ell^2 \odot \ell^2)$. Moreover, after a permutation of the basis elements, we have $\ell^2 \otimes \ell^2 \simeq \ell^2$. With this, one can then argue that $\mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K}$. More generally, we say a C*-algebra is *stable* if $A \otimes \mathcal{K} \simeq A$. (Because of nuclearity, it does not matter what tensor product we choose.)

Since $\mathcal{K}$ is stable and since $(A \otimes \mathcal{K}) \otimes \mathcal{K} \simeq A \otimes (\mathcal{K} \otimes \mathcal{K}) \simeq A \otimes \mathcal{K}$ for any C*-algebra A[19], we call $A \otimes \mathcal{K}$ the *stabilization* of A. This is a basic object in many results and theories in C*-algebras, such as multiplier algebras, K-theory and classification, and is closely tied to Morita equivalence for C*-algebras. It turns out that the stabilization of A is very similar to A from the perspective of many C*-algebraic invariants, and so replacing A by its stabilization gives one more "wiggle room" for computations without affecting the underlying structure very much.

There is another fundamental class of nuclear C*-algebras: commutative C*-algebras. This was not so hard to prove with the completely positive approximation property definition of nuclearity (Proposition 10.10). Before the Choi-Effros/Kirchberg theorem, Takesaki showed that tensor products with commutative C*-algebras always have a unique C*-norm, but the proof was much more involved.

Theorem 11.54 (Takesaki)**.** *Let A and C be* C**-algebras with C commutative. Then there is a unique* C**-tensor norm on $C \odot A$.*

11.6. **$C_0(X, A)$ as tensor products.** Let us spend a little more time on this last class of nuclear C*-algebras. Recall from the Gelfand Naimark Theorem that any commutative C*-algebra is $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. With this in mind look into another description of the tensor product of a C*-algebra with a commutative C*-algebra.

Definition 11.55. Let A be a C*-algebra and X a locally compact Hausdorff space (when X is not compact, we denote by $X \cup \{\infty\}$ its one point compactification). Just as we did for $A = \mathbb{C}$, we define

$$C_0(X, A) := \{f : X \cup \{\infty\} \to A \ : \ f \text{ continuous and } f(\infty) = 0\}.$$

When X is moreover compact, this is the same as $C(X, A)$.

Lemma 11.56. *Let A be a* C**-algebra and X a locally compact Hausdorff space. Define the $*$-homomorphism $\phi : C_0(X) \odot A \to C_0(X, A)$ on simple tensors by $f \odot a \mapsto f(\cdot)a$. This gives a $*$-homomorphism, which then extends to a surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$. Moreover, ϕ is injective on $C_0(X) \odot A$.*

The proof that the image of ϕ is dense in $C_0(X, A)$ is another example of a "partition of unity argument." We will give the argument from [7, Lemma 6.4.16] in the case where X is compact. The non-compact case amounts to identifying $C_0(X, A) = \{f \in C(X \cup \{\infty\}, A) \ : f(\infty) = 0\}$ (see [7, Lemma 6.4.16] for full details).

Recall that we take for granted the fact from topology that, given any compact Hausdorff space X with open cover $U_1, ..., U_n$, there exist continuous functions $h_1, ..., h_n : X \to [0, 1]$ so that $\operatorname{supp}(h_j) \subset U_j$ and $\sum_j h_j = 1$. (See [Theorem 2.13, Rudin, Real and Complex Analysis].) This is a *partition of unity* subordinate to $U_1, ..., U_n$ (in fact a rather nice one).

Proof of Lemma 11.56. Since there is nothing surprising in checking that it is a $*$-homomorphism, which by universality extends to a $*$-homomorphism on $A \otimes_{\max} B$, we move straight to the surjective $*$-isomorphism claim.

For the surjectivity argument, we assume X is compact (or work in its one point compactification as aforementioned). Since the image of a $*$-homomorphism from a C*-algebra is closed, it suffices to show that $C(X, A)$ is the closed linear span of functions of the form $f(\cdot)a$ for $f \in C(X)$ and $a \in A$. Let $g \in C(X, A)$ and $\varepsilon > 0$. Since X is compact and g continuous, $g(X)$ is compact, which means we can find a finite collection

[19] In fact, the associativity for the minimal and maximal tensor product norms holds for all C*-algebras, i.e. for C*-algebras A, B, C, we have $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ and $(A \otimes_{\max} B) \otimes_{\max} C \simeq A \otimes_{\max} (B \otimes_{\max} C)$. This is normally an exercise, but we have plenty already.

