5. Group C*-algebras

Preview of Lecture: Today's C*-lectures will discuss 3 classes of examples of C*-algebras: group C*-algebras, AF algbras, and Cuntz–Krieger algebras. There's a quick preview at the beginning of each section.

For the group C*-algebras, in lecture, we'll discuss Proposition 5.7 and Example 5.9. We'll save Proposition 5.10 for Wednesday.

Most of the steps of the proof of Proposition 5.17 are relatively straightforward; the one which requires the most creativity is the fact that $h(\omega) \in \widehat{C_r^*(G)}$ for all $\omega \in \widehat{G}$ so we'll discuss that in lecture.

A useful source of examples and motivation for C*-theory are the *group* C*-*algebras*. Indeed, one can view a group C*-algebra as encoding the (infinite-dimensional) representations of the group. (See Exercise 5.12.) Understanding these representations better was a main motivation for a lot of the early work on C*-algebras, and group C*-algebras are still a fundamental source of examples and inspiration for research today.

Definition 5.1. Let G be a discrete group. The *complex group algebra* $\mathbb{C}G$ is the algebra generated by $\{u_g : g \in G\}$, where $u_g u_h = u_{gh}$.

By definition, then, $\mathbb{C}G$ consists of all finite products of finite linear combinations of $\{u_g : g \in G\}$.

Observe that $\mathbb{C}G$ is always unital (what's the unit?). Moreover, we have a natural involution on $\mathbb{C}G$:

$$(a_g u_g)^* := \overline{a_g} u_{g^{-1}}.$$

(Check for yourself that this formula indeed gives an involution.)

Given two finite linear combinations of generators $\sum_{g \in G} a_g u_g, \sum_{g \in G} b_g u_g \in \mathbb{C}G$, then the formula for the multiplication of the generators $\{u_g\}_{g \in G}$ implies that

$$\left(\sum_{g \in G} a_g u_g\right)\left(\sum_{g \in G} b_g u_g\right) = \sum_{h \in G}\left(\sum_{k \in G} a_k b_{k^{-1}h}\right) u_h.$$

This multiplication may look familiar if you've seen convolution multiplication or the Fourier transform before. For functions ϕ, ψ on a discrete group G, their *convolution product* is

$$\phi * \psi(g) := \sum_{h \in G} \phi(h)\psi(h^{-1}g).$$

That is, if we think of the coefficients $(a_g)_{g \in G}$ of an element $\sum_{g \in G} a_g u_g \in \mathbb{C}G$ as a function from G to $\mathbb{C}$, then the function associated to the product $(\sum_{g \in G} a_g u_g)(\sum_{g \in G} b_g u_g)$ is precisely the convolution product of the functions $(a_g)_{g \in G}$ and $(b_g)_{g \in G}$.

If we want to complete the $*$-algebra $\mathbb{C}G$ into a C*-algebra, we first need a norm. In our case this will come from a representation.

Definition 5.2. A *representation* of a $*$-algebra A is a $*$-preserving homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If A is unital, we will assume π is *unital* in that it takes the unit of A to the unit of $B(\mathcal{H})$. If π is injective we say that it is *faithful*.

Note that if π is a representation of $\mathbb{C}G$ and $a \in \mathbb{C}G$, then the fact that $B(\mathcal{H})$ is a C*-algebra implies that

$$\|\pi(a^*a)\| = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

In particular, the norm on A induced by π, $\|a\|_\pi := \|\pi(a)\|$, satisfies the C*-identity. Therefore,

$$C^*_\pi(G) := \overline{\pi(\mathbb{C}G)}$$

is a C*-algebra.

Exercise 5.3. If π is a representation of $\mathbb{C}G$, what sort of operator will $\pi(u_g)$ be? Can you say anything about $\|\pi(u_g)\|$?

There is a natural representation of $\mathbb{C}G$ on $\ell^2(G) = \overline{\text{span}}\{\delta_g : g \in G\}$, called the *left regular representation* and often denoted by λ: On the generators, we define

$$\lambda(u_g)(\delta_h) = \delta_{gh},$$

and extend λ to $\mathbb{C}G$ by requiring it to be a linear multiplicative map.

Exercise 5.4. What is the adjoint of $\lambda(u_g)$? Is λ $*$-preserving?

Observe (check!) that λ is injective. So, we can think of $\mathbb{C}G$ as a subalgebra of $B(\ell^2(G))$. The *reduced group* C^**-algebra* $\mathrm{C}^*_r(G)$ is defined to be

$$\mathrm{C}^*_r(G) := \overline{\lambda(\mathbb{C}G)}.$$

So that we don't always have to choose a specific representation (and for abstract-nonsense reasons) we often want to work with the *universal group* C^**-algebra* $\mathrm{C}^*(G)$, which is defined to be the completion of $\mathbb{C}G$ in the *universal norm*

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G\}. \tag{5.1}$$

A reader who is familiar with set theory might notice that we have made no assertion about whether the collection of all representations of $\mathbb{C}G$ is a set. How, then, do we know that we can take the supremum in (5.1)? Recall that, for any $a \in \mathbb{C}G$ and any representation π of $\mathbb{C}G$, the quantity $\|\pi(a)\|$ is a real number, being the norm of an operator on some Hilbert space. So the collection in (5.1) is a subclass of the set of all real numbers, and basic results from set theory guarantee that a subclass of a set is still a set. It follows that the universal norm is well defined.

