Chapter 2

Borel Functional Calculus and Abelian von Neumann Algebras

Let $\mathcal{H}$ be a Hilbert space and let $x \in B(\mathcal{H})$ be a *normal* operator: $[x, x^*] = 0$. This implies that $\mathbb{C}[x, x^*]$—the set of polynomials in x, x^*, and 1—is an abelian $*$-algebra and consequently its norm-closure, which we denote by $C^*(x)$, is an abelian C^*-algebra. The Gelfand transform then yields an isometric $*$-isomorphism

$$\Gamma \colon C^*(x) \to C(\sigma(x)).$$

This gives us a way to apply continuous functions on $\sigma(x)$ to the operator x: for $f \in C(\sigma(x))$ define $f(x) := \Gamma^{-1}(f)$. Since it is an isometric $*$-isomorphism, this definition respects the $*$-algebra structure and norm of $C(\sigma(x))$:

$$(f + g)(x) = f(x) + g(x) \qquad (f \cdot g)(x) = f(x)g(x) \qquad \|f(x)\| = \|f\|_\infty.$$

We call this the *continuous functional calculus*. In this chapter, we will extend this functional calculus to bounded Borel functions on $\sigma(x)$. While $f(x) \in C^*(x)$ when f is continuous, it may not be the case if f is only assumed to be bounded and Borel. However, we do always have $f(x) \in W^*(x)$, where

$$W^*(x) := \mathbb{C}[x, x^*]''$$

is the von Neumann algebra generated by x. Recall from the Bicommutant Theorem that $W^*(x)$ is equivalent to both the SOT and WOT closures of $\mathbb{C}[x, x^*]$, and since norm-convergence implies SOT and WOT convergence we have $C^*(x) \subset W^*(x)$.

At the end of the chapter, we will then use the Borel functional calculus to produce a (partial) classification of abelian von Neumann algebras.

Lecture Preview: In the first lecture, we will prove the Borel Functional Calculus (Theorem 2.1.3) in detail. You should familiarize yourself with the following proof ingredients ahead of time: the Riesz Representation Theorem (see [Theorem 2.16, GOALS Prerequisite Notes]), Proposition 2.1.1, and Lemma 2.1.2. In the second lecture, we will give the classification of abelian von Neumann algebras (Theorem 2.2.6). It is important to be comfortablw with Definitions 2.2.1 and 2.2.3 and Corollary 2.2.5. You might also find Examples 2.2.8 and 2.2.9 illuminating.

2.1 Borel Functional Calculus

We will use Borel measures to extend from continuous functions to bounded Borel functions. Since $\sigma(x)$ for $x \in B(\mathcal{H})$ is a compact subset of $\mathbb{C}$, $C(\sigma(x))$ falls under the scope of the Riesz Representation Theorem (see [Theorem 2.16, GOALS Prerequisite Notes]), which gives us easy access to Borel measures, as seen in the following proposition.

Proposition 2.1.1. *Let $x \in B(\mathcal{H})$ be a normal operator. For any $\xi, \eta \in \mathcal{H}$, there exists a unique regular Borel measure $\mu_{\xi,\eta} \in M(\sigma(x))$ satisfying $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$ and*

$$\langle f(x)\xi, \eta\rangle = \int_{\sigma(x)} f \; d\mu_{\xi,\eta} \qquad \forall f \in C(\sigma(x)). \tag{2.1}$$

Moreover, we have $\overline{\mu_{\xi,\eta}} = \mu_{\eta,\xi}$ for all $\xi, \eta \in \mathcal{H}$ and

$$\begin{aligned} \mu_{\alpha\xi_1+\xi_2,\eta} &= \alpha\mu_{\xi_1,\eta} + \mu_{\xi_2,\eta} \qquad && \forall \alpha \in \mathbb{C}, \;\; \xi_1, \xi_2, \eta \in \mathcal{H} \\ \mu_{\xi,\beta\eta_1+\eta_2} &= \bar{\beta}\mu_{\xi,\eta_1} + \mu_{\xi,\eta_2} \qquad && \forall \beta \in \mathbb{C}, \;\; \xi, \eta_1, \eta_2 \in \mathcal{H}. \end{aligned}$$

