# Chapter 4

# Types of von Neumann Algebras

We saw back in Corollary 2.1.4 that von Neumann algebras are equal to the $C^*$-algebra generated by its projections. So it is perhaps unsurprising that much of the structure of a von Neumann algebras is determined by its projections. More precisely, there is an equivalence relation on the projections in a von Neumann algebra, and one can classify von Neumann algebras into three types according to the behavior of this equivalence relation

In the first section we will define and study this equivalence relation on projections. In the second section we study certain subalgebras related to projections called *compressions*. In the third section we will define the three types of von Neumann algebras and show how any von Neumann algebra decomposes into a direct sum of the three types. We will also consider a few examples.

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**Lecture Preview:** The content of this lecture will covered over two days: Wednesday, July 8th (p. 44–54) and Friday, July 10th (p. 55–63). The first lecture on July 8th will cover equivalence of projections (Definition 4.1.1), central supports (Definition 4.1.7), and the Comparison Theorem. We will likely forego most proofs in favor of concrete examples. Regardless, it is recommended that you skip the proof of Proposition 4.1.5. The second lecture on July 8th will cover compressions of von Neumann algebras (Definition 4.2.1) and various properties of projections (Definitions 4.2.5 and 4.3.1), and emphasis will be put on concrete examples.

For the first lecture on July 10th, we will state the type decomposition (see Theorem 4.3.7) and its refinements (see Definitions 4.3.10 and 4.3.13), though we will not prove them. Instead we will focus on the examples at the end of Section 4.3 (Examples 4.3.14, 4.3.15, and 4.3.16).

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## 4.1 Equivalence of Projections

Throughout this section, let $M \subset B(\mathcal{H})$ be a von Neumann algebra. We will write $\mathcal{P}(M)$ for the collection of projections in $M$. Also, for a subset $\mathcal{S} \subset \mathcal{H}$ we write $[\mathcal{S}]$ for the projection onto the closed span of $\mathcal{S}$; that is, $[\mathcal{S}] = P_{\overline{\text{span}}\mathcal{S}}$.

Recall that, viewing $B(\mathcal{H})$ as $C^*$-algebra, positivity gives us a partial ordering on projections: $p \leq q$ if and only if $q - p \geq 0$. In fact, $(\mathcal{P}(M), \leq)$ is a complete lattice for any von Neumann algebra $M \subset B(\mathcal{H})$ (see Exercise 4.1.1). For $\mathcal{P} \subset \mathcal{P}(M)$ a set of projections (not assumed to be pairwise orthogonal) the *infimum* and *supremum* of $\mathcal{P}$ are defined by

$$\bigwedge \mathcal{P} := \left[ \bigcap_{p \in \mathcal{P}} p\mathcal{H} \right] \qquad \bigvee \mathcal{P} := \left[ \bigcup_{p \in \mathcal{P}} p\mathcal{H} \right].$$

If $\mathcal{P} = \{p_1, \ldots, p_n\}$ is a finite subset, we also write $p_1 \wedge \cdots \wedge p_n := \bigwedge \mathcal{P}$ and $p_1 \vee \cdots \vee p_n := \bigvee \mathcal{P}$. Note that $\mathcal{P} \subset M$ implies that the subspaces used to define $\bigwedge \mathcal{P}$ and $\bigvee \mathcal{P}$ are reducing for $M'$, and consequently $\bigwedge \mathcal{P}, \bigvee \mathcal{P} \in M$ by Lemma 1.2.5.

Unfortunately, this lattice structure tends to be too rigid for our purposes. For example, in $M_2(\mathbb{C})$ the projections $E_{1,1}$ and $E_{2,2}$ have the same rank but are not comparable via $\leq$. The underlying issue is that this partial ordering it too dependent on the Hilbert space: $p \leq q$ if and only $p\mathcal{H} \subset q\mathcal{H}$. Partial isometries will be the key ingredient for loosening this dependence.

**Definition 4.1.1.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}$, we say that $p$ is **equivalent** to $q$ in $M$ and write $p \sim q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. We say that $p$ is **subequivalent** to $q$ in $M$ and write $p \preceq q$ if there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$. If $p \preceq q$ but $p \not\sim q$, we write $p \prec q$.

Note that if $p, q \in \mathcal{P}(M)$ are such that $p \leq q$, then by taking $v = p$ we see that $p \preceq q$. Thus $p \preceq q$ is a coarser relation than $p \leq q$.

**Example 4.1.2.** Consider the following projections in $M_3(\mathbb{C})$:

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we set

$$V := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then $V^*V = P$ and $VV^* = Q$, so $P \sim Q$. We also have $E_{1,1} \preceq P$ (since $E_{1,1} \leq P$) and $E_{1,1} \preceq Q$ (using either of the partial isometries $E_{2,1}$ or $E_{3,1}$). Actually, we have $E_{1,1} \prec P, Q$. To see this note that for any partial isometry $V \in M_3(\mathbb{C})$ with $V^*V = E_{1,1}$ we have

$$\mathrm{Tr}(VV^*) = \mathrm{Tr}(V^*V) = \mathrm{Tr}(E_{1,1}) = 1 < 2 = \mathrm{Tr}(P), \mathrm{Tr}(Q).$$

So $VV^*$ can never equal $P$ or $Q$. In general, a projection in $M_3(\mathbb{C})$ is equivalent to another projection if and only if they have the same trace (see Exercise 4.1.4). ■

**Remark 4.1.3.** A subtle aspect of Definition 4.1.1 is that we can only say $p$ is subequivalent to $q$ in $M$ if we can find a partial isometry $v$ in $M$ that satisfies $v^*v = p$ and $vv^* \leq q$. To emphasize this, we may write $p \preceq_M q$ or $p \sim_M q$. If $M \subset N \subset B(\mathcal{H})$ is a larger von Neumann algebra, it may be that $p \sim_N q$ but $p \not\sim_M q$. For example, $E_{1,1}$ and $E_{2,2}$ are equivalent in $M_2(\mathbb{C})$, but not in the von Neumann algebra $\mathbb{C}E_{1,1} \oplus \mathbb{C}E_{2,2}$.

**Proposition 4.1.4.** *For a von Neumann algebra $M \subset B(\mathcal{H})$, $\sim$ is an equivalence relation on $\mathcal{P}(M)$, and the relation $\preceq$ is reflexive and transitive (a preorder).*

*Proof.* The reflexivity of $\sim$ and $\preceq$ follows form the fact that a projection is also a partial isometry. The symmetry of $\sim$ is evident from the definition. The transitivity of $\sim$ will follow as a special case of the transitivity of $\preceq$, which we now show. Let $p, q, r \in \mathcal{P}(M)$ with $p \preceq q$ and $q \preceq r$. Then there exists partial isometries $u, v \in M$ so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. It follows that

$$qu = quu^*u = uu^*u = u,$$

so that

$$(vu)^*(vu) = u^*v^*vu = u^*qu = u^*u = p$$

and

$$(vu)(vu)^* = vuu^*v \leq vqv^* = v(v^*v)v^* = vv^* \leq r.$$

Thus $p \preceq r$, and $\preceq$ is transitive. □

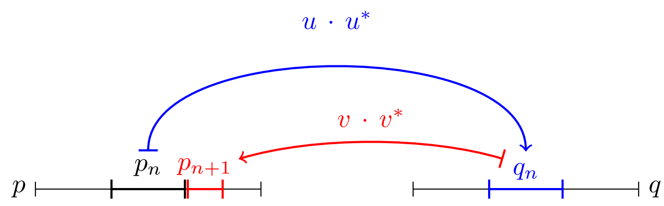
The relation $\preceq$ is **not** a partial order because $p \preceq q$ and $q \preceq p$ does not imply $p = q$. For example, in $M_n(\mathbb{C})$ we have $E_{1,1} \preceq E_{2,2}$ and $E_{2,2} \preceq E_{1,1}$, but $E_{1,1} \neq E_{2,2}$. Instead, we have $E_{1,1} \sim E_{2,2}$. We will see in the next proposition that this actually holds in general: $p \preceq q$ and $q \preceq p$ imply $p \sim q$ (it would be a crime against notation for this not to hold). Although the proof appears to be rather complicated, it more or less follows the same argument used to prove the Schröder–Berstein Theorem.

**Proposition 4.1.5.** *For a von Neumann algebra $M \subset B(\mathcal{H})$ and $p, q \in \mathcal{P}(M)$, $p \preceq q$ and $q \preceq p$ imply $p \sim q$.*

*Proof.* Let $u, v \in M$ be partial isometries so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Set $p_1 = p - vv^*$, $q_1 = up_1u^*$, and inductively define sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ by

$$p_n = vq_{n-1}v^* \qquad \text{and} \qquad q_n = up_nu^*.$$

By Exercise 4.1.8, $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ are each pairwise orthogonal families of projections, with $p_n \leq p$ and $q_n \leq q$ for each $n \in \mathbb{N}$. In particular, $p_n \leq vv^*$ for all $n \geq 2$ and $q_n \leq uu^*$ for all $n \geq 1$. The following diagram provides a rough but helpful mental picture of how these sequences are defined:




Using Proposition 1.1.5, we define projections

$$p_\infty := p - \sum_{n=1}^{\infty} p_n \qquad \text{and} \qquad q_\infty := q - \sum_{n=1}^{\infty} q_n.$$

We also define

$$w := v^*p_\infty + u\sum_{n=1}^{\infty} p_n = v^*p_\infty + \sum_{n=1}^{\infty} up_n.$$

We claim $w^*w = p$ and $ww^* = q$. The argument will be broken up into the following smaller claims:

**(I)** $(p_nu^*)(up_m) = \delta_{n=m}p_n$ and $(up_n)(up_m)^* = \delta_{n=m}q_n$ for all $m, n \in \mathbb{N}$.

**(II)** $(p_\infty v)(v^*p_\infty) = p_\infty$ and $(v^*p_\infty)(v^*p_\infty)^* = q_\infty$.

**(III)** $(p_nu^*)(v^*p_\infty) = 0$, $(p_\infty v)(up_n) = 0$, $(v^*p_\infty)(p_nu^*) = 0$, and $(up_n)(p_\infty v^*) = 0$ for all $n \in \mathbb{N}$.

Before proving these claims, observe that they are simply the multiplication rules needed to expand the products $w^*w$ and $ww^*$:

$$\begin{aligned} w^*w &= \left(p_\infty v + \sum_{m=1}^{\infty} p_mu^*\right)\left(v^*p_\infty + \sum_{n=1}^{\infty} up_n\right) \\ &= (p_\infty v)(v^*p_\infty) + \sum_{n=1}^{\infty}(p_\infty v)(up_n) + \sum_{m=1}^{\infty}(p_mu^*)(v^*p_\infty) + \sum_{m,n=1}^{\infty}(p_mu^*)(up_n) = p_\infty + \sum_{n=1}^{\infty} p_n = p \end{aligned}$$

and similarly $ww^* = q$. Thus proving these claims with complete the proof.

