Problem Session

July 8th, 2020

C*.1 Show that any positive linear functional $\phi : A \to \mathbb{C}$ is $*$-preserving, i.e. $\phi(a^*) = \overline{\phi(a)}$ for all $a \in A$.

C*.2 Show that for a unital C^*-algebra A, $\mathcal{S}(A)$ is a weak* closed convex subset of $A^*_{\leq 1}$. It follows from Alaoglu's theorem that it is weak*-compact. What does the Krein-Milman theorem say about $\mathcal{S}(A)$?

C*.3 Show that if the C*-algebra A is finite dimensional as a vector space, then we may take the Hilbert space $\mathcal{H}$ of the GNS Theorem to be finite dimensional.

[**Hint:** Show that you only need finitely many states $\phi \in F$, and that H_ϕ is finite dimensional for all ϕ.]

W*.1 Let $\mathcal{H}$ be a Hilbert space.

(a) For orthonormal sets $\{\xi_1, \dots, \xi_n\}, \{\eta_1, \dots, \eta_n\} \subset \mathcal{H}$, show that $\sum_{i=1}^n \xi_i \otimes \bar{\eta}_i$ is a partial isometry that implements the equivalence $(\sum_{i=1}^n \eta_i \otimes \bar{\eta}_i) \sim \left(\sum_{i=1}^n \xi_i \otimes \bar{\xi}\right)$.

(b) For finite-rank projections $p, q \in B(\mathcal{H})$, show that $p \sim q$ if and only if $\mathrm{Tr}(p) = \mathrm{Tr}(q)$.

(c) Let $\mathcal{E}, \mathcal{F} \subset \mathcal{H}$ be two orthonormal subsets with the same cardinality. Show that $[\mathcal{E}] \sim [\mathcal{F}]$.
[**Hint:** start with a bijection from $\mathcal{E}$ to $\mathcal{F}$ (as sets).]

W*.2 Let $M \subset B(\mathcal{H})$ be a factor. Show any two minimal projections are equivalent.

[**Hint:** use the Comparison Theorem.]

W*.3 Let (X, μ) be a positive σ-finite measure space. We call a measurable subset $A \subset X$ an **atom** of if $\mu(A) > 0$ and for all measurable subsets $E \subset A$ one has $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

(a) If $A_1, A_2 \subset X$ are atoms, show that either $1_{A_1 \cap A_2} = 0$ or $1_{A_1 \cap A_2} = 1_{A_1} = 1_{A_2}$.

(b) If $A \subset X$ is an atom, show that $f|_A$ is constant for all $f \in L^\infty(X, \mu)$.

(c) Show that 1_A is a minimal projection in $L^\infty(X, \mu)$ if and only if A is an atom.