A Crash Course in Crossed Product C^*-Algebras

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Blanket Assumptions

Throughout this talk

- A and D are a unital C^*-algebra.
- G is a discrete group.
- α is an action of G on D or A.
- The C^*-algebra crossed product is denoted $C^*(G, D, \alpha)$.
- If $B \subset A$ is said to be a unital subalgebra, then $1_B = 1_A$.

Motivational Questions-1

We want to answer questions of the type

Vague Question

Suppose

- D is MUMBLE, MUMBLE,
- G is MUMBLE, MUMBLE, and
- α satisfies MUMBLE, MUMBLE.

What can we say about $C^*(G, D, \alpha)$?

Refined Motivational Question

We want to prove theorems of the type

Theorem Type

Suppose

- D is MUMBLE, MUMBLE,
- G is MUMBLE, MUMBLE, and
- α satisfies MUMBLE, MUMBLE.

If D has stable rank 1, then $C^*(G, D, \alpha)$ has stable rank 1.
OR
If D has real rank zero, then $C^*(G, D, \alpha)$ has real rank zero.
OR
If D is $\mathcal{Z}$-absorbing, then $C^*(G, D, \alpha)$ is $\mathcal{Z}$-absorbing.

Constructing the Crossed Product

Ingredients

- A a C^*-algebra, preferably unital.
- G a locally compact discrete group.
- A group action α of G on A, $\alpha : G \to \mathrm{Aut}(A)$

Step 1 The Algebra AG

- Elements are finite sums $\sum_{g \in G} a_g u_g$
- Multiplication $u_g a u_g^{-1} = \alpha_g(a)$.
- Adjoint $u_g^* = u_{g^{-1}}$

Constructing the Crossed Product 2

Step 2 Define a norm

For $f = \sum_{g \in G} a_g u_g \in AG$, define $\|f\|_1 = \sum_{g \in G} \|a_g\|$

By GNS theorem there actually are some representations.

Step 3 Complete

As usual, let $\ell^1(G, A, \alpha)$ be the Banach $*$-Algebra obtained by completing AG in $\| \cdot \|_1$.

Constructing the Crossed Product 3

Step 4 Represent

Define the *Universal Representation* σ of $\ell^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $\ell^1(G, A, \alpha)$ on Hilbert Spaces.

Step 5 complete again

The crossed product $C^*(G, A, \alpha)$ is the norm closure of $\sigma\left(\ell^1(G, A, \alpha)\right)$

Reduced Crossed Product

To get the Reduced crossed product, use only only regular representations.

Observation about crossed products

If G is amenable, The crossed product and reduced crossed product are the same.
Remember that $C^*(G, A, \alpha)$ is generated by finite sums of the form

$$\sum_{g \in G} a_g u_g$$

where $a_g \in A$ and u_g is a unitary.
A embeds unitally into $C^*(G, A, \alpha)$ as $a \mapsto au_e$.
If A is unital, $C^*(G, A, \alpha)$ conatains a unitary subgroup isomorphic to G.

Irrational Rotation Algebras

Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

Fact

$A_\theta \cong C^*(\mathbb{Z}, C(S^1), \tau)$

Where $A_\theta = C^* \left(\{u, v \text{unitaries} : uv = e^{2\pi i\theta} vu\} \right)$

And $\tau : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ by $\tau(f)(z) = f(e^{-2\pi i\theta} z) = f \circ R_\theta^{-1}(z)$

Irrational Rotation Algebras

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Actually τ is τ_1 and for other integers $\tau_n = (\tau)^n$

Action of $Z/2\mathbb{Z}$

Fact

$C^*(\mathbb{Z}/2\mathbb{Z}, C(S^1), \alpha)$
$\cong \{f \in C([-1,1], M_2) : f(1) \text{ and } f(-1) \text{ are diagonal}\}$

Where $\alpha : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(C(S^1))$ is defined by $\alpha_1(f)(z) = f(\overline{z})$.

Better Notation

Recall the previous two examples.

$\tau : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ by $\tau(f)(z) = f(e^{-2\pi i\theta z}) = f \circ R_\theta^{-1}(z)$

$\alpha : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(C(S^1))$ is defined by $\alpha_1(f)(z) = f(\overline{z})$.

In both,

- The group is singly generated, so only need one automorphism.
- Automorphism is given by composing with a homeomorphism of the space $X = S^1$.