$a_1, ..., a_n \in g(X) \subset A$ so that $\{B_\varepsilon(a_j)\}_j$ covers $g(X)$, and hence $U_j = g^{-1}(B_\varepsilon(a_j))$ forms a finite open cover of X. Since X is compact, the aforementioned fact from topology tells us there exist continuous functions $h_j : X \to [0,1]$, $1 \leq j \leq n$ so that for each j, $\mathrm{supp}(h_j) \subset U_j$ and $\sum_j h_j(x) = 1$ for all $x \in X$. Notice that, by our choice of U_j, that means that for each $x \in X$, either $h_j(x) = 0$ or $\|g(x) - a_j\| < \varepsilon$. Then we compute for each $x \in X$,

$$\begin{aligned} \|g(x) - \sum_j h_j(x)a_j\| &= \| \left(\sum_j h_j(x)\right) g(x) - \sum_j h_j(x)a_j\| \\ &= \|\sum_j h_j(x)(g(x) - a_j)\| \leq \sum_j h_j(x)\|g(x) - a_j\| \\ &\leq \sum_j h_j(x)\varepsilon = \varepsilon. \end{aligned}$$

This establishes our claim.

For injectivity, on $C_0(X) \odot A$, suppose $c = \sum_{j=1}^n f_j \odot a_j \in \ker(\phi)$ where $f_1, ..., f_n \in C_0(X)$ and $a_1, ..., a_n$ are linearly independent elements of A. Then $\phi(c) = 0$ implies that $\sum f_j(x)a_j = 0$ for all $x \in X$. But now these $f_j(x)$ are just complex numbers, and so the linear independence of the $a_1, ..., a_n$ implies that $f_j(x) = 0$ for each $1 \leq j \leq n$ and every $x \in X$. That means $f_1 = ... = f_n = 0$ and so $c = 0$. Hence ϕ is injective on $C_0(X) \odot A$. □

Theorem 11.57. *If A is a C*-algebra and X is a locally compact Hausdorff space, then for any C*-tensor norm, we have $\overline{C_0(X) \odot A}^{\|\cdot\|} \simeq C_0(X, A)$.*

Proof. Since the map ϕ from Lemma 11.56 is injective, the pull-back of the norm from $C_0(X, A)$ (i.e. $\|c\| = \|\pi(c)\|$) gives a C*-norm on $C_0(X) \odot A$ (as opposed to just a semi-norm). By Theorem 11.54, there is a unique C*-tensor norm on $C_0(X) \odot A$, which means this norm agrees with $\| \cdot \|_{\max}$. Hence the surjective $*$-homomorphism $C_0(X) \otimes_{\max} A \to C_0(X, A)$ is isometric, and hence a $*$-isomorphism. By identifying $C_0(X) \otimes_{\max} A$ with the closure of $C_0(X) \odot A$ under any other C*-norm, the claim follows. □

Example 11.58. Three particularly interesting cases are when $X = [0,1]$, $X = (0,1]$, and $X = (0,1)$. [20] For a C*-algebra A, the *cone* over A is the C*-algebra

$$CA := C_0((0,1], A) = \{f : (0,1] \to A \ : \ f \text{ is continuous and } \lim_{t \to 0} f(t) = 0\},$$

and the *suspension*[21] over A is the C*-algebra,

$$SA := C_0((0,1), A) := \{f : (0,1)] \to A \ : \ f \text{ is continuous and } \lim_{t \to 0} f(t) = 0 = \lim_{t \to 1} f(t)\}.$$

The suspension will become very important when we get to K-theory. It is also sometimes denoted by ΣA.

11.7. **Continuous linear maps on tensor products.** In Takesaki's proof that $\| \cdot \|_{\min}$ is the smallest C*-norm, a delicate and crucial part of the argument is showing that states extend to tensor products, i.e. for $\phi_i \in S(A_i)$, $\phi_1 \odot \phi_2$ extends to a state on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ for any C*-norm $\| \cdot \|$ (mapping into $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$).

Given a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, we have a $*$-homomorphism

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

defined on the dense $*$-subalgebra $A_1 \odot A_2$ of $\overline{A_1 \odot A_2}^{\|\cdot\|}$ where $\|\cdot\|$ is any C*-norm. By Proposition 11.34, this extends to a $*$-homomorphism on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ iff $\phi_1 \odot \phi_2$ is contractive on sums of simple tensors. Naturally, this depends on the norm we put on $B_1 \odot B_2$ (e.g. if $A_i = B_i$ and we give $A_1 \odot A_2$ the minimal norm and $B_1 \odot B_2$ the maximal norm).

We already saw in Corollary 11.29 that this holds when we consider both $A_1 \odot A_2$ and $B_1 \odot B_2$ with their respective minimal tensor product norms.

[20]Depending on how we like to define our functions these intervals are sometimes replaced with homeomorphic copies, e.g., sometimes $\mathbb{R}$ is used in place of $(0,1)$. This certainly makes the "∞" notation more natural!

[21]"Cone" and "suspension" are not to be confused with the notions from topology, in case you are wondering.

Exercise 11.59. Show that for a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism on

$$\phi_1 \otimes_{\max,\beta} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_\beta B_2$$

for any C*-tensor product $B_1 \otimes_\beta B_2$.