Proof. Observe that for $f \in C(\sigma(x))$

$$|\langle f(x)\xi, \eta\rangle| \leq \|f(x)\|\|\xi\|\|\eta\| = \|f\|_\infty\|\xi\|\|\eta\|.$$

Thus $f \mapsto \langle f(x)\xi, \eta\rangle$ is a bounded linear functional on $C(\sigma(x))$ with norm at most $\|\xi\|\|\eta\|$. The Riesz Representation Theorem implies there exists $\mu_{\xi,\eta} \in M(\sigma(x))$ satisfying $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$ and (2.1). Since $M(\sigma(x)) = C(\sigma(x))^*$, this measure is uniquely determined by (2.1). Using this uniqueness, one obtains the remaining properties via the conjugate symmetry, linearity, and conjugate linearity (respectively) of the inner product. □

For a locally compact Hausdorff space X we denote by $B(X)$ the collection of bounded Borel measurable functions $f\colon X \to \mathbb{C}$, which we equip with the supremum norm $\|f\|_\infty$. Any $f \in B(X)$ is integrable with respect to any $\mu \in M(X)$. In particular, for any Borel measurable subset $S \subset X$, we have $1_S \in B(X)$ and for any $\mu \in M(X)$ we have

$$\mu(S) = \int_X 1_S \; d\mu.$$

In the context of the above proposition, any reasonable definition of $f(x) \in B(\mathcal{H})$ for $f \in B(\sigma(x))$ should satisfy

$$\langle f(x)\xi, \eta\rangle = \int_{\sigma(x)} f \; d\mu_{\xi,\eta}.$$

The above discussion tells us we can already make sense of the right-hand side, and the following lemma tells us precisely how to produce $f(x) \in B(\mathcal{H})$ satisfying the above equation.

Lemma 2.1.2. *Let $\mathcal{H}$ be a Hilbert space and suppose $q\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is linear in the first coordinate, conjugate linear in the second coordinate, and there exists $C > 0$ such that $|q(\xi, \eta)| \leq C\|\xi\|\|\eta\|$ for all $\xi, \eta \in \mathcal{H}$. Then there exists a unique $x \in B(\mathcal{H})$ satisfying*

$$\langle x\xi, \eta\rangle = q(\xi, \eta) \qquad \forall \xi, \eta \in \mathcal{H},$$

and $\|x\| \leq C$.

We leave the proof as an exercise (see Exercise 2.1.2), but remark that it is similar to the proof of [Theorem 1.36, GOALS Prerequisite Notes]. The map q is called a *bounded sesquilinear form*, and the above lemma is sometimes called the Riesz Representation Theorem (for Bounded Sesquilinear Forms).

Theorem 2.1.3 (Borel Functional Calculus)**.** *Let $x \in B(\mathcal{H})$ be a normal operator. There exists a contractive $*$-homomorphism*

$$B(\sigma(x)) \ni f \mapsto f(x) \in W^*(x).$$

In particular, for $f \in C(\sigma(x))$ the operator $f(x)$ is the same operator given by the continuous functional calculus.

Proof. Fix $f \in B(\sigma(x))$. For $\xi, \eta \in \mathcal{H}$ define

$$q(\xi, \eta) := \int_{\sigma(x)} f \; d\mu_{\xi,\eta},$$

where $\mu_{\xi,\eta}$ is as in Proposition 2.1.1. The same proposition implies q is linear in the first coordinate, conjugate linear in the second coordinate, and satisfies

$$|q(\xi,\eta)| \leq \int_{\sigma(x)} |f|\ d|\mu_{\xi,\eta}| \leq \|f\|_\infty \|\mu_{\xi,\eta}\| \leq \|f\|_\infty \|\xi\|\|\eta\|.$$

Thus Lemma 2.1.2 implies there exists $y \in B(\mathcal{H})$ with $\|y\| \leq \|f\|_\infty$ and

$$\langle y\xi, \eta\rangle = q(\xi,\eta) = \int_{\sigma(x)} f\ d\mu_{\xi,\eta} \qquad \forall \xi,\eta \in \mathcal{H}.$$

Define $f(x) := y$.

