First lecture

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What is a subfactor? It's a unital inclusion $1_M \in N \subset M$ where both N & M are II_1-factors

- Identify M with it's standard rep on $L^2(M)$
- Whenever we write N' we'll be talking about $N' \cap B(L^2(M))$ which will be a factor of type II (as N is type II_1)

In case N' is of type II_1 we'll denote it's unique trace by $\tau_{N'}$

Note that $\tau_M|_N = \tau_N$ and therefore the closure of $N\hat{1}$ in $L^2(M)$ is isomorphic to $L^2(N)$ hence we can consider

$$L^2(M) \xrightarrow{e_N} L^2(N) \subset L^2(M) \text{ be the orthogonal proj } (e_N \in B(L^2(M)))$$

in fact $e_N \in N'$.

Def: Let $1_M \in N \subset M$ be an inclusion of II_1-factors.

$$[M:N] := \frac{1}{\tau_{N'}(e_N)} \quad \text{if } N' \text{ is a } \mathrm{II}_1\text{-factor},$$

otherwise $[M:N] := \infty$

Obs: • If N' is of type $\mathrm{II}_1 \Rightarrow \tau_{N'}(e_N) \leq 1 \Rightarrow [M:N] \geq 1$

• If $[M:N] = 1$ then $\tau_{N'}(e_N) = 1$

$$\Rightarrow e_N \sim 1 \text{ in } N' \Rightarrow e_N = 1 \quad \text{hence } L^2(M) = L^2(N)$$
$$\Rightarrow M = N.$$

the index measures how much larger M is than N.

Extra

We can also obtain the index in terms of coupling constants. Given $\mathcal{H}$ a separable M-module when M is a II_1-factor we can define the coupling constant of M on $\mathcal{H}$: $\dim_M \mathcal{H} \in [0,\infty]$

Obs: $\dim_M L^2(M) = 1$.

we can define the coupling constant of M on H: $\dim_M H \in [0,\infty]$

obs: $\dim_M L^2(M) = 1$.

You can think about it as the "M-dimension" of H, then

$$[M:N] = \frac{\dim_N H}{\dim_M H}$$ for any M-module H with $\dim_M H < \infty$

in particular $[M:N] = \dim_N L^2(M)$

Using this form, consider $N \subset P \subset M$ II_1-factors, then

$$[M:N] = \dim_N L^2(M) = \dim_P L^2(M) \frac{\dim_N L^2(M)}{\dim_P L^2(M)}$$

$$= [M:P][P:N]$$

Just like groups!

(we'll explore more of this in the problem session)

Prop: $N \overset{II_1}{\subset} M$, $p \in N$, $q \in N' \cap M$ projections & $[M:N] < \infty$. Then

1) $[pMp : pNp] = [M:N]$
2) $[qMq : Nq] = [M:N]\, tr_M(q)\, tr_{N'}(q)$

($Nq \subset qMq$ is called a local subfactor and $[qMq:Nq]$ it's local index)

Corollary (local index formula): $N \overset{II_1}{\subset} M$, $[M:N] < \infty$ and let $(p_i)_{i\in I}$ $N' \cap M$ projs be s.t. $\sum_{i \in I} p_i = 1$. Then

$$[M:N] = \sum_{i \in I} \frac{[p_i M p_i : N p_i]}{tr_M(p_i)}$$

Moreover $\dim N' \cap M < \infty$ and if $[M:N] < 4$ then $N' \cap M = \mathbb{C} \cdot 1$

Proof:

- $[p_i M p_i : N p_i] = [M:N]\, tr_M(p_i)\, tr_{N'}(p_i)$

$$\Rightarrow [M:N] = \sum_{i \in I} [M:N]\, tr_{N'}(p_i) = \sum_{i \in I} \frac{[p_i M p_i : N p_i]}{tr_M(p_i)}$$

- $\dim N' \cap M < \infty$: Otherwise $\exists\, (p_i)_{i\in I}$ family of proj's in $N' \cap M$ s.t. $\sum p_i = 1$ and $|I| = \infty$. Hence $tr_M(p_i) \to 0$ as tr_M is normal

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but $$[M:N] = \sum_{i \in I} \frac{[p_i M p_i, N p_i]}{tr_M(p_i)} \geq \sum \frac{1}{tr_M(p_i)} = \infty \quad (\Rightarrow\Leftarrow).$$