However, many maps that we want to work with (e.g. states) are not necessarily $*$-homomorphisms. Hence it is important to understand which class of bounded linear maps extend to tensor products, in particular, for which bounded linear maps $\phi_i : A_i \to B_i$ does $\phi_1 \odot \phi_2$ extend to continuous linear maps

$$\phi_1 \otimes_{\max} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$$

and

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2?$$

Let us consider an example where this fails.

Example 11.60. Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. As we saw in Example 11.52, $\mathcal{K}$ is nuclear, meaning in particular that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \otimes \ell^2)$ (via the tensor product of faithful representations $id_\mathcal{K} \odot id_\mathcal{K}$). For each i, j, we define the rank one operator $P_{i,j} = \langle \cdot, e_i \rangle e_j$. (Think of these as an infinite-dimensional version of the matrix units for $\mathrm{M}_n(\mathbb{C})$.) For each $n \geq 1$, define $V_n \in \mathcal{K} \otimes \mathcal{K}$ by

$$V_n := \sum_{i,j=1}^{n} P_{i,j} \otimes P_{j,i}.$$

Then V_n is a partial isometry. (Indeed, since $P_{i,j}P_{l,k} = \delta_{j,l}P_{i,k}$, we can compute that $V_n^* V_n = P_n \odot P_n$ where P_n is the rank n projection sending $e_j \mapsto e_j$ for $1 \leq j \leq n$ and $e_j \mapsto 0$ for $j > n$.) So $\|V_n\| = 1$ for all n.

Now considering each $T = [t_{ij}] \in \mathcal{K}$ as an array, we let $Tr : \mathcal{K} \to \mathcal{K}$ denote the transpose map, which is given by $Tr([t_{ij}]) = [t_{ji}]$. This is a linear $*$-preserving isometric map (since $T^* = [\bar{t}_{ji}]$), and

$$Tr \odot 1_\mathcal{K}(V_n) = \sum_{i,j=1}^{n} e_{ji} \otimes e_{ji}.$$

Now, consider the vector $\xi = \sum_{k=1}^{n} e_k \otimes e_k$. One computes

$$\begin{aligned} \|Tr \odot 1_\mathcal{K}(V_n)\xi\| &= \| \sum_{i,j=1}^{n} \sum_{k=1}^{n} \langle e_k, e_j \rangle e_i \otimes \langle e_k, e_j \rangle e_i \| \\ &= \| \sum_{i=1}^{n} \sum_{k=1}^{n} \langle e_k, e_k \rangle e_i \otimes \langle e_k, e_k \rangle e_i \| \\ &= \| \sum_{i=1}^{n} n(e_i \otimes e_i) \| = \|n\xi\| = n\|\xi\|. \end{aligned}$$

In particular, this means that $\|Tr \odot 1_\mathcal{K}(V_n)\| \geq n$ and hence $\|Tr \odot 1_\mathcal{K}\| \geq n$ for all $n \in \mathbb{N}$. This is an unbounded operator and hence not continuous.

So what kinds of bounded linear maps on C*-algebras yield continuous tensor product maps? Notice that the above example is $*$-preserving, so that's not enough. We have remarked several times that much of the structure of the C*-algebra is preserved by positive elements. Perhaps we need to consider linear maps $\phi : A \to B$ that send positive elements in A to positive elements in B? But even that isn't enough. It turns out that the transpose map above does send positive elements to positive elements. So, what gives? This is where we finally motivate the idea of *completely* positive maps. Recall that a linear map $\phi : A \to B$ between C*-algebras is completely positive if (equivalently) the linear map

$$\phi^{(n)} : \mathrm{M}_n(\mathbb{C}) \otimes A \to \mathrm{M}_n(\mathbb{C}) \otimes B$$

is positive for all $n \geq 1$.

Theorem 11.61. *Let $\phi_i : A_i \to B_i$ be linear cp maps. Then the algebraic tensor product map*

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

extends to a linear cp map (which is then also bounded and hence continuous) map on both the maximal and minimal tensor products:

$$\begin{aligned}\phi_1 \otimes \phi_2 :& A_1 \otimes A_2 \to B_1 \otimes B_2\\ \phi_1 \otimes_{\max} \phi_2 :& A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2.\end{aligned}$$

Moreover, we have $\|\phi_1 \otimes_{\max} \phi_2\| = \|\phi_1 \otimes \phi_2\| = \|\phi_1\|\|\phi_2\|$.

Remember that we have already proved this for $*$-homomorphisms. Stinespring's Dilation theorem will allow us to transfer this fact to cpc maps.

In full disclosure, we need a stronger version of this to prove the $\otimes_{\max}$ part of Theorem 11.61, so we direct you to [3, Proposition 1.5.6] and its use in the proof of [3, Theorem 3.5.3]. But for the sake of seeing Stinespring's Theorem in action, let's prove that the algebraic tensor product of cp maps extends to a cp map between spatial tensor products.