Thus $B(\sigma(x)) \ni f \mapsto f(x)$ is contractive. For all $\xi,\eta \in \mathcal{H}$ we have

$$\begin{aligned} \langle (f+g)(x)\xi,\eta\rangle = \int_{\sigma(x)} f+g\ d\mu_{\xi,\eta} &= \int_{\sigma(x)} f\ d\mu_{\xi,\eta} + \int_{\sigma(x)} g\ d\mu_{\xi,\eta} \\ &= \langle f(x)\xi,\eta\rangle + \langle g(x)\xi,\eta\rangle = \langle (f(x)+g(x))\xi,\eta\rangle, \end{aligned}$$

which implies $(f+g)(x) = f(x)+g(x)$. It is similarly shown that $(fg)(x) = f(x)g(x)$ and $\bar{f}(x) = f(x)^*$. So $f \mapsto f(x)$ is a contractive $*$-homomorphism. Note that—by construction—if $f \in C(\sigma(x))$ then $f(x)$ agrees with the operator given by the continuous functional calculus.

It remains to show that this $*$-homomorphism is valued in $W^*(x) = \mathbb{C}[x,x^*]''$. Observe that for $y \in \mathbb{C}[x,x^*]'$, $f \in C(\sigma(x))$, and $\xi,\eta \in \mathcal{H}$ we have

$$0 = \langle (yf(x) - f(x)y)\xi,\eta\rangle = \langle f(x)\xi, y^*\eta\rangle - \langle f(x)y\xi,\eta\rangle = \int_{\sigma(x)} f\ d\mu_{\xi,y^*\eta} - \int_{\sigma(x)} f\ d\mu_{y\xi,\eta}.$$

Since $f \in C(\sigma(x))$ was arbitrary and $\mu_{\xi,y^*\eta}, \mu_{y\xi,\eta} \in M(\sigma(x)) = C(\sigma(x))^*$, we must have $\mu_{\xi,y^*\eta} = \mu_{y\xi,\eta}$. Consequently, for $f \in B(\sigma(x))$ we have

$$\langle (yf(x) - f(x)y)\xi,\eta\rangle = \int_{\sigma(x)} f\ d\mu_{\xi,y^*\eta} - \int_{\sigma(x)} f\ d\mu_{y\xi,\eta} = 0$$

for all $y \in \mathbb{C}[x,x^*]'$ and all $\xi,\eta \in \mathcal{H}$. It follows that $yf(x) - f(x)y = 0$ for all $y \in \mathbb{C}[x,x^*]'$ so that $f(x) \in \mathbb{C}[x,x^*]'' = W^*(x)$. □

For $x \in B(\mathcal{H})$ normal, let $S \subset \sigma(x)$ be Borel measurable. Then $1_S \in B(\sigma(x))$ and $1_S = \overline{1_S} = 1_S^2$ imply $1_S(x) = 1_S(x)^* = 1_S(x)^2$; that is, $1_S(x)$ is a projection. Consequently, if $f \in B(\sigma(x))$ is a simple function, then $f(x)$ is a linear combination of projections. From this we can deduce that projections are ubiquitious in von Neumann algebras:

Corollary 2.1.4. *A von Neumann algebra is the norm closure of the span of its projections.*

Proof. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra, and let $x \in M$. By considering the real and imaginary parts of x ($\text{Re }(x) = \frac{1}{2}(x+x^*)$ and $\text{Im }(x) = \frac{i}{2}(x^*-x)$) we may assume x is self-adjoint. In particular, x is normal and hence $f(x) \in W^*(x) \subset M$ for all $f \in B(\sigma(x))$ by the Borel functional calculus. Thus the discussion preceding the statement of the corollary implies that approximating the identity function on $\sigma(x)$ uniformly by simple functions gives, via the Borel functional calculus, a uniform approximation of x by linear combinations of projections in M. □

Contrast this result with the fact that there exist C^*-algebras with no non-trivial projections. Indeed, if X is compact Hausdorff space, and X is connected, then $C(X)$ has exactly two projections: 0 and 1. Non-commutative examples exist as well.

It is not true in general that the $*$-homomorphism in the Borel functional calculus is injective. For example, if there exists a subset $S \in \sigma(x)$ such that $\mu_{\xi,\eta}(S) = 0$ for all $\xi,\eta \in \mathcal{H}$ then we will have $f(x) = g(x)$ so long as f and g agree on $\sigma(x) \setminus S$. This concept is explored further in Exercise 2.1.4.