In fact, the universal norm is bounded above by the ℓ^1 norm:

Proposition 5.5. *If π is a representation of $\mathbb{C}G$, then for any $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$ we have $\|\pi(a)\| \leq \sum_{g\in F} |a_g|$.*

Proof. Since $\pi(u_g)$ is a unitary for all g, and hence has norm 1, the triangle inequality tells us that

$$\|\pi(a)\| \leq \sum_{g\in F} \|a_g u_g\| = \sum_{g\in F} |a_g|. \qquad \square$$

It follows that if a net in $\mathbb{C}G$ is Cauchy in the ℓ^1 norm, then that net is also Cauchy in $\mathrm{C}^*(G)$ (and $\mathrm{C}^*_r(G)$). In other words, we could alternatively think of $\mathrm{C}^*(G)$ and $\mathrm{C}^*_r(G)$ as completions in a C^*-norm of $\ell^1(G)$. This will come in handy sometimes, for example in Section 5.1.

Proposition 5.6. *$\mathbb{C}G$ is dense in both $\mathrm{C}^*_r(G)$ and $\mathrm{C}^*(G)$.*

Proof. The fact that $\mathbb{C}G$ is dense in $\mathrm{C}^*_r(G)$ follows from the injectivity of λ. Similarly, to see that $\mathbb{C}G$ is dense in $\mathrm{C}^*(G)$, it will suffice to show that if $a \in \mathbb{C}G$ is nonzero, then $\|a\|_u \neq 0$. Since $\|a\|_u \geq \|\lambda(a)\|$ by the definition of the universal norm, it follows that $\|a\|_u = 0$ implies $a = 0$. $\square$

The reason we call $\mathrm{C}^*(G)$ the "universal group C^*-algebra" is the following proposition. While the argument used in the proof is straightforward, it's a very powerful technique for constructing $*$-homomorphisms out of many examples of C^*-algebras, not just group C^*-algebras.

Proposition 5.7. *For any representation π of $\mathbb{C}G$, there is an associated surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(G) \to \mathrm{C}^*_\pi(G)$.*

Proof. We define $\hat{\pi}$ first for $a \in \mathbb{C}G \subseteq \mathrm{C}^*(G)$:

$$\hat{\pi}(a) := \pi(a) \in \mathrm{C}^*_\pi(G).$$

As π is a representation of $\mathbb{C}G$, in order to extend $\hat{\pi}$ to a $*$-homomorphism on all of $\mathrm{C}^*(G)$, I claim that it suffices to check that $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$. Why? Well, once we know that $\|\hat{\pi}(a)\| \leq \|a\|_u$ for all $a \in \mathbb{C}G$, then if $x \in \mathrm{C}^*(G)$ is a norm limit of elements in $\mathbb{C}G$, $x = \lim_i a_i$, then in particular, given any $\epsilon > 0$, we can find I such that $\|a_i - a_j\|_u < \epsilon$ whenever $i, j \geq I$. If $\hat{\pi}$ is norm-decreasing on $\mathbb{C}G \subseteq \mathrm{C}^*(G)$, then it follows that $(\hat{\pi}(a_i))_i$ is Cauchy in $\mathrm{C}^*_\pi(G)$. As $\mathrm{C}^*_\pi(G)$ is complete, $\lim_i(\hat{\pi}(a_i))_i$ has a limit, call it y. Defining $\hat{\pi}(x) := y$, one can check that $\hat{\pi}(x)$ is independent of the approximating Cauchy sequence $(a_i)_i \subseteq \mathbb{C}G \subseteq \mathrm{C}^*(G)$, and that this definition makes $\hat{\pi}$ into a $*$-homomorphism.

Thus, it (essentially) suffices to check that $\|\hat{\pi}(a)\| \leq \|a\|_u$ for all $a \in \mathbb{C}G \subseteq \mathrm{C}^*(G)$. However, the definition of the universal norm makes this immediate:

$$\|\hat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u. \qquad \square$$

Exercise 5.8. Fill in the gaps in the proof of Proposition 5.7. (This includes checking that $\hat{\pi}$ is surjective.)

Example 5.9. Let $G = \mathbb{Z}$ (under addition). Observe that if $u \in B(\mathcal{H})$ is a unitary, then we obtain a representation $\pi : \mathbb{C}\mathbb{Z} \to B(\mathcal{H})$ given by defining $\pi(u_0) = u$. Conversely, any representation π of $\mathbb{C}\mathbb{Z}$ arises in this way.

It follows that, for any $u \in B(\mathcal{H})$, there is a surjective $*$-homomorphism $\hat{\pi} : \mathrm{C}^*(\mathbb{Z}) \to \mathrm{C}^*(\{u\})$. In other words, $\mathrm{C}^*(\mathbb{Z})$ is the *universal* C^**-algebra generated by a unitary.*

Now, consider $\mathrm{C}^*_r(\mathbb{Z})$. The Fourier transform $\mathcal{F}$ gives us a unitary isomorphism $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$,

$$\mathcal{F}(\xi)(z) = \sum_{n\in\mathbb{Z}} \xi_n z^n,$$

which takes convolution multiplication to pointwise multiplication. That is, if we define, for $f \in C(\mathbb{T})$, the operator $M_f \in B(L^2(\mathbb{T}))$ by

$$M_f\xi(z) = f(z)\xi(z),$$

then the Fourier transform implements an isomorphism

$$\mathrm{C}^*_r(\mathbb{Z}) \cong \{M_f : f \in C(\mathbb{T})\} \subseteq B(L^2(\mathbb{T})).$$

However, one easily checks that the $*$-algebra structure on $\{M_f : f \in C(\mathbb{T})\}$ agrees with the $*$-algebra structure on $C(\mathbb{T})$, and $\|M_f\| = \|f\|_\infty$, so $\{M_f : f \in C(\mathbb{T})\} \cong C(\mathbb{T})$ as C*-algebras.