Proof of Theorem 11.61 (for spatial tensor). Let A_1, A_2, B_1, B_2 be C*-algebras and $\phi_i : A_i \to B_i$ cp maps. First, by taking faithful representations, it suffices to assume that $B_i \subset B(\mathcal{H}_i)$ for $i = 1, 2$ (why?). Then $\phi_i : A_i \to B(\mathcal{H}_i)$ are cp maps, which have Stinespring dilations $(\pi_i, \mathcal{H}_i', V_i)$ for $i = 1, 2$. Since these are $*$-homomorphisms, $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1') \odot B(\mathcal{H}_2') \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ extends to $A_1 \otimes A_2$. Define the map $\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_1 \otimes B_2 \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ by

$$\phi_1 \otimes \phi_2(x) = (V_1 \otimes V_2)^*(\pi_1 \otimes \pi_2)(x)(V_1 \otimes V_2).$$

By Example 9.9, this is a cp map. Moreover, for elementary tensors $a_1 \odot a_2 \in A_2 \odot A_2$, we have

$$\phi_1 \otimes \phi_2(a_1 \odot a_2) = (V_1^* \pi_1(a_1) V_1) \otimes (V_2^* \pi_2(a_2) V_2) = \phi_1(a_1) \odot \phi_2(a_2),$$

which means (by linearity) that $\phi_1 \otimes \phi_2|_{A_1 \odot A_2} = \phi_1 \odot \phi_2$. □

12. Amenability

Preview of Lecture: In lecture, we'll discuss the paradoxical decomposition of $\mathbb{F}_2$ (Example 12.3), but probably not the proof of Proposition 12.4 or Proposition 12.5. My goal in lecture will be to discuss the proof of Theorem 12.13; this will require also discussing Følner sets, but we won't get into the proof of Proposition 12.10 or Proposition 12.12.

The concept of amenability for groups was introduced by John von Neumann in 1929, in response to the Banach-Tarski paradox. For modern operator algebraists, amenable groups are important because these are precisely the groups G for which $C^*(G) \cong C^*_r(G)$. Another C*-algebraic characterization of amenability is that G is amenable iff $C^*_r(G)$ is nuclear – indeed, this is what underlies the use of the word "amenable" instead of "nuclear" for more general C*-algebras. More generally, if a C*-algebra A is nuclear and $\alpha : G \to \mathrm{Aut}(A)$ is an action of an amenable group on A, then the crossed product C*-algebra $C^*(G, A, \alpha)$ will be nuclear. (In particular, this is true for all of the crossed products Dawn Archey mentioned yesterday in her talk.)

There are many (many) equivalent characterizations of amenability (and they all have analogues for locally compact groups, although in these notes we'll just treat the discrete case). If you want to know more than what's presented here, [3, Section 2.6] is a good place to start. For a more exhaustive account, check out [8].

Definition 12.1. A discrete group G is *amenable* if it admits a left-invariant mean: that is, there is a state[22] μ on $\ell^\infty(G)$ such that

$$\mu(f) = \mu(g \mapsto f(s^{-1}g))$$

for all $f \in \ell^\infty$ and $s \in G$.

Example 12.2. Any finite group G is amenable. We define $\mu(\delta_g) = \frac{1}{|G|}$ for each $g \in G$. It is easy to check that if we extend μ to $\ell^\infty(G)$ by requiring it to be linear, the result is a state.

Example 12.3. The free group $\mathbb{F}_2$ is not amenable.

Recall that $\mathbb{F}_2 = \langle a, b\rangle$ is the set of all words in two noncommuting generators (here called a, b) and their inverses. We will assume that the words are *reduced* in the sense that a variable is never immediately followed by its inverse. Let A_+ denote the set of words in $\mathbb{F}_2$ whose first letter is a, and A_- denote the set of words whose first letter is a^{-1}, and note that

$$\mathbb{F}_2 = A_+ \sqcup aA_-;$$

if a reduced word w doesn't start with a, then $a^{-1}w \in \mathbb{F}_2$ lies in A_-, and so $w \in aA_-$.

Similarly, define B_+ (resp. B_-) to be the words whose first letter is b (resp. b^{-1}). So if $C = \{b^n : n \geq 0\}$, then we can also write

$$\mathbb{F}_2 = A_+ \sqcup A_- \sqcup (B_+ \backslash C) \sqcup (B_- \cup C).$$

Finally, I claim that $\mathbb{F}_2 = b^{-1}(B_+\backslash C) \sqcup (B_- \cup C)$. Why? Notice that $(B_+\backslash C)$ is the set of words whose first letter is b (so the second letter can't be b^{-1}) but which contain other letters, so $b^{-1}(B_+\backslash C)$ consists of words whose first letter is not b^{-1}, and which contain some letter that's not b. On the other hand, $(B_- \cup C)$ is the set of words which either have b^{-1} as the first letter, or contain only nonnegative powers of b.

