Second Lecture

Wednesday, May 24, 2023 4:53 PM

Q: How good is $[M:N]$ as an invariant?

We've seen that $[N \rtimes_\alpha G : N] = |G|$ so we can't expect for the index to be a good invariant for subfactors, how can we improve this?

The standard invariant of $N \subset M$ is the collection of f.d. relative commutants

$$\mathcal{G}_{N\subset M}\left\{\begin{array}{ccccccc} \mathbb{C}=N'\cap N & \subset & N'\cap M & \subset & N'\cap M_1 & \subset & \cdots \\ & & \cup & & \cup & & \\ & & \mathbb{C}=M'\cap M & \subset & M'\cap M_1 & \subset & \cdots \end{array}\right.$$

Recall: Each of these algs is finite dimensional so each inclusion can be described by a graph.

There are a number of axiomatizations of the std. inv:
Ocneanu's paragroup, Popa's λ-lattice, Jone's planar algebras, unitary tensor cats

We can at most expect these invariant to be good for hyperfinite subfs i.e. $N \cong M \cong R$ as

$$\mathcal{G}_{N\subseteq M} = \mathcal{G}_{N\otimes P \subseteq M \otimes P} \quad \text{for any } II_1\text{-factor } P.$$

Remark: These inclusions need not be trivial e.g. $R \subset M_d(R)$ has index d^2.

In this setting, the std. invariant is a complete invariant for "amenable subfactors", in particular for f.d. subfactors. (Popa, 1994)

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It can be very difficult to compute $\mathcal{G}_{N\subseteq M}$ however, is there anything in between the index and $\mathcal{G}_{N\subseteq M}$ that allows us to classify subfs?

Principal graphs: Underneath a subfactor and it's h.r.c we have plenty of bimodules, let $X = {}_N L^2(M)_M$ and $\bar{X} = {}_M L^2(M)_N$

Consider all the N-N, N-M, M-M, M-N bimodules you obtain by Connes fusion of X's and $\bar{X}$'s: $X\bar{X}$, $X\bar{X}X$, ...

Then
$$N' \cap M_{2n+1} \cong \mathrm{Hom}_{N-N}\left(\left(X\bar{X}\right)^{n+1}\right)$$
$$N' \cap M_{2n} \cong \mathrm{Hom}_{N-M}\left(\left(X\bar{X}\right)^{n} X\right)$$
$$M' \cap M_{2n} \cong \mathrm{Hom}_{M-M}\left(\left(\bar{X}X\right)^{n}\right)$$
$$M' \cap M_{2n+1} \cong \mathrm{Hom}_{M-N}\left(\left(\bar{X}X\right)^{n} \bar{X}\right)$$

(see Bisch, 97 or Ocneanu, 91)

The principal graph Γ will be a bipartite graph where

$\Gamma_{odd} = \{$ equiv classes of irred N-N bims showing in $(X\bar{X})^n$ $\}$

$\Gamma_{even} = \{$ equiv classes of irred N-M bims showing in $(X\bar{X})^n X$ $\}$

we add k vertices between $Y \in \Gamma_{odd}$ and $Z \in \Gamma_{even}$ where k is the multiplicity of Z in YX.

The dual principal graph Γ' is obtained in a similar manner but for M-M & M-N bimodules.

Def: We say $N \subset M$ is irreducible if $N' \cap M = \mathbb{C} \cdot 1$

(Note this implies that ${}_N L^2(M)_M$ is irred)

Example: $N \subset N \rtimes_\sigma G = M$ we know index $= |G|$

$\mathbb{C} = N' \cap N \subset \mathbb{C} = N' \cap M \subset N' \cap M_1$

 ?

${}_N L^2(M)_N \cong \bigoplus_g {}_N L^2(g)_N$ where $L^2(g) = {}^2(N)$

w/ $x \cdot \xi \cdot y = x J \sigma_g(y)^* J(\xi)$

then $\overline{L^2(g)} \cong L^2(g^{-1})$

Frobenius reciprocity shows each summand is irreducible

and $L^2(g) \otimes_N L^2(h) \cong L^2(gh)$ as N-N bim

Claim: $L^2(g) \not\cong L^2(N)$ for any $g \neq e$

If $\exists\, u: L^2(N) \to L^2(g)$ then $u \in B(L^2(N))$

where $x \cdot u(\xi) = u(x \cdot \xi)$ $\Rightarrow$ $xu(\xi) = ux(\xi) \Rightarrow u \in N'$

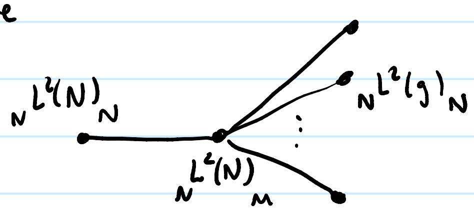
& $u(\xi) \cdot y = u(\xi \cdot y)$ $\quad J\sigma_g(y)^* J u(\xi) = u(Jy^*J)(\xi)$

$$J\sigma_g(y)^* J u = u JyJ$$

$$\sigma_g(y)^* = JuJy^*Ju^*J$$

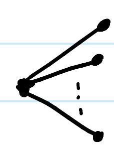
$$\sigma_g(y) = \underbrace{Ju^*J}_{\in JN'J = N} \; y \; \underbrace{JuJ}_{JN'J = N} \Rightarrow \sigma_g \text{ is inner } (\Rightarrow\Leftarrow)$$

So we have



$\Big\}$ $|G|$ many

since $[M:N] = |G|$ we conclude

$\Gamma =$ 

Fact: The dual principal graph has one odd vertex $*$

and even vertices are indexed by irreps of G π

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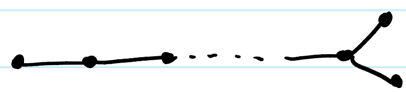
with d_π edges from $*$ to π where d_π = degree of π.

Def: We say $N \subset M$ has finite depth if Γ (or Γ') is a finite graph.

This means that $N \subset N \rtimes_\alpha G$ is a finite depth subfactor w/ index $|G|$.

Other graphs that show up as princ. graphs:

finite depth
- A series : A_n  n vertices
- D series : D_n  n vertices

infinite depth
- A_∞ : 

Q: Can any graph show up as a princ. graph of an irred hyperfinite subf?

Q: Can we obtain further obstructions (a la Jones rigidity) from looking at these?

This is partially answered in the long program of classification of small index subfs.

The current state of classification: (work of a lot of people)

A.4 The map of subfactors