Better Notation 2

If $A = C(X)$ and $h : X \to X$ is a homeomorphism, then $\alpha : C(X) \to C(X)$ given by $\alpha(f) = f \circ h$ determines an action of $\mathbb{Z}$ on $C(X)$.
Crossed product is denoted $C^*(\mathbb{Z}, C(X), h)$.

Simplicity

Definition

Let $h : X \to X$ be a homeomorphism. We say (x, h) is a *minimal dynamical system* if X has no proper closed h invariant subsets.

Theorem

Let X be a an infinite, compact, Hausdorff space. Then $C^*(\mathbb{Z}, C(X), h)$ is simple if and only if (X, h) is minimal.

Example Theorem

Theorem (Putnum 1989)

Let X be the Cantor set. Let h be a minimal homeomorphism of X. There exists an embedding of $A = C^*(\mathbb{Z}, C(X), h)$ into an AF algebra such that the induced map on K_0 is an order isomorphism. We also have

- $K_0(C^*(\mathbb{Z}, C(X), h)) \cong C(X, \mathbb{Z})/\mathrm{Im}(\mathrm{id} - h_*)$
- $K_1(C^*(\mathbb{Z}, C(X), h)) \cong \mathbb{Z}$

Example Theorem

Theorem

Let X be the Cantor set and (h, X) be minimal. Then $C^*(\mathbb{Z}, C(X), h)$ is an AT-algebra.

Pimsner-Voiculescu Exact Sequence

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{id_* - \alpha_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(C^*(\mathbb{Z}, A, \alpha)) \\
\uparrow & & & & \downarrow \\
K_1(C^*(\mathbb{Z}, A, \alpha)) & \xleftarrow[\iota_*]{} & K_1(A) & \xleftarrow[id_* - \alpha_*]{} & K_1(A).
\end{array}$$

Outer Actions

Let G be a discrete group and α an action of G on a unital C^*-Algebra A.

Definition

We say α is *inner* if there is a homeomorphism $g \mapsto u_g$, from G to the unitary group of $M(A)$, such that $\alpha_g = \mathrm{Ad}(u_g)$ for all $g \in G \setminus \{1\}$.

Definition

We say α is *(pointwise) outer* if α_g is not inner for all $g \in G \setminus \{1\}$.

Bad News

There are examples of actions which are not inner even though every α_g is inner. (Ex: using $A = M_2$ and $G = (\mathbb{Z}/2\mathbb{Z})^2$.)

Simplicity in the Finite Group Case

Theorem

Let G be a finite group, and let A be a simple unital C^*-Algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be a pointwise outer action. Then $C^*(G, A, \alpha)$ is simple.

Stable Rank One

Definition

If A is a unital, then A has stable rank one ($\text{tsr}(A) = 1$), if the invertible elements in A are dense in A.

Remark

If X is a compact metric space, then $\text{tsr}(C(X)) = [\frac{\dim X}{2}] + 1$. Stable rank is approximately the dimension of X as a complex vector space.

Long Open Problem

Question

If A is a simple unital C^*-Algebra with $\operatorname{tsr}(A) = 1$ and if G is a finite group acting on A by α, does it follow that $C^*(G, A, \alpha)$ has stable rank one?

Remark

The answer is not known even if $G = \mathbb{Z}/2\mathbb{Z}$ and A is an AF algebra.

But can we bound it?

Theorem (Jeong-Osaka-Phillips-Teruya)

Let A be a C^*-Algebra, let G be a finite group and let $\alpha : G \to \mathrm{Aut}(A)$ be an action. Then

$$\mathrm{tsr}\left(C^*\left(G, A, \alpha\right)\right) \leq \mathrm{tsr}(A) + \mathrm{card}(G) - 1$$

Let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $S \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in S$.
3. $\sum_{g \in G} e_g = 1$.

We call $(e_g)_{g \in G}$ a *family of Rokhlin projections* for α, S, and ε.

The Rokhlin Property is Strong

Crossed products by actions of finite groups with the Rokhlin prperty preserve the following classes of C^*-algebras.

- AT algebras
- D-absorbing separable unital C^*-algebras for a strongly self-absorbing C^*-algebra D.
- Unital C^*-algebras with stable rank one.
- Many more.

Too Strong

An action α on A with the Rokhlin property implies strong restrictions on the K-theory.

- There is no action of $\mathbb{Z}/2/Z$ on the 3^{∞} UHF algebra.
- There is no action of any nontrivial finite group on $\mathcal{O}_{\infty}$ which has the Rokhlin property.