Now that we have these three decompositions of $\mathbb{F}_2$, suppose that we did in fact have a left-invariant mean μ on $\ell^\infty(\mathbb{F}_2)$. Observe that $\chi_{tS} = \chi_S(t^{-1}\cdot)$, for any $t \in \mathbb{F}_2$. In other words (abusing notation and writing $\mu(S)$ rather than $\mu(\chi_S)$ for $S \subseteq \mathbb{F}_2$) we have $\mu(tS) = \mu(S)$ for any $S \subseteq \mathbb{F}$ and any $t \in \mathbb{F}$. It follows that

$$1 = \mu(\mathbb{F}_2) = \mu(A_+ \sqcup aA_-) = \mu(A_+) + \mu(A_-).$$

On the other hand, $\mu(\mathbb{F}_2) = \mu(A_+)+\mu(A_-)+\mu(B_+\backslash C)+\mu(B_-\cup C)$, so we must have $\mu(B_+\backslash C) = \mu(B_-\cup C) = 0$. However, this contradicts the fact that (by our third decomposition)

$$1 = \mu(\mathbb{F}_2) = \mu(B_+\backslash C) + \mu(B_- \cup C).$$

Notice that our decomposition $\mathbb{F}_2 = A_+ \sqcup A_- \sqcup (B_+\backslash C) \sqcup (B_- \cup C)$ thus writes $\mathbb{F}_2$ as the disjoint union of two subsets, namely $A_+ \sqcup A_-$ and $(B_+\backslash C) \sqcup (B_- \cup C)$, which both end up having the same measure as $\mathbb{F}_2$ under any translation-invariant measure (thanks to our first and last decompositions of $\mathbb{F}_2$). This is often called a *paradoxical decomposition* of $\mathbb{F}_2$, and is what underlies the Banach-Tarski paradox.

[22] We've only defined states on C*-algebras so far, but the definition in this context is the same: a linear functional of norm 1 which assigns a nonnegative real number to any nonnegative function.

Proposition 12.4. *If G is abelian then G is amenable.*

The proof uses the *Markov-Kakutani fixed point theorem:* [5, Theorem VII.2.1] if X is a topological vector space, $K \subseteq X$ is compact and convex, and T is a collection of continuous, linear, pairwise commuting maps $t : X \to X$ is such that every $t \in T$ satisfies $tK \subseteq K$, then there is a point in K which is fixed by all $t \in T$.

Proof of Proposition 12.4. The compact convex set K of interest here is the set $S(\ell^\infty(G))$ of states on $\ell^\infty(G)$; take $T = \{\lambda_s^* : s \in G\}$, where

$$\lambda_s^*(\phi)(f) = \phi(\lambda_s f) = \phi(g \mapsto f(s^{-1}g)).$$

Then one checks that every element of T is continuous, in the sense that if a net $(\phi_i)_i \in \ell^\infty(G)^*$ satisfies $\phi_i \to \phi$ in the weak-* topology, then $\lambda_s^*(\phi_i) \to \lambda_s^*(\phi)$ for all $s \in G$. The fact that G is abelian implies that T is a set of pairwise commuting maps, and one can check that T preserves $S(\ell^\infty(G))$. So, the Markov-Kakutani fixed point theorem gives us $\mu \in S(\ell^\infty(G))$ such that $\lambda_s^*(\mu) = \mu$ for all s. By construction, μ is a left-invariant mean on $\ell^\infty(G)$. $\square$

Proposition 12.5. *The class of amenable groups is closed under taking subgroups, quotients, extensions, and inductive limits.*

Proof. We will prove that the class of amenable groups is closed under extensions, and leave the rest as an exercise. So, suppose that N, H are amenable, with left invariant means μ_N, μ_H respectively, and $1 \to N \to G \to H \to 1$ is a short exact sequence of groups (so that N is normal in G and $H \cong G/N$). We define a functional μ on $\ell^\infty(G)$ by

$$\mu(f) = \mu_H(sN \mapsto \mu_N(g \mapsto f(sg))).$$

Notice that the function $sN \mapsto \mu_N(g \mapsto f(sg))$ is well defined by our hypothesis that μ_N is left invariant; we have

$$\mu_N(g \mapsto f(sg)) = \mu_N(g \mapsto f(sng)).$$

Moreover, if f is positive, then the fact that μ_H, μ_N are positive linear functionals implies that μ is also a positive linear functional. To see that μ is indeed a left invariant mean, then, it merely remains to check left invariance. If $\tilde{g} \in G$, then

$$\mu(\lambda_{\tilde{g}} f) = \mu_H(sN \mapsto \mu_N(g \mapsto (\lambda_{\tilde{g}}(sg))) = \mu_H(sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg))) = \mu_H(\tilde{g}^{-1}sN \mapsto \mu_N(g \mapsto f(\tilde{g}^{-1}sg)))$$

by the left invariance of μ_H. However, replacing the variable $s \in G$ with $\tilde{g}s$ reveals that this latter is precisely $\mu(f)$, as desired. $\square$

Exercise 12.6. Complete the proof of Proposition 12.5. Some hints:

- If $H \leq G$ is a subgroup of an amenable group, pick a set S of left coset representatives of $H \leq G$, so that you can write any $g \in G$ uniquely as $g = sh$ for $s \in S, h \in H$. Use this to embed $\ell^\infty(H)$ into $\ell^\infty(G)$.