**(I):** We compute

$$(up_n)^*(up_m) = p_nu^*up_m = p_npp_m = p_np_m = \delta_{n=m}p_n.$$

Also

$$(up_n)(up_m)^* = up_np_mu^* = \delta_{n=m}up_nu^* = \delta_{n=m}q_n.$$

**(II):** Let $v_k = v^*\left(p - \sum_{n=1}^{k} p_n\right)$. Then

$$v_kv_k^* = v^*\left(p - \sum_{n=1}^{k} p_n\right)v = v^*pv - \sum_{n=1}^{k} v^*p_nv = q - \sum_{n=2}^{k} q_{n-1} = q - \sum_{n=1}^{k-1} q_n,$$

where we we have used $v^*pv = q$, $v^*p_1v = 0$, and $v^*p_nv = q_{n-1}$ for $n \geq 2$. Also

$$v_k^*v_k = \left(p - \sum_{n=1}^{k} p_n\right) vv^* \left(p - \sum_{n=1}^{k} p_n\right) = vv^* - \sum_{n=2}^{k} p_n = p - p_1 - \sum_{n=2}^{k} p_n = p - \sum_{n=1}^{k} p_n.$$

where we have used $vv^* \leq p$, $p_1vv^* = 0$, and $p_n \leq vv^*$ for $n \geq 2$. Taking limits in the SOT we obtain

$$(p_\infty v)(v^*p_\infty) = \lim_{k\to\infty} v_k^*v_k = \lim_{k\to\infty} \left(p - \sum_{n=1}^{k} p_n\right) = p_\infty,$$

and

$$(v^*p_\infty)(p_\infty v) = \lim_{k\to\infty} v_kv_k^* = \lim_{k\to\infty} \left(q - \sum_{n=1}^{k-1} q_n\right) = q_\infty.$$

**(III):** First note that by taking adjoints, the second equality follows from the first and the fourth from the third. The third equality is simply a consequence of $p_\infty p_n = 0$. To see the first equality, note that $v^*p = v^* = v^*q$ and $v^*p_n = q_{n-1}v^*$, while $v^*p_1 = 0$. It follows that $v^*p_\infty = q_\infty v^*$, which along with $p_nu^* = u^*q_n$ imply $(p_nu^*)(v^*p_\infty) = u^*q_nq_\infty v^* = 0$. □

The next lemma is an important example of equivalence, and a nice application of the polar decomposition. Recall that for a subset $\mathcal{S} \subset \mathcal{H}$, $[\mathcal{S}]$ denotes the projection onto $\overline{\text{span}}\,\mathcal{S}$.

**Lemma 4.1.6.** *For a von Neumann algebra $M \subset B(\mathcal{H})$ and $x \in M$, $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ and $[x\mathcal{H}] \sim_M [x^*\mathcal{H}]$.*

*Proof.* Let $x = v|x|$ be the polar decomposition and recall that $v \in M$. From Theorem 3.1.1 we know that $vv^*$ is the projection onto $\overline{\text{ran}(x)} = \overline{x\mathcal{H}}$ and $v^*v$ is the projection onto

$$\overline{\text{ran}(|x|)} = \ker(|x|)^\perp = \ker(x)^\perp = \overline{\text{ran}(x^*)} = \overline{x^*\mathcal{H}}.$$

Thus $vv^* = [x\mathcal{H}]$ and $v^*v = [x^*\mathcal{H}]$, which shows the projections are equivalent and in $M$. □

Another way to see that $[x\mathcal{H}], [x^*\mathcal{H}] \in M$ is to observe that the subspaces $\overline{x\mathcal{H}}$ and $\overline{x^*\mathcal{H}}$ are reducing for $M'$ and use Lemma 1.2.5.

**Definition 4.1.7.** For $x \in M$, the **central support** of $x$ in $M$ is the projection

$$\mathbf{z}(x) := \bigwedge\{z \in \mathcal{P}(\mathcal{Z}(M)) \colon xz = zx = x\}.$$

We amy also write $\mathbf{z}_M(x) := \mathbf{z}(x)$ to emphasize the role of $M$ in the above. We say $p, q \in \mathcal{P}(M)$ are **centrally orthogonal** if their central supports are orthogonal: $\mathbf{z}(p)\mathbf{z}(q) = 0$.

Note that for $p \in \mathcal{P}(M)$, $zp = p$ for $z \in \mathcal{P}(\mathcal{Z}(M))$ implies $p \leq z$, and therefore $p \leq \mathbf{z}(p)$. So in this case we can think of $\mathbf{z}(p)$ as the smallest central projection that is larger than $p$ (*central* being the key word here). Also, if $p, q \in \mathcal{P}(M)$ are centrally orthogonal, then this shows $p$ and $q$ are also orthogonal. The next lemma provides another way to think of the central support.

**Lemma 4.1.8.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. The central support of $p \in \mathcal{P}(M)$ is*

$$\mathbf{z}(p) = \bigvee_{x\in M} [xp\mathcal{H}] = [Mp\mathcal{H}].$$

*Proof.* The second equality above follows from the definition of the supremum. Let $z = [Mp\mathcal{H}]$. Since $M$ is unital, we have $p \leq z$. Because $\overline{Mp\mathcal{H}}$ is reducing for $M$ and $M'$, we have that $z \in M \cap M' = \mathcal{Z}(M)$. Thus $\mathbf{z}(p) \leq z$. Conversely, for any $x \in M$ we have

$$xp\mathcal{H} = x\mathbf{z}(p)p\mathcal{H} = \mathbf{z}(p)xp\mathcal{H},$$

which implies $[xp\mathcal{H}] \leq \mathbf{z}(p)$. Since this holds for all $x \in M$, we have $z \leq \mathbf{z}(p)$. □

**Proposition 4.1.9.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, the following are equivalent:*

*(i) $p$ and $q$ are centrally orthogonal.*

*(ii) $pMq = \{0\}$.*

*(iii) There does not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ such that $p_0 \sim q_0$.*

*Proof.* We first show (i) and (ii) are equivalent. If $p$ and $q$ are centrally orthogonal, then for any $x \in M$ we have

$$pxq = p\mathbf{z}(p)x\mathbf{z}(q)q = px\mathbf{z}(p)\mathbf{z}(q)q = 0.$$

Thus $pMq = \{0\}$. Conversely, if $pMq = \{0\}$, then by Lemma 4.1.8 $p\mathbf{z}(q) = p[Mq\mathcal{H}] = 0$. This implies $p \leq 1 - \mathbf{z}(q)$, and since $1 - \mathbf{z}(q) \in \mathcal{Z}(M)$ we have $\mathbf{z}(p) \leq 1 - \mathbf{z}(q)$. That is, $\mathbf{z}(p)\mathbf{z}(q) = 0$. Thus (i) and (ii) are equivalent.

Next we show (ii) and (iii) are equivalent. Suppose (ii) does not hold and let $x \in M$ be such that $pxq \neq 0$. Then $qx^*p \neq 0$ and consequently, $p_0 := [pxq\mathcal{H}]$ and $q_0 := [qx^*p\mathcal{H}]$ are non-zero projections. Clearly $p_0 \leq p$ and $q_0 \leq q$, and by Lemma 4.1.6 $p_0 \sim q_0$. Conversely, suppose (iii) does not hold and $p_0 \leq p$ and $q_0 \leq q$ are non-zero projections such that $p_0 \sim q_0$. Let $v \in M$ be a partial isometry so that $v^*v = p_0$ and $vv^* = q_0$. Then $v^* = p_0v^*q_0$ so that

$$pv^*q = pp_0v^*q_0q = p_0v^*q_0 = v^* \neq 0.$$

Thus $pMq \neq \{0\}$, and we see that (ii) and (iii) are equivalent. □

Our next objective in this section is to prove the Comparison Theorem (see Theorem 4.1.11), which says that—modulo multiplying by a central projection—all projections are comparable via $\preceq$. We must first prove a lemma that will also be useful in our forthcoming classification of von Neumann algebras.

**Lemma 4.1.10.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If $\{p_i : i \in I\}, \{q_i : i \in I\} \subset \mathcal{P}(M)$ are two pairwise orthogonal families such that $p_i \preceq q_i$ for each $i \in I$, then $\sum_{i \in I} p_i \preceq \sum_{i \in I} q_i$. In particular, if $p_i \sim q_i$ for each $i \in I$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$.*

*Proof.* Let $u_i \in M$ be a partial isometry such that $u_i^*u_i = p_i$ and $u_iu_i^* \leq q_i$. Write $r_i = u_iu_i^*$ and note that $\{r_i : i \in I\}$ is pairwise orthogonal because $\{q_i : i \in I\}$ is. We have for $i \neq j$

$$u_i^*u_j = u_i^*u_iu_i^*u_ju_j^*u_j = u_i^*r_ir_ju_j = 0,$$

and

$$u_iu_j^* = u_iu_i^*u_iu_j^*u_ju_j^* = u_ip_ip_ju_j^* = 0.$$

Consequently,

$$\left(\sum_{i \in I} u_i\right)^* \left(\sum_{j \in I} u_j\right) = \sum_{i \in I} u_i^*u_i = \sum_{i \in I} p_i$$

and

$$\left(\sum_{i \in I} u_i\right) \left(\sum_{j \in I} u_j\right)^* = \sum_{i \in} u_iu_i^* = \sum_{i \in I} r_i \leq \sum_{i \in I} q_i.$$

Thus $\sum p_i \preceq \sum q_i$. The last assertion follows from the above and Proposition 4.1.5. □

**Theorem 4.1.11** (Comparison theorem)**.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. For $p, q \in \mathcal{P}(M)$, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ such that*

$$pz \preceq qz \qquad \textit{and} \qquad q(1-z) \preceq p(1-z).$$

*Proof.* By Zorn's Lemma there exists maximal families $\{p_i : i \in I\}, \{q_i : i \in I\} \subset \mathcal{P}(M)$ of pairwise orthogonal projections such that $p_i \sim q_i$ for all $i \in I$ and

$$p_0 := \sum_{i\in I} p_i \leq p$$
$$q_0 := \sum_{i\in I} q_i \leq q.$$

Note that $p_0 \sim q_0$ by Lemma 4.1.10. Choose $z := \mathbf{z}(q - q_0)$. By maximality of the families, Proposition 4.1.9 yields $\mathbf{z}(p - p_0)z = 0$. Consequently, $(p - p_0)z = 0$, or $pz = p_0z$. Now, if $v \in M$ is such that $v^*v = p_0$ and $vv^* = q_0$, then one easily checks that $p_0z \sim q_0z$ via the partial isometry $vz$. Thus

$$pz = p_0z \sim q_0z \leq qz.$$

Similarly, $p_0(1 - z) \sim q_0(1 - z)$ and since $q - q_0 \leq z$ we have

$$q(1 - z) = q_0(1 - z) \sim p_0(1 - z) \leq p(1 - z).$$

□

**Corollary 4.1.12.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If $M$ is a factor, then for $p, q \in \mathcal{P}(M)$ exactly one of the following holds:*

$$p \prec q \qquad p \sim q \qquad q \prec p.$$

*Proof.* By the Comparison Theorem, there exists $z \in \mathcal{P}(\mathcal{Z}(M))$ so that $pz \preceq qz$ and $q(1 - z) \preceq p(1 - z)$. Since $\mathcal{Z}(M) = \mathbb{C}$, we have either $z = 0$ or $z = 1$ and the result follows. □

## Exercises

**4.1.1.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. In this exercise you will show that $(\mathcal{P}(M), \leq)$ is a complete lattice.

(a) Show that $\bigwedge \mathcal{P}, \bigvee \mathcal{P} \in M$.

(b) Show that $\bigwedge \mathcal{P} \leq p \leq \bigvee \mathcal{P}$ for all $p \in \mathcal{P}$.

(c) Show $\bigwedge \mathcal{P}$ is a greatest lower bound for $\mathcal{P}$ and that $\bigvee \mathcal{P}$ is a least upper bound for $\mathcal{P}$.

**4.1.2.** Let $\mathcal{P} \subset B(\mathcal{H})$ be a set of projections

$$\bigvee \mathcal{P} = 1 - \bigwedge \mathcal{P}^\perp \qquad\qquad \bigwedge \mathcal{P} = 1 - \bigvee \mathcal{P}^\perp.$$

**4.1.3.** Let $\{\xi_1, \ldots, \xi_n\}, \{\eta_1, \ldots, \eta_n\} \subset \mathcal{H}$ be two orthornomal subsets. Show that $\sum_{i=1}^n \xi_i \otimes \bar{\eta}_i$ is a partial isometry that implements the equivalence $(\sum_{i=1}^n \eta_i \otimes \bar{\eta}_i) \sim \left(\sum_{i=1}^n \xi_i \otimes \bar{\xi}\right)$.