Finally, consider the C*-algebra $C(\mathbb{T})$. The Stone-Weierstrass Theorem (cf. [4, Theorem I.5.6]) tells us that $C(\mathbb{T})$ is generated, as a C*-algebra, by the function

$$f(z) = z.$$

It turns out that $C(\mathbb{T})$ can also be described as the universal C*-algebra generated by a unitary. That is,

$$\mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}^*_r(\mathbb{Z}) \cong C(\mathbb{T}).$$

Proposition 5.10. *If $G \leq H$ then $\mathrm{C}^*(G)$ is a norm-closed subalgebra of $\mathrm{C}^*(H)$. The same is true for the reduced* C*-*algebras.*

Proof. Let $\iota : \mathbb{C}G \to \mathbb{C}H$ denote the canonical inclusion. We first claim that if we view $\mathbb{C}G$ (respectively $\mathbb{C}H$) as a subalgebra of $\mathrm{C}^*(G)$ (resp. $\mathrm{C}^*(H)$), then ι is norm-decreasing. It then follows (using the same argument as in Proposition 5.7) that ι induces an $*$-homomorphism $\tilde{\iota} : \mathrm{C}^*(G) \to \mathrm{C}^*(H)$.

To see that ι is norm-decreasing, observe that every representation of $\mathbb{C}H$ restricts to a representation of $\mathbb{C}G$. Thus, the set used in (5.1) to compute the universal norm for G contains the set

$$\{\|\pi(a)\| : \pi \text{ a representation of } \mathbb{C}G \text{ which extends to a representation of } \mathbb{C}H\}.$$

It follows that $\|\iota(a)\|_{u,H} \leq \|a\|_{u,G}$ for all $a \in \mathbb{C}G$.

The proof that $\tilde{\iota}$ is injective will be relatively straightforward once we've proved the Gelfand-Naimark-Segal Theorem, so we'll come back to it. ∎

Here are two more structural results about $\mathrm{C}^*(G)$.

Proposition 5.11.

(1) $\mathrm{C}^*(G)$ is never simple unless $G = \{e\}$ is trivial.
(2) If $|G| = n$ and G is abelian, then $\mathrm{C}^*(G) \cong \mathbb{C}^n$.

Proof. (1) For any group G, there is a representation π of $\mathbb{C}G$ on $\mathbb{C}$, given by

$$\pi(u_g) = 1, \qquad \forall\, g \in G.$$

Observe that π is onto. If $G \neq \{e\}$, then we can choose $g \neq h \in G$, and

$$u_g - u_h \in \ker \pi.$$

Thus, $\ker \pi$ is a nontrivial ideal in $\mathrm{C}^*(G)$.

(2) As a vector space, $\mathbb{C}G = \mathbb{C}^{|G|}$, which is already complete, so $\mathbb{C}G \cong \mathrm{C}^*(G)$ is a finite dimensional vector space. Notice also (Exercise 5.14) that if G is abelian, so is $\mathbb{C}G$ and hence $\mathrm{C}^*(G)$. Since every finite dimensional C*-algebra is a direct sum of matrix algebras by Proposition 6.1 and any nontrivial matrix algebra is nonabelian, the result follows. ∎

Exercise 5.12. Recall that the set $U(\mathcal{H})$ of unitaries in $B(\mathcal{H})$ is a group under multiplication. A *unitary representation* of a group G is a group homomorphism $\rho : G \to U(\mathcal{H})$. Show that representations of $\mathbb{C}G$ are in bijection with unitary representations of G.

Remark 5.13. In this section we've focused on discrete groups and their C*-algebras. However, one can also define the group C*-algebra for any group G which has a locally compact Hausdorff topology with respect to which multiplication and inversion are continuous (for short, these are called *locally compact* groups). While a lot of the theory of (discrete) group C*-algebras goes through smoothly in the locally compact setting, Proposition 5.10 is a major exception: it is not true for locally compact groups. For example, consider $\mathbb{R}$ under addition. It turns out that $C^*(\mathbb{R}) = C_0(\mathbb{R})$, and $\mathbb{Z}$ is a subgroup of $\mathbb{R}$, but $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ is not a subalgebra of $C_0(\mathbb{R})$. This example highlights the other major exception: Proposition 5.6. Notice that $C_0(\mathbb{R})$ is not unital. In particular, it contains no units, let alone a copy of $\mathbb{R}$– that's right, $C^*(\mathbb{R})$ does not contain $\mathbb{R}$.

5.1. **Abelian group C*-algebras.** If G is abelian, then $u_g u_h = u_h u_g$ for all $g, h \in G$, and so $\mathbb{C}G$ is also abelian.

Exercise 5.14. Show that any C*-completion of $\mathbb{C}G$ is an abelian C*-algebra.

By Exercise 5.14 and the Gelfand-Naimark Theorem (Theorem 2.11), it follows that $C_r^*(G) = C_0(\widehat{G})$ for some locally compact Hausdorff space $\widehat{G}$. In fact, $\widehat{G}$ must be compact since $\mathbb{C}G$ (hence $C_r^*(G)$) is unital. So what is this space $\widehat{G}$ exactly?

From the Gelfand-Naimark Theorem, we know we have $\widehat{G} = \widehat{C_r^*(G)}$, the spectrum of $C_r^*(G)$. However, I've used the new symbol $\widehat{G}$ deliberately.