Definition

Let G be a finite group, let A be an infinite dimensional simple unital C^*-Algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are nonzero mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is MvN equivalent to a projection in the hereditary subalgbra of A generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

Permenance Theorems

Theorem (H. Osaka and N.C. Phillips, 2006)

Let D be a stably finite simple unital C^*-algebra, and let α be an action of $\mathbb{Z}$ on D which has the tracial Rokhlin property. Let $A = \mathbb{C}^*(\mathbb{Z}, D, \alpha)$.

$\mathrm{RR}(D) = 0$ and order on projections over D determined by traces	$\Rightarrow$	$\mathrm{RR}(A) = 0$ and order on projections over A determined by traces
If also $\mathrm{tsr}(D) = 1$	then	$\mathrm{tsr}(A) = 1$

Theorem (D.A. 2008)

The above results hold if $\mathbb{Z}$ is replaced by a finite group.

Real Rank Zero and Order on Projections

Definition

Let A be a C^*-algebra. We say that A has *real rank zero* if the invertible selfadjoint elements are dense in the selfadjoint part of A.

Definition

Let A be a simple exact unital C^*-algebra. The order on projections over A is determined by traces if, as happens for type II_1 factors, whenever $p, q \in M_\infty(A)$ are projections such that for all $\tau \in T(A)$ we have $\tau(p) < \tau(q)$, then $p \precsim q$.

Example Theorem

Theorem (Putnum 1989)

Let X be the Cantor set. Let h be a minimal homeomorphism of X. There exists an embedding of $A = C^*(\mathbb{Z}, C(X), h)$ into an AF algebra such that the induced map on K_0 is an order isomorphism. We also have

- $K_0(C^*(\mathbb{Z}, C(X), h)) \cong C(X, \mathbb{Z})/\mathrm{Im}(\mathrm{id} - h_*)$
- $K_1(C^*(\mathbb{Z}, C(X), h)) \cong \mathbb{Z}$

Example of a Large Subalgebra

Let $A = C^*(\mathbb{Z}, C(X), h)$.
Putnum used the Y-**Orbit breaking subalgebra**
$A_Y = C^*(C(X) \cup \{fu : f \in C(X) \text{ and } f(y) = 0 \text{ for all } y \in Y\})$
where u is the standard unitary implementing h.

Theorem (N.C. Phillips)

If $h^n(Y) \cap Y = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$, then the Y-orbit breaking subalgebra is large in the crossed product.

Theorem (Osaka-Phillips, DA)

Let G be a finite group or $\mathbb{Z}$. Let D be a stably finite simple unital C^*-algebra, and let α be an action of G on D which has the tracial Rokhlin property. Let $A = \mathbb{C}^*(G, D, \alpha)$.

$\text{RR}(D) = 0$ and order on projections over D determined by traces	$\Rightarrow$	$\text{RR}(A) = 0$ and order on projections over A determined by traces
If also $\text{tsr}(D) = 1$	then	$\text{tsr}(A) = 1$

Osaka-Phillips and I used a collection of subalgebras each isomorphic to $M_n(fDf)$, where f is a projection in D.

Large Subalgebra Approach

- Abstraction to hide irrelevant details.

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- Abstraction to hide irrelevant details.
- Lets us provide proofs that are more generalizable.
- Lets us prove new theorems.

Stable Rank 1 and Real Rank 0

Theorem (D.A., N.C. Phillips)

Suppose A is an infinite dimensional simple separable unital C^*-algebra. Let $B \subset A$ be a centrally large subalgebra. Then

$$\mathrm{tsr}(B) = 1 \quad \Rightarrow \quad \mathrm{tsr}(A) = 1.$$

Theorem (D.A., N.C. Phillips)

Suppose A is an infinite dimensional simple separable unital C^*-algebra. Let $B \subset A$ be a centrally large subalgebra. Then
Suppose A has a centrally large subalgebra B

$$\mathrm{tsr}(B) = 1 \quad \text{and} \quad \mathrm{RR}(B) = 0 \quad \Rightarrow \quad \mathrm{RR}(A) = 0.$$

$\mathcal{Z}$-stability

Theorem (D.A., J. Buck, N.C. Phillips)

Let A be a simple separable infinite dimensional nuclear unital C^*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then

$$\mathcal{Z} \otimes B \cong B \iff \mathcal{Z} \otimes A \cong A.$$