**4.1.4.** Let $p, q \in (B(\mathcal{H}))$ be finite-rank projections. Show that $p \sim q$ if and only if $\mathrm{Tr}(p) = \mathrm{Tr}(q)$.

**4.1.5.** Let $\mathcal{E}, \mathcal{F} \subset \mathcal{H}$ be two orthonormal subsets with the same cardinality. Show that $[\mathcal{E}] \sim [\mathcal{F}]$. [**Hint:** start with a bijection from $\mathcal{E}$ to $\mathcal{F}$ (as sets).]

**4.1.6.** Let $A \subset B(\mathcal{H})$ be an abelian von Neumann algebra. For $p, q \in \mathcal{P}(A)$, show that $p \sim_A q$ if and only if $p = q$.

**4.1.7.** For $p \preceq q$, let $v$ be a partial isometry satisfying $v^*v = p$ and $vv^* \leq q$. Show that $qvp = v$.

**4.1.8.** Let $p, q$ be projections, and let $u, v$ be partial isometries so that $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq p$. Set $p_1 := p - vv^*$, $q_1 = up_1u^*$, and inductively define sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ by

$$p_n = vq_{n-1}v^* \qquad \text{and} \qquad q_n = up_nu^*.$$

(a) For each $n \in \mathbb{N}$, show that $p_n = (vu)^{n-1}p_1((vu)^*)^{n-1}$ and $q_n = (uv)^{n-1}q_1((uv)^*)^{n-1}$.

(b) For each $n \in \mathbb{N}$, show that $(vu)^n$ and $(uv)^n$ are partial isometries. In particular, show

$$((vu)^*)^n(vu)^n = p \qquad (vu)^n((vu)^*)^n \leq vv^*$$
$$((uv)^*)^n(uv)^n = q \qquad (uv)^n((uv)^*)^b \leq uu^*.$$

(c) For each $n \in \mathbb{N}$, show that $p_n$ and $q_n$ are projections satisfying $p_n \leq p$ and $q \leq q$.

(d) For $m < n$, show that

$$((vu)^*)^m(vu)^n = (vu)^{n-m} \qquad \text{and} \qquad ((uv)^*)^m(uv)^n = (uv)^{n-m}.$$

(e) For $m < n$, show that $p_m p_n = 0$ and $q_m q_n = 0$. [**Hint:** first check that $p_1 v = 0$ and $q_1 uv = 0$.]

**4.1.9.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $p, q \in \mathcal{P}(M)$ satisfy $p \preceq q$. Show that $\mathbf{z}(p) \leq \mathbf{z}(q)$. [**Hint:** use Lemma 4.1.8.]

**4.1.10.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $p, q \in \mathcal{P}(M)$. In this exercise you will prove **Kaplansky's formula**:

$$(p \vee q - p) \sim (q - p \wedge q).$$

(a) For $x := (1-p)q$, show that $[x^*\mathcal{H}] = q - p \wedge q$.
[**Hint:** first compute $[\ker(x)]$.]

(b) For $x$ as above, show that $[x\mathcal{H}] = p \vee q - p$.
[**Hint:** use the previous part and Exercise 4.1.2.]

(c) Use Lemma 4.1.6 to deduce the desired equivalence.

## 4.2 Compressions

Before we can continue our study of projections, it is necessary to understand an important operation on von Neumann algebras.

**Definition 4.2.1.** For a von Neumann algebra $M \subset B(\mathcal{H})$ and $p \in B(\mathcal{H})$ a projection,

$$pMp := \{pxp \colon x \in M\}$$

is called a **compression** (or **corner**) of $M$.

The terminology comes from the fact that under the identification $\mathcal{H} \cong p\mathcal{H} \oplus (1-p)\mathcal{H}$, $pxp$ for $x \in M$ is identified with

$$\begin{pmatrix} pxp & 0 \\ 0 & 0 \end{pmatrix} \in B(p\mathcal{H} \oplus (1-p)\mathcal{H}),$$

where we view $pxp$ as an operator on $p\mathcal{H}$. In fact, for $M = B(\mathcal{H})$ we have $pB(\mathcal{H})p \cong B(p\mathcal{H})$

Note that $pMp$ is a subspace and is closed under taking adjoints. There are two cases where $pMp$ is actually a $*$-algebra. The first is if $p \in M$, in which case $pMp$ is actually a $*$-subalgebra of $M$. The second is if $p \in M'$, where $pxp = xp$ for all $x \in M$ implies $pMp = Mp$. In both cases $p$ is the unit of the $*$-algebra, so if $p < 1$ then they cannot be von Neumann algebras in $B(\mathcal{H})$. However, $p$ is the identity operator on $B(p\mathcal{H})$, and by the above identification we can view $pMp$ as operators on $p\mathcal{H}$.

**Theorem 4.2.2.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$. Then $pMp$ and $M'p$ are von Neumann algebras in $B(p\mathcal{H})$ and are commutants of one another.*

*Proof.* From the discussion preceding the theorem, we see that $pMp$ and $M'p$ are both unital $*$-subalgebras of $B(p\mathcal{H})$. So it suffices to show $(pMp)'' = pMp$ and $(M'p)'' = M'p$, where the commutants here are taken in $B(p\mathcal{H})$ (rather than $B(\mathcal{H})$). Toward this end we will show the following equalities:

$$(M'p)' \cap B(p\mathcal{H}) = pMp$$
$$(pMp)' \cap B(p\mathcal{H}) = M'p.$$

The inclusion $pMp \subset (M'p)' \cap B(p\mathcal{H})$ is immediate. Conversely, suppose $x \in (M'p)' \cap B(p\mathcal{H})$. Define $\tilde{x} \in B(\mathcal{H})$ by

$$\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, $\tilde{x} = p\tilde{x}p$, and for $p\xi \in p\mathcal{H}$ we have $\tilde{x}p\xi = xp\xi$. If $y \in M'$, then for $\xi \in \mathcal{H}$ we have

$$y\tilde{x}\xi = yp\tilde{x}p\xi = ypxp\xi = xyp\xi = xpy\xi = \tilde{x}py\xi = \tilde{x}y\xi.$$

So $y\tilde{x} = \tilde{x}y$ and hence $\tilde{x} \in M'' = M$. As operators on $p\mathcal{H}$ we have $x = p\tilde{x}p \in pMp$.

The inclusion $M'p \subset (pMp)' \cap B(p\mathcal{H})$ is immediate. Suppose $y \in (pMp)' \cap B(p\mathcal{H})$. Using the functional calculus to write $y$ as a linear combination of four unitaries, we may assume $y = u$ is a unitary. We will extend $u$ to an element $\tilde{u} \in B(\mathcal{H})$. Define $\tilde{u}$ on $Mp\mathcal{H}$ by

$$\tilde{u}\left(\sum_{i=1}^n x_i p\xi_i\right) = \sum_i x_i up\xi_i,$$

for $x_1, \ldots, x_n \in M$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$. Observe that

$$\begin{aligned}
\left\|\tilde{u}\sum_{i=1}^n x_i p\xi_i\right\|^2 &= \sum_{i,j=1}^n \langle x_i up\xi_i, x_j up\xi_j\rangle \\
&= \sum_{i,j=1}^n \left\langle px_j^* x_i pu\xi_i, up\xi_j\right\rangle \\
&= \sum_{i,j=1}^n \left\langle upx_j^* x_i p\xi_i, up\xi_j\right\rangle \\
&= \sum_{i,j=1}^n \left\langle px_j^* x_i p\xi_i, p\xi_j\right\rangle = \left\|\sum_{i=1}^n x_i p\xi_i\right\|^2.
\end{aligned}$$

Thus $\tilde{u}$ is well-defined and an isometry, which we extend to $\overline{Mp\mathcal{H}}$. Observe that $\tilde{u}$ commutes with $M$ on $Mp\mathcal{H}$ by definition of $\tilde{u}$, and consequently they commute on $\overline{Mp\mathcal{H}}$. Recall that $\mathbf{z}(p) = [Mp\mathcal{H}]$ by Lemma 4.1.8. So if we extend $\tilde{u}$ to $\mathcal{H}$ by setting $\tilde{u}|_{(Mp\mathcal{H})^\perp} \equiv 0$, then $\tilde{u} = \tilde{u}\mathbf{z}(p)$. It follows that for $x \in M$ and $\xi \in \mathcal{H}$ we have

$$x\tilde{u}\xi = x\tilde{u}\mathbf{z}(p)\xi = \tilde{u}\mathbf{z}(p)x\xi = \tilde{u}x\xi.$$

That is, $\tilde{u} = M' \cap \mathcal{B}(\mathcal{H})$. By definition $\tilde{u}$, we have $\tilde{u}p = u$ and so $u \in M'p$. □

**Corollary 4.2.3.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p \in \mathcal{P}(M)$. If $M$ is a factor then $pMp$ and $M'p$ are factors.*

*Proof.* Since $pMp$ and $M'p$ are each commutants of one another in $B(p\mathcal{H})$ by Theorem 4.2.2, they have the same center and so it suffices to show $M'p$ is a factor. First note that for $y \in M'$, if $yp = 0$ then for all $x \in M$ and $\xi \in \mathcal{H}$ we have

$$yxp\xi = xyp\xi = 0.$$

Since $M$ is a factor, we have $[Mp\mathcal{H}] = \mathbf{z}(p) = 1$ by Lemma 4.1.8. This means $Mp\mathcal{H}$ is dense in $\mathcal{H}$ and consequently the above implies $y = 0$. Now, if $zp \in \mathcal{Z}(M'p)$ for $z \in M'$, then for all $y \in M'$ we have $[z, y]p = [zp, yp] = 0$. By what we just argued, $[z, y] = 0$ and so $z \in \mathcal{Z}(M')$. Since $M'$ is a factor (by virtue of $M$ being a factor), we have $z \in \mathbb{C}$ and $zp \in \mathbb{C}p$. Thus $\mathcal{Z}(M'p) = \mathbb{C}p$ and $M'p$ is a factor. □

The next proposition shows that a compression depends only on the equivalence class of $p$ in $M$.

**Proposition 4.2.4.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. If $p, q \in \mathcal{P}(M)$ are equivalent in $M$, then $pMp$ and $qMq$ are spatially isomorphic.*

*Proof.* Let $v \in M$ be a partial isometry satisfying $v^*v = p$ and $vv^* = q$. We will show that $v|_{p\mathcal{H}}$ is a unitary from $p\mathcal{H}$ to $q\mathcal{H}$ that implements the spatial isomorphism. Note that $v = qvp$. This implies $v|_{p\mathcal{H}}$ is indeed valued in $q\mathcal{H}$, and is surjective since $q\xi = vv^*\xi = vpv^*\xi$ for any $x \in \mathcal{H}$. For $p\xi, p\eta \in p\mathcal{H}$, we have

$$\langle vp\xi, vp\eta\rangle = \langle v^*vp\xi, p\eta\rangle = \langle p\xi, p\eta\rangle .$$

Thus $v|_{p\mathcal{H}} \colon p\mathcal{H} \to q\mathcal{H}$ is a unitary. Using $v = qvp$ again, we have for any $x \in M$

$$vpxpv^* = vxv^* = q(vxv^*)q.$$

and

$$qxq = vv^*xvv^* = v(pv^*xvp)v^*.$$

Thus $v(pMp)v^* = qMq$. □

Note that in the above proof, we used $v \in M$ to guarantee $vxv^* \in M$ and $v^*xv \in M$ for all $x \in M$. Also note that if $y \in M'$, then $vypv^* = yvpv^* = vq$, which shows the spatial isomorphism sends $M'p$ to $M'q$.