Definition 5.15. For an abelian group G, $\widehat{G}$ denotes the *Pontryagin dual* of G:

$$\widehat{G} = \{\omega : G \to \mathbb{T} \text{ group homomorphism}\}. \tag{5.2}$$

Exercise 5.16. Show that $\widehat{G}$ is also a group, under pointwise multiplication. Do you need to assume G is abelian?

Our next main goal is to prove Proposition 5.17, which shows that $\widehat{G}$ and $\widehat{C_r^*(G)}$ are homeomorphic. In order to do that, we need to identify the topology on $\widehat{G}$.

The topology on $\widehat{G}$ (when G is discrete) is the *point-norm topology*: a net $(\omega_i)_{i\in\Lambda} \subseteq \widehat{G}$ is Cauchy iff, for all $g \in G$, the nets $(\omega_i(g))_{i\in\Lambda} \subseteq \mathbb{T}$ are Cauchy.[3] Equivalently, a basis for the topology on $\widehat{G}$ consists of the sets

$$B_{\epsilon,F}(\omega) := \{\eta \in \widehat{G} : |\eta(g) - \omega(g)| < \epsilon \ \forall \ g \in F \text{ finite}\}.$$

Proposition 5.17. *The map $h : \widehat{G} \to \widehat{C_r^*(G)}$ given by, for $\omega \in \widehat{G}$ and $a = \sum_{g\in F} a_g u_g \in \mathbb{C}G$,*

$$h(\omega)(a) = \sum_{g\in G} a_g \omega(g), \tag{5.3}$$

is a homeomorphism of topological spaces.

Proof. We first need to show that the formula for $h(\omega)$ given in Equation (5.3) does indeed define an element of $\widehat{C_r^*(G)}$. We begin by showing that $h(\omega)$ is a $*$-algebra homomorphism. If $b = \sum_{g\in G} b_g u_g$ is another element of $\mathbb{C}G$,

$$h(\omega)(ab) = \sum_{g\in G} \left(\sum_{h\in G} a_h b_{h^{-1}g} \right) \omega(g),$$

whereas the fact that ω is a group homomorphism implies that

$$h(\omega)(a) \cdot h(\omega)(b) = \left(\sum_{g\in G} a_g \omega(g) \right) \left(\sum_{h\in G} b_h \omega(h) \right) = \sum_{k\in G} \left(\sum_{h\in G} a_{kh^{-1}} b_h \right) \omega(k).$$

[3]If G abelian but not discrete, its Pontryagin dual still exists, but the topology is that of uniform convergence on compact sets. For discrete groups, these are the same.

Making the change of variable $h \mapsto h^{-1}k$, we see that $h(\omega)(ab) = h(\omega)(a) \cdot h(\omega)(b)$ as claimed. Similarly, since $\omega(g^{-1}) = \omega(g)^{-1} = \overline{\omega(g)}$,

$$h(\omega)(a^*) = \sum_{g \in G} \overline{a_g} \omega(g^{-1}) = \overline{\sum_{g \in G} a_g \omega(g)} = (h(\omega)(a))^*.$$

To see that our formula for $h(\omega)$ extends to a bounded linear functional on $\mathrm{C}^*_r(G)$, we need to show that $|h(\omega)a| \leq \|a\|_r$ for all $a \in \mathbb{C}G$. To that end, we first observe that for any $\chi \in \widehat{\mathrm{C}^*_r(G)}$, if we define

$$\tilde{a} = \sum_{g \in G} a_g \omega(g) \overline{\chi(u_g)} u_g,$$

then $h(\omega)(a) = \chi(\tilde{a})$. Since the Gelfand transform is isometric, it follows that

$$\|\tilde{a}\|_r = \sup\{|\eta(\tilde{a})| : \eta \in \widehat{\mathrm{C}^*_r(G)}\} \geq |\chi(\tilde{a})| = |h(\omega)(a)|.$$

We will therefore show that $\|\tilde{a}\|_r = \|a\|_r$. To that end, given $\xi \in \ell^2(G)$, define $\tilde{\xi}$ by

$$\tilde{\xi}_h = \chi(u_h^{-1}) \overline{\omega(h)}.$$

Since u_h is a unitary for each $h \in G$, and χ is a $*$-homomorphism, it follows that $\|\tilde{\chi}\|_2^2 = \|\xi\|_2^2$. Moreover,

$$\lambda(\tilde{a})\tilde{\xi}(g) = \sum_{k \in G} a_k \omega(k) \overline{\chi(u_k)} \tilde{\xi}_{k^{-1}g} = \sum_k a_k \omega(k) \overline{\chi(u_k)} \chi(u_{g^{-1}k}) \overline{\omega(k^{-1}g)} \xi_{k^{-1}g},$$

and since both χ and ω are multiplicative, we see that

$$\lambda(\tilde{a})\tilde{\xi}(g) = \omega(g)\chi(u_g^{-1}) \sum_k a_k \xi_{k^{-1}g} = \omega(g)\chi(u_g^{-1})(\lambda(a)\xi)(g).$$

As $|\omega(g)| = |\chi(u_g^{-1})| = 1$, we have $\|\lambda(\tilde{a})\tilde{\xi}\|_2^2 = \|\lambda(a)\xi\|_2^2$. It follows that

$$\|\tilde{a}\|_r \leq \sup\{\|\lambda(\tilde{a})\tilde{\xi}\|_2 : \|\xi\|_2 = 1\} = \sup\{\|\lambda(a)\xi\|_2 : \|\xi\|_2 = 1\} = \|a\|_r.$$

(A symmetric argument shows the other inequality, so that $\|\tilde{a}\|_r = \|a\|_r$.) In other words,

$$|h(\omega)a| \leq \|\tilde{a}\|_r = \|a\|_r,$$

so our formula for $h(\omega)$ determines an element of $\widehat{\mathrm{C}^*_r(G)}$ as claimed.