  (Side question: Why can't we just define μ by $\frac{\mu|_H}{\mu(H)}$?)
- To show that $G = \varinjlim G_n$ is amenable whenever all the groups G_n are, you'll need to take a weak-$*$ cluster point of the left invariant means witnessing amenability of the G_ns.

In particular, Proposition 12.5 implies that $\mathbb{F}_n$ is not amenable for any $n \geq 2$: Each such $\mathbb{F}_n$ contains $\mathbb{F}_2$ as a subgroup.

Theorem 12.7. *G is amenable iff $\mathrm{C}_r^*(G) \cong \mathrm{C}^*(G)$.*

Proof. We will prove the backwards direction; the forwards direction (cf. [5, Theorem VII.2.8] or [3, Theorem 2.6.8]) uses a lot of machinery that we don't have time to introduce.

Suppose $\mathrm{C}_r^*(G) \cong \mathrm{C}^*(G)$. Note that the universal property of $\mathrm{C}^*(G)$ means that it always admits a one-dimensional representation χ, arising from the unitary representation $\pi(u_g) = 1$ for all $g \in G$. Then, since we assumed that the canonical surjection $\pi_\lambda : \mathrm{C}^*(G) \to \mathrm{C}_r^*(G)$ is an isomorphism, χ becomes a 1-dimensional representation on $\mathrm{C}_r^*(G)$.

By the Hahn-Banach Theorem, extend χ to a norm-1 bounded linear functional (also called χ) on $B(\ell^2(G))$, and then restrict it to a bounded linear functional on $\ell^\infty(G)$ (viewed as a subalgebra of $B(\ell^2(G))$, acting by left multiplication). If $f \in \ell^\infty(G)$ is positive, $f = \sup\{f|_F : F \subseteq G \text{ finite}\}$, and as each $f|_F$ is positive in $\mathrm{C}_r^*(G)$, the fact that $\chi|_{\mathrm{C}_r^*(G)}$ is a $*$-homomorphism (and hence positive) implies $\chi(f) \geq 0$ for all $f \geq 0$ in $\ell^\infty(G)$.

It's straightforward to check **Exercise: do it!** that if $f \in \ell^\infty(G)$, $f = \sum_{g\in G} a_g u_g$, then $\lambda_s(f) = u_s f u_s^*$ as operators on $\ell^2(G)$. Moreover, as χ is a $*$-homomorphism on $C_r^*(G) \ni u_g$, these elements are in the multiplicative domain of χ (see Proposition 9.26). Therefore, $\chi(\lambda_s f) = \chi(f)$ for any $f \in \ell^\infty(G)$, so χ is our left-invariant mean. □

We would also like to prove that G is amenable iff $C_r^*(G)$ is nuclear. To do this, it will be easier to work with a different characterization of amenability. To introduce it, recall that if S, T are sets, then $S\Delta T = (S \cup T)\backslash(S\cap T)$ is the set of elements which are in precisely one of S, T.

Definition 12.8. A discrete group G satisfies the *Følner condition* if for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a finite subset $F \subseteq G$ such that

$$\frac{|sF\Delta F|}{|F|} < \epsilon \text{ for all } s \in E.$$

It is a fact (although not one we'll prove here) that G satisfies the Følner condition iff G is amenable. However, we can prove that satisfying the Følner condition is equivalent to the following property, which is hopefully sufficiently reminiscent of the definition of amenability that you're willing to believe said fact. If you recall that $\ell^1(G)$ is the predual of $\ell^\infty(G)$ and hence is dense in $\ell^\infty(G)^*$, you may be even more credulous.

Definition 12.9. A discrete group G admits an *approximate invariant mean* if, for any finite subset $E \subseteq G$ and any $\epsilon > 0$, there is a positive function $m = m(E,\epsilon) \in \ell^1(G)$ with $\sum_{s\in G} m(s) = 1$ and such that

$$\sup_{s\in E} \sum_{t\in G} |m(s^{-1}t) - m(t)| < \epsilon.$$

Proposition 12.10. *G satisfies the Følner condition iff G admits an approximate invariant mean.*

Proof. Suppose G satisfies the Følner condition. Given a finite set E and $\epsilon > 0$, let $F \subseteq G$ be the finite set guaranteed by the Følner condition and let $m = \frac{1}{|F|}\chi_F$. Note that

$$\chi_F(s^{-1}t) = 1 \Leftrightarrow s^{-1}t \in F \Leftrightarrow t \in sF,$$

so $\sum_{t\in G} |m(s^{-1}t) - m(t)| = \frac{|sF\Delta F|}{|F|} < \epsilon$ for all $s \in E$.