**Definition 4.2.5.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. We say $p \in \mathcal{P}(M)$ is **minimal** in M if $p \neq 0$ and $pMp = \mathbb{C}p$. We say $p$ is **abelian** in M if $pMp$ is abelian.

Note that a minimal projection is also abelian.

**Example 4.2.6.**

(1) Let $p \in B(\mathcal{H})$. Then $pB(\mathcal{H})p \cong B(p\mathcal{H})$. Since $B(p\mathcal{H})$ is always a factor, it can only be abelian if $B(p\mathcal{H}) \cong \mathbb{C}$. This holds off and only if $p\mathcal{H} \cong \mathbb{C}$; that is, if and only if $p$ is a rank 1 projection.

(2) Let $(X, \mu)$ be a $\sigma$-finite measure space. Recall $f \in \mathcal{P}(L^\infty(X, \mu))$ if and only if $f = 1_E$ for some measurable $E \subset X$ (see Exercise 1.3.3). Consequently, all compressions of $L^\infty(X, \mu)$ are of the form $L^\infty(E, \mu|_E)$ for some measurable $E \subset X$, and so all projections in $L^\infty(X, \mu)$ are abelian. If $1_E$ is minimal, then $1_E \neq 0$ and $L^\infty(E, \mu|_E) = \mathbb{C}1_E$. The former holds if and only if $\mu(E) \neq 0$ and the latter holds if and only if for all measurable subsets $F \subset E$ we have $\mu(F) \in \{0, \mu(E)\}$ (see Exercise 4.2.3). We call such a subset $E$ an *atom* of $\mu$. Thus $L^\infty(X, \mu)$ has minimal projections if and only if $\mu$ has atoms. ■

If $p \in \mathcal{P}(M)$ is minimal, then whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$ we must have $q \in \{0, p\}$ since $q = pqp \in pMp = \mathbb{C}p$. Conversely, if $p \in \mathcal{P}(M)$ is such that $q \in \{0, p\}$ whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$, then $p$ and 0 are the only projections in $pMp$. Since von Neumann algebras are equal to the $C^*$-algebras generated by their projections (see Corollary 2.1.4), we must have $pMp = \mathbb{C}p$ and so $p$ is minimal. Thus, "$q \in \{0, p\}$ whenever $q \in \mathcal{P}(M)$ satisfies $q \leq p$" is an equivalent definition of being minimal, and this is non-commutative analogue of an atom for a measure.

Proposition 4.2.4 implies that if $p$ is minimal (resp. abelian) and $q \in \mathcal{P}(M)$ satisfies $q \sim_M p$, then $q$ is also minimal (resp. abelian). In fact, if $q \neq 0$ and $q \preceq p$ then it is minimal (resp. abelian). For $p$ minimal, this is simply because $q \preceq p$ implies $q \sim p$ by the above characterization of minimality. For $p$ abelian, suppose $v \in M$ is a partial isometry satisfying $v^*v = q$ and $vv^* \leq p$. Then $(vv^*)M(vv^*)$ is abelian as a subalgebra of $pMp$, and hence $qMq(\cong (vv^*)M(vv^*))$ is abelian. We record these observations in the following proposition.

**Proposition 4.2.7.** *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $q \preceq p$. If $p$ is minimal (resp. abelian), then $q$ is minimal (resp. abelian).*

We conclude this section by using compressions to prove that the image of a von Neumann algebra under a normal unital $*$-homomorphism is again a von Neumann algebra.

**Theorem 4.2.8.** *Let $M \subset B(\mathcal{H})$ and $N \subset B(\mathcal{K})$ be von Neumann algebras. If $\pi \colon M \to N$ is a normal unital $*$-homomorphism, then $\pi(M) \subset B(\mathcal{K})$ is a von Neumann algebra.*

*Proof.* We first prove a special case: assume $\pi$ is injective. Because $\pi$ is a unital $*$-homorphism, $\pi(M)$ is a unital $*$-subalgebra of $B(\mathcal{K})$ and so by Corollary 3.4.8 we just need to check that $(\pi(M))_1$ is $\sigma$-WOT closed. Because $*$-homomorphisms preserve positivity, for $x \in M$ we have

$$\pi(x)^*\pi(x) = \pi(x^*x) \leq \pi(\|x^*x\|1) = \|x^*x\|\pi(1) = \|x^*x\|,$$

and hence $\|\pi(x)\| = \|\pi(x)^*\pi(x)\|^{1/2} \leq \|x^*x\|^{1/2} = \|x\|$. The same argument applied to $\pi^{-1}\colon \pi(M) \to M$ gives $\|\pi(x)\| = \|x\|$ for all $x \in M$. Thus $(\pi(M))_1 = \pi((M)_1)$. The duality $M \cong (M_*)^*$ and the Banach–Alaoglue theorem imply $(M)_1$ is $\sigma$-WOT compact, and consequently so is its $\sigma$-WOT continuous image $\pi((M)_1) = (\pi(M))_1$. In particular, $(\pi(M))_1$ is $\sigma$-SOT closed and therefore $\pi(M)$ is a von Neumann algebra.

Now suppose $\pi$ is not injective. Consider $p := [\ker(\pi)M]$ and note that $\overline{\ker(\pi)\mathcal{H}}$ is reducing for $M$ since $\ker(\pi)$ is an ideal, and is reducing for $M'$ since $\ker(\pi) \subset M$. Thus $p \in M \cap M' = \mathcal{Z}(M)$ by Lemma 1.2.5. We will show that $\pi(M)$ is the injective image of $(1-p)M(1-p) = M(1-p)$, which is a von Neumann algebra by Theorem 4.2.2, and hence $\pi(M)$ is a von Neumann algebra by the first part of the proof. Our first step, is to show that $p \in \ker(\pi)$.

Since $\pi$ is $*$-homomorphism, $\ker(\pi)$ is a $*$-subalgebra of $M$, and it is norm closed by virtue of being $\sigma$-WOT closed. Consequently, $\ker(\pi)$ is a $C^*$-algebra and therefore has an approximate identity $(e_i)_{i\in I}$ by [Theorem 4.2, $C^*$-Algebras Mini-course]. We claim that $(e_i)_{i\in I}$ converges to $p$ in the $\sigma$-WOT, and consequently $p \in \ker(\pi)$ since $\ker(\pi)$ is $\sigma$-WOT closed. Note that $x = pxp$ for all $x \in \ker(\pi)$, and so it suffices to check $\sigma$-WOT convergence on $p\mathcal{H}$. Moreover, because $(e_i)_{i\in I}$ is uniformly bounded, it not only suffices to show WOT convergence on $p\mathcal{H}$, it suffices to show this on the dense subset $\ker(\pi)\mathcal{H}$. For $x, y \in \ker(\pi)$ and $\xi, \eta \in \mathcal{H}$ we have

$$|\langle (e_i - p)x\xi, y\eta\rangle| = |\langle (e_i x - px)\xi, y\eta\rangle| = |\langle (e_i x - x)\xi, y\eta\rangle| \leq \|e_i x - x\|\|\xi\|\|y\eta\| \to 0$$

by definition of the approximate identity. Thus $p$ is the $\sigma$-WOT limit of $(e_i)_{i\in I}$.

Since $p \in \ker(\pi)$, for $x \in M$ we have

$$\pi(x(1-p)) = \pi(x)(\pi(1) - \pi(p)) = \pi(x)(1-0) = \pi(x).$$

Thus $\pi(M)$ is the image of $M(1-p)$ under $\pi$. This also shows $x(1-p) \in \ker(\pi)$ if and only if $x \in \ker(\pi)$, but in this case $x(1-p) = x - xp = x - x = 0$. Thus $\pi|_{M(1-p)}$ is injective and so $\pi(M)$ is a von Neumann algebra by the first part of the proof. □

**Remark 4.2.9.** There is a partial converse to the above theorem: if $\pi\colon M \to B(\mathcal{K})$ is an injective $*$-homomorphism such that $\pi(M)$ is a von Neumann algebra, then $\pi$ is normal. That is, $*$-isomorphisms between von Neumann algebras are automatically normal (compare this to how $*$-isomorphisms between $C^*$-algebras are automatically isometric). This follows from a characterization of normality in terms of the increasing but uniformly bounded nets of positive operators (see Section III.2.2 in *Operator Algebras: Theory of $C^*$-Algebras and von Neumann Algebras* by Bruce Blackadar).

## Exercises

**4.2.1.** Let $p \in B(\mathcal{H})$ be a rank $n$ projection for $n \in \mathbb{N}$. Show that $pB(\mathcal{H})p \cong M_n(\mathbb{C})$.

**4.2.2.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $z \in \mathcal{P}(\mathcal{Z}(M))$. Show that $M$ is spatially isomorphic to the direct sum of compressions $Mz \oplus M(1-z)$ (see Exercise 1.2.8).

**4.2.3.** Let $(X, \mu)$ be a positive $\sigma$-finite measure space. We call a measurable subset $A \subset X$ an **atom** of if $\mu(A) > 0$ and for all measurable subsets $E \subset A$ one has $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

(a) If $A_1, A_2 \subset X$ are atoms, show that either $1_{A_1\cap A_2} = 0$ or $1_{A_1\cap A_2} = 1_{A_1} = 1_{A_2}$.

(b) If $A \subset X$ is an atom, show that $f|_A$ is constant for all $f \in L^\infty(X, mu)$.

**4.2.4.** Let $(X, \mu)$ be a positive $\sigma$-finite measure space. Show that $L^\infty(X, \mu)$ is finite dimensional (as a vector space) if and only if $X$ can be partitioned into a finite union of atoms. Also show that in this case the dimension is given by the number of distinct atoms.

**4.2.5.** Let $M \subset B(\mathcal{H})$ be a factor. Show that any abelian projection in $M$ is either zero or minimal. [**Hint:** use Corollary 4.2.3.]

**4.2.6.** Let $M \subset B(\mathcal{H})$ be a factor. Show any two minimal projections are equivalent. [**Hint:** use the Comparison Theorem.]

**4.2.7.** Let $\pi\colon M \to N$ be a $*$-homomorphism between von Neumann algebras.

(a) Show that $\pi(\mathcal{P}(M)) \subset \mathcal{P}(N)$.

(b) For $p, q \in \mathcal{P}(M)$, show that $p \preceq q$ implies $\pi(p) \preceq \pi(q)$.

(c) Show that if $p$ is minimal (resp. abelian) in $M$, then $\pi(p)$ is minimal (resp. abelian) in $\pi(M)$. Show that $\pi(p)$ need not be minimal (resp. abelian) in $N$.

**4.2.8.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{I} \subset M$ be a $\sigma$-WOT closed subspace.

(a) Show that if $\mathcal{I}$ is a left ideal then there exists $p \in \mathcal{P}(M)$ so that $\mathcal{I} = Mp$.

(b) Show that if $\mathcal{I}$ is a right ideal then there exists $p \in \mathcal{P}(M)$ so that $\mathcal{I} = pM$.

(c) Show that if $\mathcal{I}$ is a (two-sided) ideal then there exists $p \in \mathcal{P}(\mathcal{Z}(M))$ so that $\mathcal{I} = Mp$.

## 4.3 The Type Decomposition

The following definition highlights some additional important properties of projections, which will be needed in the statement of the type decomposition of von Neumann algebras.

**Definition 4.3.1.** For $M \subset B(\mathcal{H})$ be a von Neumann algebra, $p \in \mathcal{P}(M)$ is said to be

- **finite** in $M$ if $q \leq p$ and $q \sim_M p$ implies $p = q$ for $q \in \mathcal{P}(M)$.
- **semi-finite** in $M$ if there exists a family $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ of pairwise orthogonal, finite projections such that $p = \sum_{i\in I} p_i$.
- **purely infinite** in $M$ if $p \neq 0$ and there does not exist any non-zero finite projections $q \in \mathcal{P}(M)$ with $q \leq p$.
- **properly infinite** in $M$ if $p \neq 0$ and for all non-zero $z \in \mathcal{P}(\mathcal{Z}(M))$ the projection $zp$ is not finite.