The fact that h is continuous is a fairly straightforward argument using the definition of the weak-$*$ topology. Suppose $(\omega_i)_{i \in \Lambda} \subseteq \widehat{G}$ is Cauchy. We need to see that $(h(\omega_i))_{i \in \Lambda}$ is Cauchy, i.e. we need to show that for any $a \in \mathrm{C}^*(G)$ the net $(h(\omega_i)(a))_{i \in \Lambda} \subseteq \mathbb{C}$ is Cauchy. If $a \in \mathbb{C}G$, so that $a = \sum_{g \in G} a_g u_g$ and $a_g = 0$ for all but finitely many g, choose

$$\epsilon < \frac{1}{|\{g : a_g \neq 0\}|} \min\{\frac{1}{|a_g|} : a_g \neq 0\}.$$

Since $(\omega_i)_{i \in \Lambda}$ is Cauchy, and $a_g \neq 0$ for only finitely many g, we can choose I such that if $i, j \geq I$ then

$$|\omega_i(g) - \omega_j(g)| < \epsilon \text{ whenever} a_g \neq 0.$$

For $i, j \geq I$, we have $|h(\omega_i)(a) - h(\omega_j)(a)| < \epsilon$.

If $a \in \mathrm{C}^*(G)$ is the limit of a sequence $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}G$, then an $\epsilon/3$ argument and the fact that each $h(\omega_i)$ is norm-decreasing will tell us that again, $(h(\omega_i)(a))_{i \in \Lambda}$ is Cauchy. It follows that $(h(\omega_i))_{i \in \Lambda}$ is Cauchy, as desired.

Checking that h is bijective is also straightforward. Given $\phi \in \widehat{\mathrm{C}^*_r(G)}$, define $\omega_\phi : G \to \mathbb{C}$ by

$$\omega_\phi(g) := \phi(u_g).$$

Observe first that since ϕ is a $*$-homomorphism, $\phi(u_g) \in \mathbb{T}$ for all g, so in order to show that $\omega \in \widehat{G}$ we only need to show that ω is multiplicative. But this follows immediately from the fact that ϕ is a $*$-homomorphism:

$$\omega_\phi(g)\omega_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = \omega_\phi(gh).$$

It is similarly immediate to check that for a fixed $\omega \in \widehat{G}$, $\omega_{h(\omega)} = \omega$, and that $h(\omega_\phi) = \phi$. It follows that $\omega \mapsto h(\omega)$ is a bijection.

Finally, we conclude the proof by showing that the inverse function $h^{-1} : \widehat{C^*_r(G)} \to \widehat{G}$, given by $h^{-1}(\phi) = \omega_\phi$, is continuous. Suppose that $(\phi_i)_i \subseteq \widehat{C^*_r(G)}$ is Cauchy – that is, for any $a \in C^*_r(G)$ the net $(\phi_i(a))_i \subseteq \mathbb{C}$ is Cauchy. In particular, the net

$$(\phi_i(u_g))_i = (\omega_{\phi_i}(g))_i \subseteq \mathbb{T}$$

is Cauchy for each $g \in G$. By definition, then, h^{-1} is continuous. □

6. AF algebras

Preview of Lecture: By definition, AF algebras are inductive limits. So, before reading this section, it would probably be a very good idea to review the section about inductive limits from the Prerequisite Notes.

The first page of this section will be touched on very lightly in lecture – which is to say, you should work through this material for yourself, and ask questions in office hours or lecture about any points where you get stuck.

We will talk about Bratteli diagrams in lecture, probably via Example 6.9.

The last three paragraphs of this section are meant to provide inspiration for future reading or research; no need to read them now (unless you're bored) and we won't discuss them in lecture.

Proposition 6.1. *If A is a C^*-algebra which is finite dimensional as a vector space, then*

$$A \cong \bigoplus_{s=1}^{j} M_{n(s)}(\mathbb{C})$$

is a finite direct sum of matrix algebras.

This proof is surprisingly intricate, and relies on the Gelfand-Naimark-Segal Theorem, which we'll see on Wednesday. So we'll postpone the proof for now.

Definition 6.2. A C^*-algebra A is an *AF algebra* or *approximately finite dimensional C^*-algebra* if A is the inductive limit of a sequence of finite-dimensional C^*-algebras.

The following Proposition was mentioned in the Prerequisite Notes, but not proved there.

Proposition 6.3. *If $A = \overline{\bigcup_n A_n}$ is the norm closure of an increasing union of subalgebras $A_n \subseteq A_{n+1} \subseteq \cdots \subseteq A$, then A is the inductive limit of the directed system (A_n, ι_{mn}) where $\iota_{mn} : A_n \to A_m$ is the inclusion map.*

Proof. It suffices to check that A satisfies the universal property of the inductive limit. So, suppose that B is a C^*-algebra and that we have $*$-homomorphisms $\psi_n : A_n \to B$ such that $\psi_m \circ \iota_{mn} = \psi_n$ whenever $n \leq m$. Given $a \in A$, write $a = \lim_{n\to\infty} a_n$ where $a_n \in A_n$. The fact that our connecting maps are inclusions means that if $m \geq n$, $a_n = \iota_{mn}(a_n) \in A_m$. Thus, if N is large enough that $\|a_m - a_n\| < \epsilon$ if $m \geq n \geq N$, then

$$\|\psi_m(\iota_{mn}a_n) - \psi_m(a_m)\| = \|\psi_m(a_n - a_m)\| < \epsilon.$$

As $\psi_m \circ \iota_{mn} = \psi_n$, it follows that $(\psi_n(a_n))_n$ is Cauchy in B. We define $\psi : A \to B$ by $\psi(a) = \lim_n \psi_n(a_n)$ if $a = \lim_n a_n$ with $a_n \in A_n$. □

Exercise 6.4. Complete the proof of Proposition 6.3 by showing that ψ is well-defined (independent of the choice of sequence $(a_n)_n$); $*$-preserving; and multiplicative.