Exercises

2.1.1. Let $x \in B(\mathcal{H})$ be normal. For $\xi \in \mathcal{H}$, let $\mu_{\xi,\xi}$ be as in Proposition 2.1.1. Show that $\mu_{\xi,\xi}$ is a positive measure.

2.1.2. Prove Lemma 2.1.2: First fix $\xi \in \mathcal{H}$ and show for all $\eta \in \mathcal{H}$ that $q(\xi, \eta) = \langle \xi_1, \eta \rangle$ for some $\xi_1 \in \mathcal{H}$. Then show that $x(\xi) := \xi_1$ defines a bounded operator $x \in B(\mathcal{H})$.

2.1.3. Let $x \in B(\mathcal{H})$ be a normal operator and let Ω be the Borel σ-algebra on $\sigma(x)$.

(a) Show that $1_\emptyset = 0$ and $1_{\sigma(x)} = 1$.

(b) Show that $1_{S \cap T} = 1_S 1_T$ for all $S, T \in \Omega$.

(c) Let $\{S_n \in \Omega : n \in \mathbb{N}\}$ be a collection of pairwise disjoint subsets and let $S := \bigcup_{n=1} \infty S_n$. Show that

$$1_S = \sum_{n=1}^{\infty} 1_{S_n},$$

where series is the SOT-limit of the net of partial sums (see Proposition 1.1.5).

(The map $S \mapsto 1_S$ is called a **projection valued measure**.)

2.1.4. Let $x \in B(\mathcal{H})$ be a normal operator. We say a Borel measurable subset $S \subset \sigma(x)$ is x**-null** if $1_S(x) = 0$. For $f \in B(\sigma(x))$, define

$$x.im(f) := \{z \in \mathbb{C} : \text{for all } \epsilon > 0,\ \{w \in \sigma(x) : |f(w) - z| \leq \epsilon\} \text{ is not } x\text{-null}\}.$$

and

$$\|f\|_{\infty,x} := \sup_{z \in x.im(f)} |z|.$$

(a) Show that S is x-null if and only if $\mu_{\xi,\eta}(S) = 0$ for all $\xi, \eta \in \mathcal{H}$.

(b) Show that $f(x) = 0$ if and only if $\|f\|_{\infty,x} = 0$.

(c) Show that $\|f(x)\| = \|f\|_{\infty,x}$.

(d) Show that $\sigma(f(x)) \subset x.im(f)$.

2.2 Abelian von Neumann Algebras

In this section we will prove that abelian von Neumann algebras are of the form $L^\infty(X, \mu)$ for some measure space (X, Ω, μ). This result often inspires the following platitude: "Von Neumann algebras are noncommutative measure spaces." Nevertheless, this perspective is quite helpful in developing one's intuition for von Neumann algebras, and by the end of GOALS you will probably be like

For the sake of simplicity, we will restrict ourselves the case when the Hilbert space contains a *cyclic* vector.

Definition 2.2.1. Let $A \subset B(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is said to be **cyclic** for A if the subspace $A\xi$ is dense in $\mathcal{H}$.

To motivate this definition, suppose $x \in B(\mathcal{H})$ is normal and $\xi_0 \in \mathcal{H}$ is cyclic for $\mathbb{C}[x, x^*]$. Let $\mu := \mu_{\xi_0,\xi_0}$ be as in Proposition 2.1.1. Note that μ is a positive measure (see Exercise 2.1.1). For any $a, b \in \mathbb{C}[x, x^*]$ and any $S \subset \sigma(x)$ have

$$\mu_{a\xi_0,b\xi_0}(S) = \langle 1_S(x)a\xi_0, b\xi_0\rangle = \langle (b^* 1_S(x)a)\xi_0, \xi_0\rangle = \int_X \bar{q} 1_S p \; d\mu$$

where p and q are polynomials such that $p(x, x^*) = a$ and $q(x, x^*) = b$. Thus if $\mu(S) = 0$, then the above computation implies $\mu_{a\xi_0,b\xi_0}(S) = 0$. That is, $\mu_{a\xi_0,b\xi_0} \ll \mu$. Furthermore, since ξ_0 is cyclic for $\mathbb{C}[x, x^*]$, given any $\xi, \eta \in \mathcal{H}$ and any $\epsilon > 0$ we can find $a, b \in \mathbb{C}[x, x^*]$ so that $\|a\xi_0 - \xi\|, \|b\xi_0 - \eta\| < \epsilon$. Proposition 2.1.1 implies