On the other hand, suppose that G admits an approximate invariant mean. We first make a helpful technical observation. Given a positive function $f \in \ell^1(G)$ and $r \geq 0$, set $F(f, r) = \{t : f(t) > r\}$. Notice first that $F(f, r)$ must be finite for each fixed r, in order to have $f \in \ell^1(G)$. We now observe that if f, h are two such functions, both bounded above by 1, then

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr.$$

To see this, suppose without loss of generality that $f(t) = x, h(t) = y$ with $x \leq y$. Then $\chi_{F(f,r)}(t) = 1$ iff $r < x$ and $\chi_{F(h,r)}(t) = 1$ iff $r < y$, so the integrand is 1 precisely on the interval $[x, y)$.

Now, supposing G admits an approximate invariant mean, fix a finite subset $E \subseteq G$ and $\delta > 0$; write $\epsilon = \delta/|E|$, and let $m \in \ell^1(G)$ be a norm-1 positive function such that $\sum_{t\in G} |m(t) - m(s^{-1}t)| < \epsilon$ for all $s \in E$. Applying our above observation to the functions $f = m, h = (t \mapsto m(s^{-1}t))$, we have

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \sum_{t\in G} \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr = \int_0^1 \sum_{t\in G} |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)|\, dr$$

(as the integrand is positive we can exchange the integral and the sum). Moreover, we have $t \in F(h, r)$ precisely if $m(s^{-1}t) > r$, that is, if $t \in sF(m, r)$. It follows that

$$\sum_{t\in G} |m(t) - m(s^{-1}t)| = \int_0^1 |F(m, r)\,\Delta\, sF(m, r)|\, dr < \epsilon$$

for all $s \in G$. Furthermore, as m is positive, $1 = \sum_{t\in G} m(t) = \int_0^1 |F(m, r)|\, dr$. It follows that

$$\sum_{s\in E} \int_0^1 |sF(m, r)\,\Delta\, F(m, r)|\, dr < \int_0^1 |E|\epsilon|F(m, r)|\, dr,$$

and so we must have

$$\sum_{s\in E} |sF(m,r)\,\Delta\,F(m,r)| < |E|\epsilon|F(m,r)|$$

for some r. Then, in particular, for each $s \in E$ we have

$$\frac{|sF(m,r)\,\Delta\,F(m,r)|}{|F(m,r)|} < |E|\epsilon = \delta,$$

so $F(m,r)$ satisfies the Følner condition for the given E and $\delta > 0$. □

The proof of the following Proposition can be found in [3, Theorem 2.6.8] (see also [5, Theorem VII.2.8]). It uses a lot more Banach space theory than one might expect.

Proposition 12.11. *G is amenable iff G admits an approximate invariant mean (iff G satisfies the Følner condition).*

Before proving our next theorem, we need the following useful fact about completely positive maps.

Proposition 12.12. *A map $\phi : M_n(\mathbb{C}) \to A$ is completely positive iff $[\phi(E_{ij})] \in M_n(A)$ is positive.*

Proof. We prove the backwards direction and leave the forwards direction as an easy **exercise** to the reader. So, suppose $a = [\phi(E_{ij}] \in M_n(A)$ is positive; write $[b_{ij}] := a^{1/2}$, so that

$$a_{ij} = \phi(E_{ij}) = (b^*b)_{ij} = \sum_{k=1}^{n} b_{ki}^* b_{kj}.$$

Without loss of generality, assume $A \subseteq B(\mathcal{H})$, so that each entry b_{ij} of $b \in M_n(A)$ lies in $B(\mathcal{H})$. Define $V : \mathcal{H} \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H}$ by

$$V(\xi) = \sum_{j,k=1}^{n} e_j \otimes e_k \otimes b_{k,j}\xi.$$

Then we compute that if $T = [t_{ij}] \in M_n(\mathbb{C})$,

$$\begin{aligned}\langle V^*(T\otimes 1\otimes 1)V\eta, \xi\rangle &= \langle (T\otimes 1\otimes 1)(V\eta), V\xi\rangle \\ &= \langle \sum_{i,j,k=1}^{n} t_{ij} e_i \otimes e_k \otimes b_{k,j}\eta, \sum_{\ell,m=1}^{n} e_\ell \otimes e_m \otimes b_{m,\ell}\xi\rangle \\ &= \sum_{i,j,k=1}^{n} t_{ij}\langle b_{k,j}\eta, b_{k,i}\xi\rangle = \sum_{i,j,k=1}^{n} t_{ij}\langle b_{k,i}^* b_{k,j}\eta, \xi\rangle \\ &= \langle \phi([t_{ij}])\eta, \xi\rangle.\end{aligned}$$

In other words, $\phi(T) = V^*(T\otimes 1\otimes 1)V$ is a compression of the $*$-homomorphism $\psi : M_n(\mathbb{C}) \to B(\mathbb{C}^n\otimes\mathbb{C}^n\otimes\mathcal{H})$ given by $\psi(T) = T \otimes 1 \otimes 1$, so ϕ is cp. □

Finally, we can prove our second marquee theorem.