Furthermore, $M$ is said to be **finite, semi-finite, purely infinite**, or **properly infinite** if $1 \in M$ has the corresponding property in $M$.

Recall that in an abelian von Neumann algebra, projections are equivalent if and only if they are equal (see Exercise 4.1.6). This implies abelian projections (and consequently minimal ones) are necessarily finite, and all abelian von Neumann algebras are finite. We also have a number of implications that follow from the above definitions:

$$\text{finite} \implies \text{semi-finite} \implies \text{not purely infinite},$$

and

$$\text{purely infinite} \implies \text{properly infinite}.$$

Also note that a factor is either finite or properly infinite.

**Example 4.3.2.** In each of the examples below, we consider $M = B(\mathcal{H})$ and $p \in \mathcal{P}(B(\mathcal{H}))$.

(1) If $p$ is finite-rank then it is finite in the above sense. Suppose $q \leq p$. Then $q\mathcal{H} \subset p\mathcal{H}$ and so $q$ is finite-rank. Suppose $q \sim p$ and let $v$ be partial isometry satisfying $v^*v = q$ and $vv^* = p$. Then by Exercise 3.1.9 we have

$$\dim(q\mathcal{H}) = \mathrm{Tr}(q) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = \mathrm{Tr}(p) = \dim(p\mathcal{H}).$$

Thus $q\mathcal{H} = p\mathcal{H}$ and $q = p$. If $\dim(H) < \infty$, then $1 \in B(\mathcal{H})$ is a finite-rank projection and hence finite, so $B(\mathcal{H})$ is finite.

(2) If $\dim(p\mathcal{H})$ is infinite, then $p$ is **not** finite. Let $\mathcal{E} \subset p\mathcal{H}$ be an orthonormal basis. Since it is an infinite set by assumption, we can partition it into disjoint subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ so that $|\mathcal{E}| = |\mathcal{E}_1| = |\mathcal{E}_2|$. If $q := [\mathcal{E}_1]$, then $q \sim p$ (see Exercise 4.1.5), but $q < p$ since $p - q = [\mathcal{E}_2] \neq 0$. Since $B(\mathcal{H})$ is a factor, these projections are also properly infinite.

(3) $p$ is always semi-finite, and consequently never purely infinite. Let $\mathcal{E} \subset p\mathcal{H}$ be an orthonormal basis. Then
$$p = \sum_{\xi \in \mathcal{E}} \xi \otimes \bar{\xi}$$
and each $\xi \otimes \bar{\xi}$ is finite by part (1). In particular, $1 \in B(\mathcal{H})$ is semi-finite and so $B(\mathcal{H})$ is semi-finite. ■

You probably learned in linear algebra that a matrix $A \in M_n(\mathbb{C})$ is left (or right) invertible if and only if it is invertible. In particular, any isometry in $M_n(\mathbb{C})$ is necessarily a unitary. Not only does this latter fact hold in *any* finite von Neumann algebra (which $M_n(\mathbb{C})$ is by Example 4.3.2.(1)), it actually characterizes them.

**Proposition 4.3.3.** *A von Neumann algebra $M \subset B(\mathcal{H})$ is finite if and only if all isometries are unitaries.*

*Proof.* Suppose $M$ is finite and let $v \in M$ be an isometry: $v^*v = 1$. Then $vv^* \leq 1$ and so by finiteness $vv^* = 1$. That is, $v$ is a unitary. Conversely, assume every isometry is a unitary, and suppose $p \leq 1$ satisfies $p \sim 1$. Let $v \in M$ satisfy $v^*v = 1$ and $vv^* = p$. Then $v$ is an isometry and hence a unitary, and therefore $p = vv^* = 1$. Thus 1 is finite in $M$. □

We will need the next two propositions in proving the type decomposition.

**Proposition 4.3.4.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Let $p, q \in \mathcal{P}(M)$ be non-zero projections that satisfy $p \preceq q$. If $q$ is finite (resp. purely infinite), then $p$ is also finite (resp. purely infinite).*

*Proof.* Suppose $q$ is finite, and further suppose $p \sim q$. Let $v \in M$ be such that $v^*v = p$ and $vv^* = q$. If $u \in M$ satisfies $u^*u = p$ and $uu^* \leq p$, then
$$(vuv^*)^*(vuv^*) = vu^*v^*vuv^* = vu^*puv^* = vu^*uv^* = vpv^* = vv^* = q$$
and
$$(vuv^*)(vuv^*)^* = vuv^*vu^*v^* = vupu^*v^* = vuu^*v^* \leq vpv^* = q.$$
Since $q$ is finite, we must have $(vuv^*)(vuv^*)^* = q$. But then
$$uu^* = pupu^*p = v^*(vuv^*)(vuv^*)^*v = v^*qv = p.$$
Thus $p$ is finite.

Now assume $p \leq q$. If $u \in M$ is such that $u^*u = p$ and $uu^* \leq p$, then for $w = u + (q - p)$ we have
$$w^*w = u^*u + u^*(q-p) + (q-p)u + (q-p) = p + (q-p) = q,$$
and
$$ww^* = uu^* + u(q-p) + (q-p)u^* + (q-p) = uu^* + (q-p) \leq q.$$
Since $q$ is finite, we have $uu^* + (q-p) = ww^* = q$ or $uu^* = p$. Thus $p$ is finite. In general, if $p \preceq q$, then there exists $q_0 \in \mathcal{P}(M)$ such that $p \sim q_0 \leq q$. By the two previous arguments we see that $p$ is finite.

Finally, if $q$ is purely infinite then it has no finite subprojections. If $p \preceq q$ had a finite subprojection $p_0 \leq p$, then $p_0 \preceq q$. In particular, $p_0 \sim q_0 \leq q$, which is finite by the above arguments, a contradiction. □

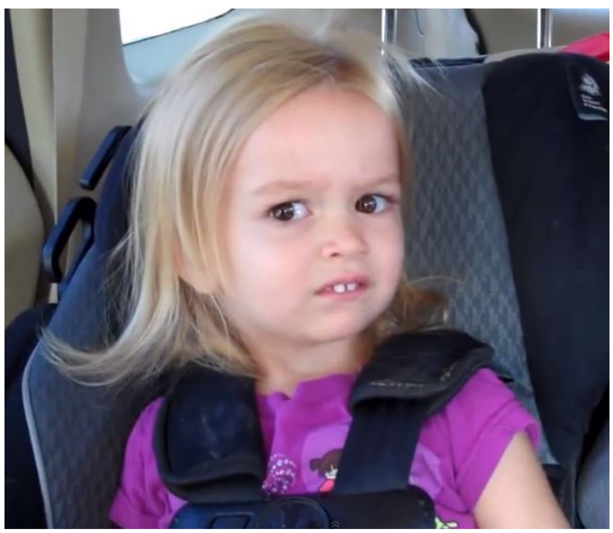
**Proposition 4.3.5.** *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. A projection $p \in \mathcal{P}(M)$ is semi-finite if and only if it is a supremum of finite projections. In particular, the supremum of semi-finite projections is again semi-finite. Moreover, any subprojection of a semi-finite projection is also semi-finite.*

*Proof.* If $p \in \mathcal{P}(M)$ is semi-finite, then by definition it is the sum (hence supremum) of pairwise orthogonal finite projections. Conversely, suppose $p = \bigvee_i p_i$ for $\{p_i\}_{i\in I} \subset \mathcal{P}(M)$ finite projections. Let $\{q_j\}_{j\in J}$ be a maximal family of pairwise orthogonal finite subprojections of $p$. Suppose, towards a contradiction, that $q := p - \sum_{j\in J} q_j \neq 0$. Then, by definition of the supremum, there exists $i \in I$ so that $q$ and $p_i$ are not orthogonal. In particular, they are not centrally orthogonal and so by Proposition 4.1.9 there exists non-zero $q_0 \leq q$ so that $q_0 \preceq p_i$. Thus $q_0$ is finite by Proposition 4.3.4, which contradicts the maximality of $\{q_j\}_{j\in J}$. The final observation follows from the fact that the above argument also works if $p \leq \bigvee_i p_i$. □

**Definition 4.3.6.** A von Neumann algebra $M \subset B(\mathcal{H})$ is said to be

- **type** I if every non-zero projection has a non-zero abelian subprojection.
- **type** II if it is semi-finite and has no non-zero abelian projections.
- **type** III if it is purely infinite.

We can see immediately from the definition that any abelian von Neumann algebra is type I. We also have $B(\mathcal{H})$ is it type I, because a non-zero projection $p$ has minimal (and hence abelian) subprojections of the form $\xi \otimes \bar{\xi}$ for any unit vector $\xi \in p\mathcal{H}$. On the other hand, group von Neumann algebras for i.c.c. groups give type II von Neumann algebras (see Example 4.3.14). Unfortunately, type III von Neumann algebras are beyond the scope of these notes. But Brent is a big fan and would love to tell you about them!



A von Neumann algebra need not be of any type. For example, if $M_1$ is type I and $M_2$ is type II, then their direct sum $M_1 \oplus M_2$ (see Exercise 1.2.8) has no type. Indeed, it is not type I because any non-zero projection $p \in \mathcal{P}(M_2)$ yields a non-zero projection $0 \oplus p \in \mathcal{P}(M_1 \oplus M_2)$ lacking non-zero abelian subprojections. It is not type II since any non-zero abelian projection $p \in M_1$ yields a non-zero abelian projection $p \oplus 0 \in \mathcal{P}(M_1 \oplus M_2)$. Since $p \oplus 0$ is finite by virtue of being abelian, we see that $M_1 \oplus M_2$ also not type III. However, note that $z_1 := 1 \oplus 0$ and $z_2 := 0 \oplus 1$ are central projections and the compressions $(M_1 \oplus M_2)z_1 = M_1 \oplus 0$ and $(M_1 \oplus M_2)z_2 = 0 \oplus M_2$ are type I and type II, respectively. The Type Decomposition tells us that this can always be done.

**Theorem 4.3.7** (Type Decomposition). *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra. Then there exists unique pairwise orthogonal central projections $\mathbf{z}_{\mathrm{I}}, \mathbf{z}_{\mathrm{II}}, \mathbf{z}_{\mathrm{III}} \in \mathcal{P}(\mathcal{Z}(M))$ such that $\mathbf{z}_{\mathrm{I}} + \mathbf{z}_{\mathrm{II}} + \mathbf{z}_{\mathrm{III}} = 1$ and the compression $M\mathbf{z}_T$ is type $T$ for each $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.*

*Proof.* Let $\mathbf{z}_{\mathrm{I}}$ be the supremum of all abelian projections in $M$. Conjugating an abelian projection in $M$ by a unitary in $M$ yields another abelian projection in $M$. It follows that $u\mathbf{z}_{\mathrm{I}}u^* = \mathbf{z}_{\mathrm{I}}$ or $u\mathbf{z}_{\mathrm{I}} = \mathbf{z}_{\mathrm{I}}u$ for all unitaries $u \in M$. Since every element in $M$ can be written as a linear combination of four unitaries, this implies $\mathbf{z}_{\mathrm{I}} \in M \cap M' = \mathcal{Z}(M)$. To see that $M\mathbf{z}_{\mathrm{I}}$ is type I, suppose $p \leq \mathbf{z}_{\mathrm{I}}$ is non-zero. Then by definition of the supremum there exists an abelian projection $r \in M$ so that $pr \neq 0$. Consequently, $pMr \neq \{0\}$ and

Proposition 4.1.9 tells us there exists non-zero $p \geq p_0 \sim r_0 \leq r$. Proposition 4.2.7 implies that $p_0$ is abelian and so $M\mathbf{z}_{\mathrm{I}}$ is type I.