$$\|\mu_{\xi,\eta} - \mu_{a\xi_0,b\xi_0}\| \leq \|\mu_{\xi-a\xi_0,\eta}\| + \|\mu_{a\xi_0,\eta-b\xi_0}\| < \epsilon\|\eta\| + \|a\xi_0\|\epsilon < \epsilon(\|\eta\| + \|\xi\| + \epsilon),$$

and it follows that $\mu_{\xi,\eta} \ll \mu$. One consequence of this is that $1_S(x) = 0$ for $S \subset \sigma(x)$ Borel if and only if $\mu(S)$ (see Exercise 2.1.4.(a)). Another consequence (which we will prove below) is that $W^*(x)$ can be identified with $L^\infty(\sigma(x), \mu)$, where a bounded Borel function $f \in L^\infty(\sigma(x), \mu)$ is identified with $f(x)$.

Example 2.2.2. Let Γ be a discrete group and let $\lambda, \rho \colon \Gamma \to B(\ell^2(\Gamma))$ be the left and right regular representations. Define algebras $A := \text{span}\lambda(\Gamma)$ and $B := \text{span}\rho(\Gamma)$ Then $\delta_e \in \ell^2(\Gamma)$ is cyclic for both A and B since $\lambda(g)\delta_e = \delta_g = \rho(g)\delta_e$ for all $g \in \Gamma$. Moreover since A and B commute, if $a \in A$ and $a\delta_e = 0$ then $a = 0$. Indeed, for any $b \in B$ we have

$$ab\xi_0 = ba\xi_0 = 0.$$

Since $B\xi_0$ is dense in $\mathcal{H}$, it must be that $a = 0$. ■

The previous example highlights a related concept:

Definition 2.2.3. Let $A \subset B(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is said to be **separating** for A if $x\xi = 0$ for $x \in A$ implies $x = 0$.

The observation we made in Example 2.2.2 is an instance of a more general fact.

Proposition 2.2.4. *Let $A \subset B(\mathcal{H})$ be a subalgebra. If $\xi \in \mathcal{H}$ is cyclic for A, then it is separating for its commutant A'. If A is a unital $*$-subalgebra and ξ is separating for A', then ξ is cyclic for A. Consequently, for a von Neumann algebra $M \subset B(\mathcal{H})$, a vector is cyclic (resp. separating) for M if and only if it is separating (resp. cyclic) for M'.*

Proof. Let $\xi \in \mathcal{H}$ be cylic for A and suppose $y \in A'$ is such that $y\xi = 0$. Then for all $x \in A$ we have

$$yx\xi = xy\xi = 0.$$

Since ξ is cyclic for A, $\{x\xi : x \in A\}$ is dense in $\mathcal{H}$. Thus $y = 0$, and so ξ is separating for A'.

Now suppose A is a unital $*$-subalgebra and ξ is separating for A'. Let $p \in B(\mathcal{H})$ be the projection onto $\mathcal{K} := (A\xi)^\perp$. To see that ξ is cyclic for A it suffices to show $p = 0$. Indeed, $p = 0$ is equivalent to $\mathcal{K} = \{0\}$ and therefore

$$\overline{A\xi} = ((A\xi)^\perp)^\perp = \mathcal{K}^\perp = \{0\}^\perp = \mathcal{H}$$

(see [Exercise 1.18, GOALS Prerequisite Notes]). Now, for $x_1, x_2 \in A$ and $\eta \in \mathcal{K}$ we have

$$\langle x_1\eta, x_2\xi\rangle = \langle \eta, x_1^* x_2 \xi\rangle = 0,$$

since $x_1^* x_2 \in A$. Thus $x_1\eta \in \mathcal{K}$, and hence $A\mathcal{K} \subset \mathcal{K}$. That is, $\mathcal{K}$ is reducing for A and so Lemma 1.2.5 implies $p \in A'$. Note that $\xi \in A\xi$ since A is unital, and hence $p\xi = 0$. Since ξ is separating for A', this implies $p = 0$. The final observations follow from M being a unital $*$-subalgebra and $M = (M')'$. ☐