- $[M:N] < 4 \Rightarrow N' \cap M = \mathbb{C} \cdot 1$: Assume $\dim N' \cap M \geq 2$ then $\exists\, p \in N' \cap M$ s.t. $0 \neq p \neq 1$. Then

$$[M:N] = \frac{[pMp : Np]}{tr(p)} + \frac{[(1-p)M(1-p) : N(1-p)]}{tr(1-p)}$$

$$\geq \frac{1}{t} + \frac{1}{1-t} = \frac{1}{t(1-t)} \qquad \text{where } t = tr(p)$$

but $\frac{1}{t(1-t)} \geq 4$ if $t \in (0,1)$ $(\Rightarrow\Leftarrow)$

Exercise: $N \overset{\mathrm{II}_1}{\subset} M$, $[M:N] < \infty \Rightarrow \dim N' \cap M \leq [M:N]$

Example: Let $G \overset{\sigma}{\curvearrowright} N$ be a properly outer (trace-preserving) action of a finite group onto M. where N is a II_1-factor.

Let $\mathcal{H} = L^2(N)$ and consider $\sigma_s(\hat{x}) := \widehat{\sigma_s(x)}$ for $x \in N$, $s \in G$, since σ_s is trace-preserving, extends to a unitary u_s on $L^2(N)$.

Consider
$$N[G] = \left\{ \sum x_s u_s \,/\, s \in G,\ x_s \in N \right\} \subset B(L^2(N))$$
and let
$$N \rtimes_\sigma G = N[G]''$$
observe that $u_s x u_s^* = \sigma_s(x)$

Now $N \subset N \rtimes_\sigma G = M$ will be a unital inclusion of II_1-factors, moreover $[M:N] = |G|$

$$L^2(M) = \overline{N[G]\hat{1}} \approx \bigoplus_{s \in G} \overline{N u_s \hat{1}} \approx \bigoplus_{s \in G} \underbrace{L^2(N) u_s}_{L^2(s)}$$

We'll see in the problem session how can we prove that

$$[M:N] = |G|$$

This means that all natural numbers occur as indices of subfactors, but there's more:

Theorem (Jones, 1983) $N \overset{\mathrm{II}_1}{\subset} M$

$$[M:N] \in \{4\cos^2(\pi/n) : n \geq 3\} \cup [4,\infty]$$

Every value occurs as the index of some $N \overset{\mathrm{II}_1}{\subset} M$.

A key step in the proof of this result is the construction of the "Jones tower"

Let $M_1 := \langle M, e_N \rangle$ be the vNa generated by e_N & M in $B(L^2(M))$
we call this the <u>basic construction</u> for $N \subseteq M$

In fact, $\langle M, e_N \rangle = JN'J \leftarrow$ (Exercise) and therefore

M_1 will be a II_1-factor iff $[M:N] < \infty$

Extra

- $(JN'J)' = JNJ \leftarrow$ type II_1 $\Rightarrow$ $\underset{M_1}{\underset{\shortparallel}{JN'J}}$ is type II

- $(JN'J)' \cap JN'J$

  $\quad \shortparallel$

  $JNJ \cap JN'J = J(N \cap N')J = \mathbb{C} \cdot 1$ $\Rightarrow$ $\overset{M_1}{\overset{\shortparallel}{JN'J}}$ is a factor

- $[M:N] < \infty \Leftrightarrow N'$ is finite $\Leftrightarrow N'$ has a trace $\tau_{N'}$

  If N' has a trace: $\tau_{M_1}(x) := \tau_{N'}(JxJ) \Rightarrow M_1$ is finite

  If M_1 has a trace: $\tau_{N'}(y) := \tau_{M_1}(JyJ) \Rightarrow N'$ is finite

Prop: If $[M:N] < \infty$ then $[M_1 : M] = [M:N]$ and

$$\tau_{M_1}(x e_N) = \frac{1}{[M:N]} \tau_M(x) \quad \forall x \in M \quad (\text{Markov trace})$$

$\ldots \; [M:N] \; \ldots$ (Markov trace)

This means that starting with $N \overset{\mathrm{II}_1}{\subseteq} M$ of fin. index

$$N \subset M \overset{e_N}{\subset} M_1 \overset{e_M}{\subset} M_2 \overset{e_{M_1}}{\subset} M_3 \subset \cdots\cdots \subset \overline{\bigcup_{i \geq 1} M_i}^{tr} =: M_\infty$$

fin index   fin index   Another II_1-factor

It's in M_∞ where the obstructions for the index start to manifest.

(Jones rigidity)