Next, let $\mathbf{z}_{\mathrm{II}}$ be the supremum of all finite $p \in \mathcal{P}(M)$ such that $p \leq 1 - \mathbf{z}_{\mathrm{I}}$. By the same argument as above, we have $\mathbf{z}_{\mathrm{II}} \in \mathcal{Z}(M)$. Also, $\mathbf{z}_{\mathrm{II}}$ is semi-finite by Proposition 4.3.5. Since $\mathbf{z}_{\mathrm{II}} \leq 1 - \mathbf{z}_{\mathrm{I}}$, it has no non-zero abelian subprojections. Thus $M\mathbf{z}_{\mathrm{II}}$ is type II.

Finally, we let $\mathbf{z}_{\mathrm{III}} = 1 - \mathbf{z}_{\mathrm{I}} - \mathbf{z}_{\mathrm{II}}$. Note that any finite projection in $M$ lies under $\mathbf{z}_{\mathrm{I}}$ if it is also abelian and otherwise lies under $\mathbf{z}_{\mathrm{II}}$. Consequently, $\mathbf{z}_{\mathrm{III}}$ has no finite subprojections and so $M\mathbf{z}_{\mathrm{III}}$ is type III.

Towards showing this decomposition is unique, suppose $p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}} \in \mathcal{P}(\mathcal{Z}(M))$ are pairwise orthogonal projections summing to one and satisfy $Mp_R$ is type $R$ for each $R \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then $p_{\mathrm{III}}\mathbf{z}_{\mathrm{I}}$ and $p_{\mathrm{III}}\mathbf{z}_{\mathrm{II}}$ are both finite and purely infinite by Proposition 4.3.4. That is, $p_{\mathrm{III}}\mathbf{z}_{\mathrm{I}} = p_{\mathrm{III}}\mathbf{z}_{\mathrm{II}} = 0$, and consequently $p_{\mathrm{III}} \leq \mathbf{z}_{\mathrm{III}}$. Reversing the roles of $\mathbf{z}$ and $p$ yields $p_{\mathrm{III}} = \mathbf{z}_{\mathrm{III}}$. Next, $p_{\mathrm{II}}\mathbf{z}_{\mathrm{I}}$ is an abelian subprojection of $p_{\mathrm{II}}$ by Proposition 4.2.7. Since $Mp_{\mathrm{II}}$ is type II, we must therefore have $p_{\mathrm{II}}\mathbf{z}_{\mathrm{I}} = 0$. Thus $p_{\mathrm{II}} \leq \mathbf{z}_{\mathrm{II}}$ and by symmetry we obtain $p_{\mathrm{II}} = \mathbf{z}_{\mathrm{II}}$. Finally

$$p_{\mathrm{I}} = 1 - p_{\mathrm{II}} - p_{\mathrm{III}} = 1 - \mathbf{z}_{\mathrm{II}} - \mathbf{z}_{\mathrm{III}} = \mathbf{z}_{\mathrm{I}}.$$

So the decomposition is unique. □

Since $\mathbf{z}_{\mathrm{I}}, \mathbf{z}_{\mathrm{II}}, \mathbf{z}_{\mathrm{III}}$ are all central projections, Exercise 4.2.2 tells that $M \cong M\mathbf{z}_{\mathrm{I}} \oplus M\mathbf{z}_{\mathrm{II}} \oplus M\mathbf{z}_{\mathrm{III}}$. So even thoughh all von Neumann algebras need not have a type, they can all be written as a direct sums of type I, type II, and type III von Neumann algebras.

If $M$ is a factor, then the only central projections are 0 and 1. Consequently, in the type decomposition for a factor the summation condition $\mathbf{z}_{\mathrm{I}} + \mathbf{z}_{\mathrm{II}} + \mathbf{z}_{\mathrm{III}} = 1$ implies $\mathbf{z}_T = 1$ for some $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and the rest are zero. This yields the following corollary.

**Corollary 4.3.8.** *A factor is either type* I*, type* II*, or type* III*.*

**Remark 4.3.9.** We remark here on some important (but non-trivial) facts whose proofs we have omitted from these notes. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Then $M$ is type $T$ if and only if $M'$ is type $T$. Additionally, if $M$ is type $T$ then $pMp$ is type $T$ for any $p \in \mathcal{P}(M)$. If $\mathbf{z}(p) = 1$, then the converse also holds.

Each of the three types can be further refined. We begin with type I.

**Definition 4.3.10.** Let $M \subset B(\mathcal{H})$ be a type I von Neumann algebra. For $n \in \mathbb{N}$, we say $M$ is **type** $\mathrm{I}_n$ if there exists non-zero pairwise orthogonal and equivalent abelian projections $p_1, \ldots, p_n \in \mathcal{P}(M)$ satisfying $p_1 + \cdots p_n = 1$. We say $M$ is **type** $\mathrm{I}_\infty$ if there is an infinite family of non-zero pairwise orthogonal and equivalent abelian projections that sum to 1.

A von Neumann algebra can only be type $\mathrm{I}_n$ for one $n \in \mathbb{N} \cup \{\infty\}$. Each type I von Neumann algebra uniquely decomposes into a direct sum of type $\mathrm{I}_1$, type $\mathrm{I}_2$,..., and type $\mathrm{I}_\infty$ von Neumann algebras, and consequently a type I factor is type $\mathrm{I}_n$ for exactly one $n \in \mathbb{N} \cup \{\infty\}$. The proofs of these facts are not terribly difficult, but we have omitted them from these notes.

**Example 4.3.11.**

(1) An abelian von Neumann algebra $A \subset B(\mathcal{H})$ is type $\mathrm{I}_1$. Indeed, $1 \in A$ is an abelian projection and this cannot be further decomposed into a sum of pairwise orthogonal and equivalent projections, because in an abelian von Neumann algebra projections are eqiuvalent if and only if they are equal (see Exercise 4.1.6).

(2) $M_n(\mathbb{C})$ is type $\mathrm{I}_n$. The projections $E_{1,1}, \ldots, E_{n,n} \in M_n(\mathbb{C})$ are non-zero pairwise orthogonal projections that sum to one. They are pairwise equivalent via the partial isometries $E_{i,j}$, and they are minimal (hence abelian) projections.

(3) $B(\mathcal{H})$ for $\dim(\mathcal{H}) = \infty$ is type $\mathrm{I}_\infty$. Let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal basis. Then the projections $\{\xi \otimes \bar{\xi} \colon \xi \in \mathcal{E}\}$ are non-zero pairwise orthogonal projections that sum to one. They are pairwise equivalent via the partial isometries $\xi \otimes \bar{\eta}$ for $\xi, \eta \in \mathcal{E}$, and they are minimal projections. ■

**Theorem 4.3.12.** *If $M \subset B(\mathcal{H})$ is a finite type* I *factor, then $M \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

*Proof.* Since $M$ is type $I$, $1 \in M$ has a non-zero abelian subprojection, and since $M$ is a factor this abelian projection is minimal by Exercise 4.2.5. Thus $M$ has non-zero minimal projections.

Let $\{p_i : i \in I\} \subset \mathcal{P}(M)$ be a maximal family of pairwise orthogonal minimal projections (note $I \neq \emptyset$ by the above). Consider

$$q := 1 - \sum_{i \in I} p_i.$$

Suppose $q \neq 0$. Then the Comparison Theorem and the factoriality of $M$ imply either $q \preceq p_i$ or $p_i \preceq q$ for $i \in I$. The former implies $q \sim p_i$ since $p_i$ is minimal, but then $q$ is minimal by Proposition 4.2.7 and this contradicts the maximality of the $\{p_i : i \in I\}$. The latter implies $p_i \sim q_0 \leq q$ and the same argument shows $q_0$ contradicts the maximality of $\{p_i : i \in I\}$. So we must have $q = 0$, and therefore

$$\sum_{\in I} p_i = 1$$

Now, the factoriality of $M$ implies $p_i \sim p_j$ for all $i, j \in I$ by Exercise 4.2.6. We claim that $I$ is finite. If not, then let $I = I_1 \sqcup I_2$ be a partition of $I$ satisfying $|I| = |I_1| = |I_2|$, which implies there is a bijection $\sigma : I \to I_1$. Setting $q_i := p_{\sigma(i)}$, we have $p_i \sim q_i$ for all $i \in I$ and so by Lemma 4.1.10

$$\sum_{i \in I} q_i \sim \sum_{i \in I} p_i = 1.$$

But

$$1 = \sum_{i \in I} p_i = \left(\sum_{i \in I_1} p_i\right) + \left(\sum_{i \in I_2} p_i\right) = \left(\sum_{i \in I} q_i\right) + \left(\sum_{i \in I_2} p_i\right) > \sum_{i \in I} q_i,$$

and so we have contradicted 1 being finite. Thus $n := |I| < \infty$, and so we can relabel $\{p_i : i \in I\} =: \{p_1, p_2, \ldots, p_n\}$. Since $p_1 \sim p_i$ for each $i = 1, \ldots, n$, we can find $v_i \in M$ satisfying $v_i^* v_i = p_i$ and $v_i v_i^* = p_1$. Using $v_i = p_1 v_i$ for each $i = 1, \ldots, n$ we have for any $x \in M$

$$x = \left(\sum_{i=1}^n p_i\right) x \left(\sum_{j=1}^n p_j\right) = \sum_{i,j=1}^n p_i x p_j = \sum_{i,j=1}^n v_i^* v_i x v_j^* v_j = \sum_{i,j=1}^n v_i^* p_i v_i x v_j^* p_1 v_j = \sum_{i,j=1}^n v_i^* (p_1 v_i x v_j^* p_1) v_j.$$

Because $p_1$ is minimal there exists a scalar $x_{i,j} \in \mathbb{C}$ so that $p_1 v_i x v_j^* p_1 = x_{i,j} p_1$. Thus we have

$$x = \sum_{i,j=1}^n v_i^* x_{i,j} p_1 v_j = \sum_{i,j=1}^n x_{i,j} v_i^* p_1 v_j = \sum_{i,j=1}^n x_{i,j}.$$

This computation shows that the map

$$\pi : M \ni x \mapsto \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \in M_n(\mathbb{C})$$

is injective. Since

$$p_1 v_i (v_k^* v_\ell) v_j p_1 = \delta_{i=k} \delta_{j=\ell} p_1 v_i v_i^* v_j v_j^* p_1 = \delta_{i=k} \delta_{j=\ell} p_1 p_1 p_1 p_1 = \delta_{i=k} \delta_{j=\ell} p_1,$$

we see that $\pi(v_k^* v_\ell) = E_{k,\ell} \in M_n(\mathbb{C})$. Thus $\pi$ is a bijection, and we leave it for Exercise 4.3.5 to check that it is also a $*$-homomorphism. □

While we only considered *finite* type I factors in the above theorem, a similar proof (see Exercise 4.3.6) shows that properly infinite (i.e. non-finite) type I factors are of the form $B(\mathcal{H})$ for $\mathcal{H}$ infinite dimensional. Moreover, the form of *any* type I von Neumann algebra $M \subset B(\mathcal{H})$ can be given by a *tensor product* (see Exercise 4.3.7): $M \cong \mathcal{Z}(M) \bar{\otimes} B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Thus the theory of type I von Neumann algebras reduces to measure theory and functional analysis, and consequently researchers today focus their efforts on type II or type III von Neumann algebras.

We move on to the refinement of type II von Neumann algebras.