Example 6.5 (cf. Example 6.2 from the Prerequisite Notes)**.** $K(\ell^2)$ is an AF algebra. To see this, write P_n for the projection onto $\text{span}\{e_1, \ldots, e_n\}$ and observe that $M_n \cong P_n K(\ell^2) P_n$. Since $\overline{\bigcup_n P_n K(\ell^2) P_n} = \overline{FR(\ell^2)} = K(\ell^2)$, the result follows by applying the previous Proposition.

Remark 6.6. In the above example, we were discussing the compact operators on a fixed $\mathcal{H} = \ell^2$. However, (cf. Exercise 7.54 from Day 1) if two Hilbert spaces $\mathcal{H}, \mathcal{K}$ have the same dimension, with orthonormal bases $\{\xi_n\}_n, \{\eta_n\}_n$ respectively, then the map $U : \mathcal{H} \to \mathcal{K}$ given by $U(\xi_n) = \eta_n$ is a unitary. In particular (this is another **exercise**) the map $\text{Ad}(U) : B(\mathcal{H}) \to B(\mathcal{K})$ given by

$$\text{Ad}(U)(T) = UTU^*$$

is a C*-algebra isomorphism. In particular, it takes $FR(\mathcal{H})$ to $FR(\mathcal{K})$ and $K(\mathcal{H})$ to $K(\mathcal{K})$.

So, if $\mathcal{H}$ is *any* Hilbert space with a countable orthonormal basis, then $K(\mathcal{H})$ is isomorphic to $K(\ell^2)$ (and in particular is an AF algebra). Because of this, and the fact that algebras of compact operators are (as we'll see) both ubiquitous and indispensable, we often talk about "the compact operators" as shorthand for $K(\ell^2)$, or $K(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$. In the literature, the Hilbert space is often dropped altogether, and the compact operators are denoted $\mathcal{K}$ (not to be confused with the Hilbert space $\mathcal{K}$ that we have occasionally used in these notes).

By construction, Example 6.3 of the Prerequisite Notes describes an AF algebra. Here it is again.

Example. Let $A_n = M_{2^n}(\mathbb{C})$ be the algebra of $2^n \times 2^n$ matrices with maps $\phi_{n,n+1} : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Letting $\phi_{n,m} := \phi_{m,m-1} \circ \cdots \circ \phi_{n,n+1}$ whenever $m > n$, we see that by construction this forms a directed system. Since these are inclusions, one can identify the inductive limit with $\bigcup_{n\in\mathbb{N}} A_n$. ∎

This is a particularly important one, known as M_{2^∞} or the CAR algebra. In fact, it's an example of a UHF algebra.

Definition 6.7. An AF algebra A is a *UHF* or *uniformly hyperfinite* algebra if A is the inductive limit of a sequence of full matrix algebras, where the connecting maps are unital embeddings.

Exercise 6.8. Is $K(\ell^2)$ a UHF algebra?

Example 6.9. [5, Example III.3.7] One can obtain quite different C*-algebras from the same sequence of finite-dimensional C*-algebras (A_n), if one uses different connecting maps.

For example, let $A_n = \mathbb{C}^{2^n}$. On the one hand, let X denote the standard middle-third Cantor set, so that $X = \bigcap_n C_n$, where $C_n \subseteq [0,1]$ is the collection of 2^n intervals that remain after step n in the construction of X. We can construct $C(X)$ as an inductive limit of the algebras A_n, by identifying A_n with the set of functions on C_n that are locally constant.

In this case, since $C_n \supseteq C_{n+1}$, the connecting maps $\iota_n : A_n \to A_{n+1}$, and the structure maps $\phi^n : A_n \to C(X)$, are given by restriction. It follows that the connecting maps are injective, so $\varinjlim(A_n, \iota_n) = \overline{\bigcup_n A_n}$ by Proposition 6.3. And a straightforward $\epsilon - \delta$ proof will show you that the set of functions which are constant on some C_n is dense in $C(X)$ – that is, $C(X) = \overline{\bigcup_n A_n} = \varinjlim(A_n, \iota_n)$.

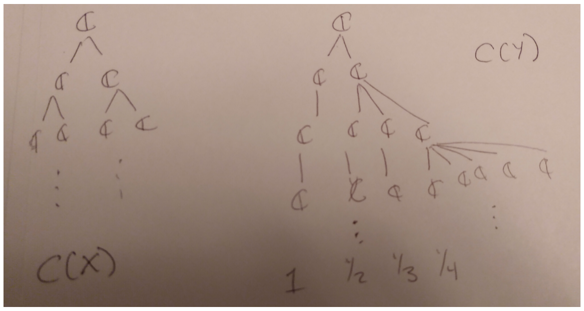
On the other hand, consider the space $Y = \{0\} \cup \{1/n : n \in \mathbb{Z}_{>0}\}$. Write $B_n \subseteq C(Y)$ for the set of functions which are constant on $[0, 2^{-n}]$. Then $B_n \cong C(\{1/k : 1 \le k \le 2^n\}) \cong \mathbb{C}^{2^n} \cong A_n$. Again, the connecting maps $\jmath_n : B_n \to B_{n+1}$ are given by inclusion, and $\bigcup_n B_n$ is dense in $C(Y)$, so $C(Y) = \varinjlim(B_n, \jmath_n)$. But clearly $C(Y) \not\cong C(X)$.