Theorem 12.13. *G is amenable iff $\mathrm{C}_r^*(G)$ is nuclear.*

Proof. Suppose G is amenable (and, for simplicity, countable, so that we can enumerate the elements of G). By Proposition 12.11, we can assume that G satisfies the Følner condition. Choose, then, a sequence of finite sets F_n such that F_n satisfies the Følner condition for $\epsilon = 1/n$ and the finite set consisting of the first n elements of G. Let $P_n \in B(\ell^2(G))$ be the projection onto the subspace spanned by $\{\delta_g : g \in F_n\}$, so that we can identify $P_nB(\ell^2(G))P_n$ with $M_{F_n}(\mathbb{C})$. Define $\phi_n : \mathrm{C}_r^*(G) \to M_{F_n}(\mathbb{C})$ by $\phi_n(x) = P_nxP_n$. Example 9.9 shows that ϕ_n is ccp.

To define $\psi_n : M_{F_n}(\mathbb{C}) \to \mathrm{C}_r^*(G)$, write E_{pq} for the matrix unit in $M_{F_n}(\mathbb{C})$ such that $E_{pq}(\delta_q) = \delta_p$. Then define

$$\psi_n(E_{pq}) = \frac{1}{|F_n|} u_p u_q^*,$$

and extend ψ_n to be a linear map on $M_{F_n}(\mathbb{C})$. If we enumerate the elements of F_n as $p_1, \ldots, p_{|F_n|}$, then $[\psi_n(E_{pq})]$ satisfies

$$[\psi_n(E_{pq})] = \frac{1}{|F_n|^2} \begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_{p_1} & 0 & \cdots & 0 \\ u_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ u_{p_{|F_n|}} & 0 & \cdots & 0 \end{bmatrix}^* \geq 0,$$

so Proposition 12.12 tells us that ψ_n is also cp. In fact, ψ is ucp: our choice of scaling factor and the fact that each u_p is a unitary means that

$$\psi_n(1) = \sum_{p \in F_n} \psi_n(E_{pp}) = 1.$$

To complete the proof that $\mathrm{C}^*_r(G)$ is nuclear when G is amenable, it remains to show that for any $a \in \mathrm{C}^*_r(G)$ we have $\lim_{n\to\infty} \|a - \psi_n(\phi_n(a))\| = 0$. In fact, since the generators u_s densely span $\mathrm{C}^*_r(G)$, it suffices to show that $\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = 0$ for all $s \in G$.

One quickly computes that $\phi_n(u_s) = \sum_{p:p,s^{-1}p \in F_n} E_{p,s^{-1}p}$, and therefore

$$\psi_n(\phi_n(u_s)) = \frac{1}{|F_n|} \sum_{p:p,s^{-1}p\in F_n} u_p u^*_{s^{-1}p} = \frac{1}{|F_n|} \sum_{p:p,s^{-1}p\in F_n} u_s = u_s \frac{|F_n \cap sF_n|}{|F_n|}.$$

As $|F_n \,\Delta\, sF_n| = 2|F_n| - 2|F_n \cap sF_n|$, our choice of the sets F_n implies that

$$0 = \lim_{n\to\infty} \frac{|F_n \,\Delta\, sF_n|}{|F_n|} = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|}$$

for any $s \in G$. In particular,

$$\lim_{n\to\infty} \|u_s - \psi_n(\phi_n(u_s))\| = \lim_{n\to\infty} 1 - \frac{|F_n \cap sF_n|}{|F_n|} = 0,$$

as desired.

Now, for the converse. Assume $\mathrm{C}^*_r(G)$ is nuclear, so that we have cpc maps $\phi_n : \mathrm{C}^*_r(G) \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to \mathrm{C}^*_r(G)$. By Arveson's Extension Theorem, we might as well assume that ϕ_n is defined on all of $B(\ell^2(G))$, so that the composition $\Phi_n = \psi_n \circ \phi_n$ is a cpc map from $B(\ell^2(G))$ to $\mathrm{C}^*_r(G)$, such that $\Phi_n(x) \to x$ for all $x \in \mathrm{C}^*_r(G)$. Take a point-ultraweak limit of the maps Φ_n (ask Brent and Rolando), and we end up with a cpc map $\Phi : B(\ell^2(G)) \to L(G)$ which restricts to the identity on $\mathrm{C}^*_r(G)$.

Recall from your von Neumann algebra lectures that there is a canonical trace τ on $L(G)$, given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. Define $\mu = \tau \circ \Phi$; we claim that μ is a left invariant mean. To see this, we again use that the left translation action λ_s on functions in $\ell^\infty(G) \subseteq B(\ell^2(G))$ is given by $\lambda_s(f) = u_s f u_s^*$. Since $\Phi|_{\mathrm{C}^*_r(G)} = \mathrm{id}$, we have u_g in the multiplicative domain of Φ for all g. Consequently, for any $f \in \ell^\infty(G)$,

$$\mu(\lambda_s(f)) = \tau(\Phi(u_s f u_s^*)) = \tau(u_s \Phi(f) u_s^*) = \tau(\Phi(f)),$$

since τ is a trace and u_s is a unitary. $\square$

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