**Definition 4.3.13.** A type II von Neumann algebra $M \subset B(\mathcal{H})$ is said to be **type** $\mathrm{II}_1$ if it is finite, and is said to be **type** $\mathrm{II}_\infty$ if $M$ is properly infinite.

Equivalently, a von Neumann algebra if type $\mathrm{II}_1$ if it is finite but has no non-zero abelian projections, and a von Neumann algebra is type $\mathrm{II}_\infty$ if it is properly infinite but semi-finite and has no non-zero abelian projections. Each type II von Neumann algebra uniquely decomposes into a direct sum of type $\mathrm{II}_1$ and type $\mathrm{II}_\infty$ von Neumann algebras, and consequently each type II factor is either type $\mathrm{II}_1$ or type $\mathrm{II}_\infty$.

**Example 4.3.14.** $L(\Gamma)$ for a countable i.c.c. group $\Gamma$ is type $\mathrm{II}_1$ factor. First note that $L(\Gamma)$ is a factor by Exercise 1.3.7. It is also finite by Exercise 4.3.1. So it remains to show it has no non-zero abelian projections. Suppose, towards a contradiction, that $p \in \mathcal{P}(L(\Gamma))$ is non-zero and abelian. Then $p$ is actually minimal by Exercise 4.2.5. Let $\{p_i\}_{i\in I} \subset \mathcal{P}(L(\Gamma))$ is a maximal family of pairwise orthogonal minimal projections. Then $I \neq \emptyset$ by the above and the exact same argument as in the proof of Theorem 4.3.12 shows $n := |I| < \infty$ and $L(\Gamma) \cong M_n(\mathbb{C})$. Note that $M_n(\mathbb{C})$ is finite dimensional as a vector space. On the other hand, $\Gamma$ is necessarily infinite as an i.c.c. group and so $\{\lambda(g) : g \in \Gamma\}$ is an infinite linearly independent set (just apply any linear combination to the vector $\delta_e$). So $L(\Gamma) \cong M_n(\mathbb{C})$ yields a contradiction and hence $L(\Gamma)$ has no non-zero abelian projections. ∎

**Example 4.3.15.** In this example we will construct an important type $\mathrm{II}_1$ factor $\mathcal{R}$ called the **hyperfinite** $\mathbf{II_1}$ **factor**. Observe that for any $n \in \mathbb{N}$ we can embed $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{C})$ via

$$M_n(\mathbb{C}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

These inclusions preserve the norm (since they are injective $*$-homomorphisms) and the normalized trace:

$$\frac{1}{2n}\mathrm{Tr}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \frac{1}{n}\mathrm{Tr}(A) \qquad A \in M_n(\mathbb{C}).$$

Thus if we consider the sequence of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow M_{2^n}(\mathbb{C}) \hookrightarrow \cdots .$$

and define $\mathcal{R}_0 := \bigcup_{n\geq 1} M_{2^n}(\mathbb{C})$, then $\mathcal{R}_0$ is a $*$-algebra with a norm (although it is not complete) and a linear functional $\tau_0 : \mathcal{R}_0 \to \mathbb{C}$ defined by $\tau_0(x) = \frac{1}{2^n}\mathrm{Tr}(x)$ when $x \in M_{2^n}(\mathbb{C})$. From the properties of the trace, it follows that $\tau_0$ is

- **unital**: $\tau_0(1) = 1$;
- **positive**: $\tau_0(x^*x) \geq 0$ for all $x \in \mathcal{R}_0$;
- **faithful**: $\tau_0(x^*x) = 0$ if and only if $x = 0$;
- **tracial**: $\tau_0(xy) = \tau_0(yx)$ for all $x, y \in \mathcal{R}_0$.

We can therefore consider the GNS representation $(\mathcal{H}, \pi)$ for $(\mathcal{R}_0, \tau_0)$, and $\mathcal{R}_0$ gives a dense subspace of $\mathcal{H}$. Define

$$\mathcal{R} := \pi(\mathcal{R}_0)'' \subset B(\mathcal{H}).$$

We will show that $\mathcal{R}$ is a $\mathrm{II}_1$ factor. We must first show it admits a WOT continuous faithful tracial state.

Viewing $1 \in \mathcal{R}_0$ as a vector in $\mathcal{H}$, we see that it is cyclic for $\mathcal{R}$ by construction. It is also separating for $\mathcal{R}$: it is separating for $\pi(\mathcal{R}_0)$ since $\tau_0$ is faithful, so it is cyclic for $\pi(\mathcal{R}_0)'$ and hence separating for $\pi(\mathcal{R}_0)'' = \mathcal{R}$ by Proposition 2.2.4. Thus the linear functional $\tau : \mathcal{R} \to \mathbb{C}$ defined by $\tau(x) = \langle x1, 1\rangle$ is faithful, and as a vector state it WOT continuous. Using $\tau(\pi(x)) = \tau_0(x)$ for $x \in \mathcal{R}_0$, it can be shown that $\tau$ also tracial (see Exercise 4.3.10).

Now, suppose $z \in \mathcal{Z}(\mathcal{R}))$. Define $\varphi : \mathcal{R} \to \mathbb{C}$ by $\varphi(x) := \tau(xz)$, which is still tracial since $z$ commutes with everything in $\mathcal{R}$. Consequently, restricting $\varphi \circ \pi$ to $M_{2^n}(\mathbb{C})$ gives a tracial linear functional, and thus Exercise 1.3.2 implies

$$\varphi \circ \pi(x) = \varphi \circ \pi(1)\frac{1}{2^n}\mathrm{Tr}(x) = \tau(z)\tau(\pi(x))$$

for all $x \in M_{2^n}(\mathbb{C})$. Since this holds for all $n \in \mathbb{N}$, we have $\varphi(x) = \tau(z)\tau(x)$ for all $x \in \pi(\mathcal{R}_0)$. The WOT density of $\pi(\mathcal{R}_0)$ along with the WOT conintuity of $\tau$ implies this holds for all $x \in \mathcal{R}$. Thus $\tau(xz) = \tau(z)\tau(x)$, or equivalently $\tau(x(z - \tau(z))) = 0$ for all $x \in \mathcal{R}$. In particular, letting $x = (z - \tau(z))^*$ we see that the faithfulness of $\tau$ implies $z - \tau(z) = 0$ or $z = \tau(z) \in \mathbb{C}$. Thus $\mathcal{R}$ is a factor.

To see that it is finite, suppose $v \in \mathcal{R}$ is a partial isometry satisfying $v^*v = 1$ and $vv^* \leq 1$, then

$$\tau((1 - vv^*)^*(1 - vv^*)) = \tau(1 - vv^*) = \tau(1) - \tau(vv^*) = \tau(1) - \tau(v^*v) = \tau(1) - \tau(1) = 0.$$

Since $\tau$ is faithful, we must have $vv^* = 1$ and so $\mathcal{R}$ is finite.

It remains to show that $\mathcal{R}$ has no non-zero abelian projections. Proceeding exactly as in Example 4.3.14, we see that if this is not the case then $\mathcal{R} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. This is a contradiction because $M_n(\mathbb{C})$ is finite dimensional while $\mathcal{R}$ is infinite dimensional since $\pi(\mathcal{R}_0)$ is infinite dimensional. Thus $\mathcal{R}$ is a type $\mathrm{II}_1$ factor. ■

The term *hyperfinite* refers to the fact that $\mathcal{R}$ is generated by the finite dimensional algebras $\pi_\tau(M_{2^n}(\mathbb{C}))$. Alain Connes showed in 1976 that $\mathcal{R}$ is the *unique* $\mathrm{II}_1$ factor with this property. Moreover, this same work, as mentioned back in Section 1.3.3, shows that the two previous examples coincide when $\Gamma$ is an amenable i.c.c. group.

**Example 4.3.16.** Let $(X, \Omega, \mu)$ be a probability space and let $\Gamma$ be a countable discrete group. Suppose there is a homomorphism $\alpha\colon \Gamma \to \mathrm{Aut}(L^\infty(X, \mu))$, where $\mathrm{Aut}(L^\infty(X, \mu))$ is the set of (normal) $*$-isomorphisms. In this case we call $\alpha$ an **action** of $\Gamma$ on $L^\infty(X, \mu)$ and write $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$. We say the action is

- **probability measure preserving (p.m.p.)** if $\int_X \alpha_g(f)\, d\mu = \int_X f\, d\mu$ for all $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.
- **free** if $f \in L^\infty(X, \mu)$ is such that $f\alpha_g(h) = fh$ for all $g \in \Gamma$ and $h \in L^\infty(X, \mu)$ then $f = 0$.
- **ergodic** if $f \in L^\infty(X, \mu)$ is such that $\alpha_g(f) = f$ for all $g \in \Gamma$ then $f = c$ for some $c \in \mathbb{C}$.

For $f \in L^\infty(X, \mu)$, define a linear operator $\pi_\alpha(f)$ on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ by

$$\pi_\alpha(f)\left(\sum_{g \in \Gamma} \delta_g \otimes f_g\right) = \sum_{g \in \Gamma} \delta_g \otimes [\alpha_{g^{-1}}(f) f_g] \qquad f_g \in L^2(X, \mu).$$

Then one can show that $\pi_\alpha(f) \in B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ and $\pi_\alpha\colon L^\infty(X, \mu) \to B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ is a normal unital injective $*$-homomorphism (Exercise 4.3.11). For $g \in \Gamma$, we define

$$\lambda(g)\left(\sum_{h \in \Gamma} \delta_h \otimes f_h\right) = \sum_{h \in \Gamma} \delta_{gh} \otimes f_h \qquad f_h \in L^2(X, \mu).$$

Note that $\lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha(f))$ (Exercise 4.3.11.(c)). This implies the $*$-algebra generated by $\pi_\alpha(L^\infty(X, \mu))$ and $\lambda(\Gamma)$ is the set

$$\mathbb{C}\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma)\rangle := \left\{ \sum_{j=1}^d \pi_\alpha(f_j)\lambda(g_j)\colon d \in \mathbb{N},\ f_1, \ldots, f_d \in L^\infty(X, \mu),\ g_1, \ldots, g_d \in \Gamma \right\}.$$

Note that $\mathbb{C}\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma)\rangle$ is unital. The von Neumann algebra

$$L^\infty(X, \mu) \rtimes_\alpha \Gamma := \mathbb{C}\langle \pi_\alpha(L^\infty(X, \mu)), \lambda(\Gamma)\rangle''$$

is called the **crossed product** of $L^\infty(X, \mu)$ by $\Gamma$. You should think of it as a von Neumann algebra containing both $L^\infty(X, \mu)$ and $L(\Gamma)$ with the action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ encoded via commutation relations. Consider the normal linear functional $\tau\colon L^\infty(X, \mu) \rtimes_\alpha \Gamma \to \mathbb{C}$ defined by $\tau(x) = \langle x(\delta_e \otimes 1), \delta_e \otimes 1\rangle$. Since $\delta_e \otimes 1$ is a unit vector and separating for $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ (see Exercise 4.3.13), $\tau$ is a unital and faithful.

Assume $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ is a free ergodic p.m.p. action and that $\Gamma$ is an infinite group. The freeness and ergodicity imply $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is a factor by Exercise 4.3.15, while the action being p.m.p implies $\tau$

is tracial by Exercise 4.3.16. Consequently, by the same arguement as in the previous two examples we see that $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is finite and therefore either a type $\mathrm{I}_n$ or type $\mathrm{II}_1$ factor. Since $L(\Gamma) \subset L^\infty(X,\mu) \rtimes_\alpha \Gamma$ and $\Gamma$ is infinite, we see that the crossed product is not finite dimensional. Thus $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is a type $\mathrm{II}_1$ factor. ■

We only considered type $\mathrm{II}_1$ factors in the examples above, but for any type $\mathrm{II}_1$ von Neuman algebra $M$ the tensor product $M \bar{\otimes} B(\mathcal{H})$ for $\mathcal{H}$ infinite dimensional yields a type $\mathrm{II}_\infty$ von Neumann algebra. In fact, all type $\mathrm{II}_\infty$ factors are of this form.