What do the connecting maps $\iota_n, \jmath_n$ look like when we identify both A_n and B_n with $\mathbb{C}^{2^n}$? We have

$$\iota_n(f)(z_1, \ldots, z_{2^{n+1}}) = f(z_1, z_3, \ldots, z_{2^{n+1}-1}), \qquad \text{and} \qquad \jmath_n(f)(z_1, \ldots, z_{2^{n+1}}) = f(z_1, \ldots, z_{2^n}).$$

In other words, $\iota_n(z_1, z_2, \ldots, z_{2^n}) = (z_1, z_1, z_2, z_2, \ldots, z_{2^n}, z_{2^n})$ and $\jmath_n(z_1, \ldots, z_{2^n}) = (z_1, z_2, \ldots, z_{2^n}, z_{2^n}, \ldots, z_{2^n})$.

One sees the difference even more clearly via the *Bratteli diagram* of the AF algebras. If $A = \varinjlim(A_n, \phi_n)$, with $A_n = \bigoplus_{j=1}^{k(n)} M_{r(j)}$, and the connecting maps $\phi_n : A_n \to A_{n+1}$ are inclusions, the Bratteli diagram consists of $\mathbb{N}$ levels, with $k(n)$ nodes at each level, and an edge from a node v at level n to a node w at level $n+1$ if ϕ_n maps the vth matrix algebra into the wth matrix algebra. For example, below are the Bratteli diagrams for $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, \jmath_n)$.



Exercise 6.10. Show that any AF algebra has an approximate identity which consists of an increasing sequence of projections.

Exercise 6.11. Show that any AF algebra is isomorphic to a direct limit of finite-dimensional C*-algebras with *injective* connecting maps.

Because AF algebras are quite tractable, it's natural to ask which C*-algebras are subalgebras of AF algebras. That is, given a C*-algebra A, when can we find an injective $*$-homomorphism $\phi : A \to B$ for some AF algebra B? This simple-seeming question was only answered recently [Schafhauser 2018], under mild assumptions on A.

Exercise 6.12.

(1) Prove that $C([0,1])$ is not an AF algebra.
(2) If X is the Cantor set, show that $C(X)$ is AF.
(3) Show that a subalgebra of an AF algebra needn't be AF, by constructing an embedding of $C([0,1])$ into $C(X)$.

However, despite the intricacy of the structure of the subalgebras of AF algebras, the lattice of ideals of an AF algebra is easy to describe: [5, Theorem III.4.2] the ideals of an AF algebra are in bijection with directed hereditary subsets of its Bratteli diagram.

One can have two different directed systems that give rise to the same C*-algebra. An example is the UHF algebra $M_{2^\infty 3^\infty} = \varinjlim(A_n, \iota_n) = \varinjlim(B_n, \iota_n)$, where

$$A_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n+1)/2}3^{(n-1)/2}}, & n \text{ odd}; \end{cases} \qquad B_n = \begin{cases} M_{2^{n/2}3^{n/2}}, & n \text{ even} \\ M_{2^{(n-1)/2}3^{(n+1)/2}}, & n \text{ odd}. \end{cases}$$

The nodes at odd levels in the Bratteli diagrams of $\varinjlim(A_n, \iota_n)$ and $\varinjlim(B_n, \iota_n)$ are not isomorphic, nor is the number of edges between levels.

Fortunately, there is a complete invariant for AF algebras – a way to tell whether or not two AF algebras are isomorphic. G. Elliott proved in 1978 that the ordered K-theory $(K_0(A), K_0(A)_+, [1])$ of an AF algebra is a classifying invariant for A, in that given two AF algebras A, B, their K-theory groups are order isomorphic – $(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B])$ – if and only if $A \cong B$. You'll hear about K-theory from Mark Tomforde next week, and [5, Chapter IV] has a proof of Elliott's classification theorem for AF algebras.

7. Cuntz–Krieger Algebras

Preview of Lecture: This section is a quick introduction to a class of (I think) fascinating C*-algebras. Unfortunately, a lot of what makes them so fascinating is beyond the scope of GOALS, but if you want to learn more, I'd recommend picking up Raeburn's book [11] on graph algebras.

For today, try to get a feel for the algebraic consequences of the relations (7.1) defining a Cuntz–Krieger algebra; you may want to pick a (small-ish) matrix B and think about what the associated C*-algebra might look like. Infinite and purely infinite C*-algebras show up in a lot of places, so it's also a good idea to build an understanding of these by playing with some examples and non-examples (cf. Exercise 7.4.)

Again, the last five paragraphs of this section are in there to inspire you to dig deeper into these Cuntz–Krieger algebras in the future;[4] don't worry too much about them now.

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[4] A word of warning, though: in the literature, Cuntz–Krieger algebras are usually denoted $\mathcal{O}_A$. I broke with tradition in these notes because we wanted to continue to reserve the letter A for C*-algebras.

In this section, B will denote an $n \times n$ matrix with entries from $\{0,1\}$. The *Cuntz–Krieger algebra* $\mathcal{O}_B$ [Cuntz-Krieger 1981] associated to B is the universal C*-algebra generated by n partial isometries $s_1, \ldots, s_n$ such that, for each $1 \leq i \leq n$, we have

$$s_i^* s_i = \sum_{j=1}^n B_{ij} s_j s_j^* \qquad \text{and} \qquad s_i^* s_j = 0 \text{ if } i \neq j. \tag{7.1}$$

What do I mean by the "universal C*-algebra"? As with the group C*-algebra, $\mathcal{O}_B$ is the "largest" C*-algebra generated by n partial isometries which satisfy (7.1). That is, if $S_1, \ldots, S_n$ are partial isometries in a C*-algebra A which satisfy Equation (7.1), then there is a surjective $*$-homomorphism $\pi_S : \mathcal{O}_B \to C^*(\{S_1, \ldots, S_n\})$ such that $\pi_S(s_i) = S_i$. One can prove (cf. [11, Proposition 1.21] or [2, II.8.3]) that this universal object exists.

Proposition 7.1. *If B is a finite matrix, the Cuntz–Krieger algebra $\mathcal{O}_B$ is unital.*

Proof. Let $S = \sum_{i=1}^n s_i s_i^*$. Observe that, for any i,

$$Ss_i = s_i + \sum_{j \neq i} s_j s_j^* s_i = s_i, \qquad s_i S = s_i s_i^* s_i S = s_i \left(\sum_{j=1}^n B_{ij} s_j s_j^* \right) \left(\sum_{k=1}^n s_k s_k^* \right) = s_i \left(\sum_{j=1}^n B_{ij} s_j s_j^* \right) = s_i.$$

The fact that S is a projection (**Exercise:** check this!) implies that we consequently have, for any word w in the generators s_i and their adjoints, $Sw = wS$. In other words, S is the unit of $\mathcal{O}_B$. □

One can define a Cuntz–Krieger algebra for an infinite matrix, too, as long as the matrix is *row-finite* – for each i, the entries in row i of B have a finite sum. We need B to be row-finite because otherwise the first equation in (7.1) would involve an infinite sum of projections, which are mutually orthogonal by the second condition of (7.1). But an infinite sum of mutually orthogonal projections cannot converge in norm, yet the first equation in (7.1) requires that.

Example 7.2. If B is the $n \times n$ matrix of all 1s, then $s_i^* s_i = S$ for all i. That is, each s_i is an isometry, not merely a partial isometry, and $\sum_{i=1}^n s_i s_i^* = 1$. In this case, $\mathcal{O}_B$ is the *Cuntz algebra* $\mathcal{O}_n$.

The Cuntz algebras were introduced by J. Cuntz in 1977 as the first explicit examples of separable simple infinite C*-algebras.

Definition 7.3. A unital C*-algebra A is *infinite* if there exists $a \in A$ with $a^*a = 1$ but $aa^* \neq 1$.

Exercise 7.4.

(1) Is $B(\ell^2)$ infinite? What about $\mathcal{K}(\ell^2)$?
(2) If a unital C*-algebra A is infinite, when can it have a trace?

Cuntz showed that, moreover, the algebras $\mathcal{O}_n$ are all *purely infinite*: for any nonzero $x \in \mathcal{O}_n$, there exist $a, b \in \mathcal{O}_n$ with $axb = 1$. (Observe that any unital purely infinite C*-algebra is *a fortiori* simple.)

In addition to being separable and purely infinite, the algebras $\mathcal{O}_n$ have a lot of other intriguing properties that you'll learn about in the coming weeks (or in your future classes on C*-algebras): they're nuclear, they can be realized as a crossed product of a UHF algebra, they're not inductive limits of type I C*-algebras. $\mathcal{O}_2$ and $\mathcal{O}_\infty$ (defined to be the universal C*-algebra generated by infinitely many isometries $s_i, i \in \mathbb{N}$, such that for any n we have $\sum_{i=1}^n s_i s_i^* \leq 1$) behave particularly nicely with respect to tensor products.

Some of above properties are shared by general Cuntz–Krieger algebras $\mathcal{O}_B$. They are again nuclear, for example – the proof of this is based on the description of $\mathcal{O}_B$ as a groupoid C*-algebra. (You'll see more about groupoid C*-algebras in Robin Deeley's expository talk next week.) The groupoid picture of $\mathcal{O}_B$ arises from a certain type of dynamical system, called a *shift of finite type*, associated to B, and it turns out [Cuntz-Krieger 1981; Franks 1984; Rørdam 1995] that the K-theory of $\mathcal{O}_B$ is a classifying invariant for these shifts of finite type. That is, the shifts of finite type associated to matrices B_1, B_2 are flow equivalent iff $K_0(\mathcal{O}_{B_1}) \cong K_0(\mathcal{O}_{B_2})$.

Another useful perspective on $\mathcal{O}_B$ is as a *graph* C*-*algebra*. One can think of B as being the adjacency matrix of a directed graph E_B on n vertices: in E_B, there is an edge from vertex i to vertex j iff $B_{ij} \neq 0$. The graph C*-algebra (cf. [11]) $C^*(E_B)$ is isomorphic to $\mathcal{O}_B$.

It turns out [11, Theorem 4.9], as for AF algebras, the ideals in a Cuntz–Krieger algebra $\mathcal{O}_B$ are in bijection with hereditary saturated subsets of the vertices of E_B.

Cuntz–Krieger algebras, graph C*-algebras, and generalizations such as higher-rank graph algebras and groupoid C*-algebras, are very active areas of current research.