The class of type III factors can also be further decomposed into types $\mathrm{III}_\lambda$ for $\lambda \in [0,1]$. This classification is achieved via some very beautiful mathematics known as *Tomita–Takesaki theory*. Essentially, von Neumann algebras of this type have intrinsic dynamical systems which determine the parameter $\lambda \in [0,1]$.

We conclude this chapter with a summary of types for factors. Recall that factor is either finite or properly infinite. We will also say a factor is **atomic** if it contains a minimal projection, and otherwise say it is **diffuse**.

<table>
<tr><td></td><td>atomic</td><td colspan="2">diffuse</td></tr>
<tr><td>finite</td><td>type $\mathrm{I}_n$, $n \in \mathbb{N}$</td><td>type $\mathrm{II}_1$</td><td></td></tr>
<tr><td>properly infinite</td><td>type $\mathrm{I}_\infty$</td><td>type $\mathrm{II}_\infty$</td><td>type III</td></tr>
<tr><td></td><td colspan="2">semi-finite</td><td>purely infinite</td></tr>
</table>

## Exercises

**4.3.1.** Let $\Gamma$ be a countable discrete group. Show that all projections in $L(\Gamma)$ are finite. [Hint: use the trace.]

**4.3.2.** Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and let $p, q \in \mathcal{P}(M)$ satisfy $p \preceq q$. Show that if $q$ is semi-finite then $p$ is semi-finite.

**4.3.3.** Let $\pi\colon M \to N$ be a $*$-isomorphism between von Neumann algebras and let $p \in \mathcal{P}(M)$.

(a) Show $p$ is finite in $M$ if and only if $\pi(p)$ is finite in $N$.

(b) Assuming $\pi$ is normal, show $p$ is semi-finite in $M$ if and only if $\pi(p)$ is finite in $N$.

(c) Show $p$ is purely infinite in $M$ if and only if $\pi(p)$ is finite in $N$.

(d) Show $p$ is properly infinite in $M$ if and only if $\pi(p)$ is finite in $N$.

**4.3.4.** Let $\pi\colon M \to N$ be a normal $*$-isomorphis between von Neumann algebras. Show that $M$ has type $T$ for $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ if and only if $N$ has type $T$.

**4.3.5.** Let $\pi\colon M \to M_n(\mathbb{C})$ be the map defined at the end of the proof of Theorem 4.3.12. Show that $\pi$ is a unital $*$-homomorphism.

**4.3.6.** Let $M \subset B(\mathcal{H})$ be properly infinite type I factor. In this exercise, you will show that $M \cong B(\mathcal{K})$ for some infinite dimensional Hilbert space $\mathcal{K}$.

(a) Show that $M$ admits an infinite family $\{p_i\colon i \in I\}$ of pairwise orthogonal and equivalent minimal projections satisfying

$$\sum_{i\in I} p_i = 1.$$

(b) Fix $i_0 \in I$ and let $v_i \in M$ be a partial isometry satisfying $v_i^* v_i = p_i$ and $v_i v_i^* = p_{i_0}$. For each $x \in M$ and $i, j \in I$, show that there is a scalar $x_{i,j} \in \mathbb{C}$ so that $p_i x p_j = x_{i,j} v_i^* v_j$.

(c) Denote $\mathcal{K}_0 := \operatorname{span}\{p_i\colon i \in I\}$. Show

$$\left\langle \sum_{k=1}^m \alpha_k p_{i_k}, \sum_{\ell=1}^n \beta_\ell p_{j_\ell} \right\rangle := \sum_{k=1}^m \sum_{\ell=1}^n \alpha_k \bar{\beta}_\ell \delta_{i_k = j_\ell}$$

defines an inner product on $\mathcal{K}_0$.

(d) Let $\mathcal{K}$ be the completion of $\mathcal{K}_0$ with respect to this inner product. For each $x \in M$ show that

$$\pi(x) := \sum_{i,j\in I} x_{i,j} p_i \otimes \bar{p}_j \in B(\mathcal{K}),$$

where here we are viewing $p_i, p_j \in \mathcal{K}$ so that $p_i \otimes \bar{p}_j \in FR(\mathcal{K})$.

(e) Show that $\pi\colon M \to B(\mathcal{K})$ is normal $*$-isomorphism.

**4.3.7.** Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces, and for each $j = 1, \ldots, n$ and $x \in B(\mathcal{H}_j)$ define a a linear operator $\pi_j(x)$ on $\mathcal{H}_1 \odot \cdots \odot \mathcal{H}_n$ by

$$\pi_j(x)(\xi_1 \otimes \cdots \otimes \xi_j \otimes \cdots \otimes \xi_n) := \xi_1 \otimes \cdots \otimes (x\xi_j) \otimes \cdots \otimes \xi_n \qquad \xi_1 \in \mathcal{H}_1, \ldots, \xi_n \in \mathcal{H}_n.$$

(a) For $j = 1, \ldots, n$ and $x \in B(\mathcal{H}_j)$, show that $\pi_j(x)$ extends to a bounded operator on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with $\|\pi_j(x)\| = \|x\|$.

(b) Show that $\pi_j\colon B(\mathcal{H}_j) \to B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ is a unital $*$-homomorphism for each $j = 1, \ldots, n$.

(c) Show that $\pi_j(B(\mathcal{H}_j))$ and $\pi_k(B(\mathcal{H}_k))$ commute for $j \neq k$.

(d) Let $M_j \subset B(\mathcal{H}_j)$ be a von Neumann algebra for each $j = 1, \ldots, n$. Show that

$$M_1 \otimes \cdots \otimes M_n := \text{span}\left\{\pi_1(x_1)\cdots\pi_n(x_n)\colon x_1 \in M_1, \ldots, x_n \in M_n\right\}.$$

is a unital $*$-algebra.

(e) The **tensor product** of $M_1, \ldots, M_n$ is the von Neumann algebra

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n := (M_1 \otimes \cdots M_n)''$$

Show that if $M_2 = \cdots = M_n = \mathbb{C}$, then $M_1 \bar{\otimes} \cdots \bar{\otimes} M_n \cong M_1$.

**4.3.8.** Using the notation from Example 4.3.15, show that $\mathcal{R}_0$ can be viewed as an inductive limit (see [Definition 6.1, GOALS Prerequisite Notes]).

**4.3.9.** Using the notation from Example 4.3.15, show that for verify that $\tau_0$ is unital, positive, faithful, and tracial.

**4.3.10.** Using the notation from Example 4.3.15, show that $\tau$ is tracial. [**Hint:** first show $\tau(xy) = \tau(yx)$ for $x \in \mathcal{R}$ and $y \in \pi(\mathcal{R}_0)$ using the SOT density of $\pi(\mathcal{R}_0)$.]

**4.3.11.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ for a countable discrete group $\Gamma$ and a probability space $(X, \mu)$.

(a) For $f \in L^\infty(X, \mu)$, show that $\pi_\alpha(f)$ is a bounded operator on $\ell^2(\Gamma) \otimes L^2(X, \mu)$ with $\|\pi_\alpha(f)\| = \|f\|_\infty$.

(b) Show that $\pi_\alpha\colon L^\infty(X, \mu) \to B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ is a unital $*$-homomorphism.

(c) Show that $\lambda(g)\pi_\alpha(f)\lambda(g^{-1}) = \pi_\alpha(\alpha(f))$ for all $g \in \Gamma$ and $f \in L^\infty(X, \mu)$.

**4.3.12.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X, \mu)$ for a countable discrete group $\Gamma$ and a probability space $(X, \mu)$. For $f \in L^\infty(X, \mu)$, define $\phi_\alpha(f) \in B(\ell^2(\Gamma) \otimes L^2(X, \mu))$ by

$$\phi_\alpha(f)\left(\sum_{g\in\Gamma} \delta_g \otimes f_g\right) = \sum_{g\in\Gamma} \delta_g \otimes f f_g \qquad f_g \in L^2(X, \mu),$$

and define $\rho(g)$ for $g \in \Gamma$ by

$$\rho(g)\left(\sum_{h\in\Gamma} \delta_h \otimes f_h\right) = \sum_{h\in\Gamma} \delta_{hg^{-1}} \otimes \alpha_g(f_h) \qquad f_h \in L^2(X, \mu).$$

Show that $\phi_\alpha(L^\infty(X, \mu)) \cup \rho(\Gamma) \subset (L^\infty(X, \mu) \rtimes_\alpha \Gamma)'$.

**4.3.13.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group $\Gamma$ and a probability space $(X,\mu)$.

(a) Show that $\delta_e \otimes 1$ is a cyclic vector for $L^\infty(X,\mu) \rtimes_\alpha \Gamma$.

(b) Show that $\delta_e \otimes 1$ is a separating vector for $L^\infty(X,\mu) \rtimes_\alpha \Gamma$.
[**Hint:** use Exercise 4.3.12.]

**4.3.14.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group $\Gamma$ and a probability space $(X,\mu)$. For $x \in L^\infty(X,\mu) \rtimes_\alpha \Gamma$, define a linear operator $x_g$ on $L^2(X,\mu)$ by

$$x_g(f) = [x(\delta_{g^{-1}} \otimes f)](e).$$

Show that $x_g \in L^\infty(X,\mu)$. [**Hint:** show that $x_g \in L^\infty(X,\mu)'$ by using $\phi_\alpha$ as in Exercise 4.3.12.]

**4.3.15.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group $\Gamma$ and a probability space $(X,\mu)$.

(a) Show that $L^\infty(X,\mu)' \cap L^\infty(X,\mu) \rtimes_\alpha \Gamma = L^\infty(X,\mu)$ if and only if the action is free.
[**Hint:** using the notation from Exercise 4.3.14, compare $(xf)_g$ and $(fx)_g$ for $x \in L^\infty(X,\mu)' \cap L^\infty(X,\mu) \rtimes_\alpha \Gamma$ and $f \in L^\infty(X,\mu)$.]

(b) Assuming the action is free, show that $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ is a factor if and only if the action is ergodic.

**4.3.16.** Suppose $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ for a countable discrete group $\Gamma$ and a probability space $(X,\mu)$. Let $\tau\colon L^\infty(X,\mu) \rtimes_\alpha \Gamma \to \mathbb{C}$ be as in Example 4.3.16.

(a) Show that $\tau(\lambda(g)) = \delta_{g=e}$ for $g \in \Gamma$.

(b) Show that $\tau(\pi_\alpha(f)) = \int_X f\, d\mu$ for $f \in L^\infty(X,\mu)$.

(c) Assume that the action is probability measure preserving. Show that $\tau$ is a tracial.

**4.3.17.** In this exercise, you will show that $M_n(\mathbb{C})$ can be realized via a crossed-product construction. Consider $\Gamma := \mathbb{Z}_n$, the countable cyclic group of order $n$, and also set $X := \mathbb{Z}_n$ which we view as simply a space and equip with the counting (probability) measure.

(a) Show that $\alpha_g(f) := f(\cdot - g)$ for $g \in \Gamma$ defines an action $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$.

(b) Show that $\Gamma \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ is free, ergodic, and probability measure preserving.

(c) Show that $1_{\{1\}}, \dots, 1_{\{n\}} \in L^\infty(X,\mu)$ are pairwise orthogonal and equivalent minimal projections.

(d) Show that $L^\infty(X,\mu) \rtimes_\alpha \Gamma \cong M_n(\mathbb{C})$. What is the preimage of $E_{i,j}$ under this isomorphism?

(e) Explain why there does not exist a discrete group $\Gamma$ such that $L(\Gamma) \cong M_n(\mathbb{C})$.