Corollary 2.2.5. *If $A \subset B(\mathcal{H})$ is an abelian algebra, then every cyclic vector for A is also separating for A.*

Proof. If $\xi \in \mathcal{H}$ is cyclic for A, then by the proposition it is separating for A'. In particular, it is separating for $A \subset A'$. ☐

Recall that for a an abelian $C*$-algebra A, the Gelfand transform gives an isometric $*$-isomorphism

$$\Gamma \colon A \to C_0(\sigma(A)),$$

where $\sigma(A)$ is locally compact Hausdorff space formed by the spectrum of A: the set of all $*$-homomorphims from A to $\mathbb{C}$. In particular, if A is unital then $\sigma(A)$ is compact and the image of the Gelfand transform is $C(\sigma(A))$.

Theorem 2.2.6. *Let $A \subset B(\mathcal{H})$ be an abelian von Neumann algebra with a cyclic vector $\xi_0 \in \mathcal{H}$. For any SOT dense unital C^*-subalgebra $A_0 \subset A$, there exists a positive regular Borel measure $\mu \in M(\sigma(A_0))$ and a spatial isomorphism*

$$\Gamma^* \colon A \to L^\infty(\sigma(A_0), \mu)$$

satisfying

$$\langle x\xi_0, \xi_0\rangle = \int_{\sigma(A_0)} \Gamma^*(x)\ d\mu \qquad \forall x \in A.$$

Moreover, Γ^* extends the Gelfand transform $\Gamma \colon A_0 \to C(\sigma(A_0))$.

Proof. Let $\Gamma : A_0 \to C(\sigma(A_0))$ be the Gelfand transform. Define $\phi \colon A \to \mathbb{C}$ by $\phi(x) = \langle x\xi_0, \xi_0\rangle$ for $x \in A$. For $f \in C(\sigma(A_0))$ we have

$$|\phi(\Gamma^{-1}(f))| = |\left\langle \Gamma^{-1}(f)\xi_0, \xi_0 \right\rangle| \leq \|\Gamma^{-1}(f)\|\|\xi_0\|^2 = \|f\|_\infty \|\xi_0\|^2.$$

Thus $\phi \circ \Gamma^{-1} \in C(\sigma(A_0))^*$, and so the Riesz Representation Theorem implies there exists a regular Borel measure $\mu \in M(\sigma(A_0))$ so that

$$\phi \circ \Gamma^{-1}(f) = \int_{\sigma(A_0)} f\ d\mu.$$

Observe that for a positive function $f \in C(\sigma(A_0))$, we have

$$\int_{\sigma(A_0)} f\ d\mu = \int_{\sigma(A_0)} \sqrt{f}^2\ d\mu = \ \phi \circ \Gamma^{-1}(\sqrt{f}^2)) = \left\langle \Gamma^{-1}(\sqrt{f}^2)\xi_0, \xi_0 \right\rangle = \left\| \Gamma^{-1}(\sqrt{f})\xi_0 \right\|^2 \geq 0.$$

Hence μ is a positive measure.

Define $U_0 \colon A_0\xi_0 \to C(\sigma(A_0)) \subset L^2(\sigma(A_0),\mu)$ by

$$U_0(x\xi_0) = \Gamma(x) \qquad x \in A_0.$$

Since ξ_0 is separating for A by Corollary 2.2.5, this is well-defined. Moreover, for $x, y \in A_0$

$$\langle U_0(x\xi_0), U_0(y\xi_0)\rangle_{L^2(\sigma(A_0),\mu)} = \int_{\sigma(A_0)} \Gamma(x)\overline{\Gamma(y)}\ d\mu = \int_{\sigma(A_0)} \Gamma(y^*x)\ d\mu = \phi(y^*x) = \langle y^*x\xi_0, \xi_0\rangle = \langle x\xi_0, y\xi_0\rangle .$$

Thus U_0 is an isometry on $A_0\xi_0$. Note that ξ_0 is cyclic for A_0 because it is cyclic for A and A_0 is SOT dense in A. Hence $A_0\xi_0$ is dense in $\mathcal{H}$ and so we can extend U_0 to an isometry $U \colon \mathcal{H} \to L^2(\sigma(A_0),\mu)$. Since $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0),\mu)$, U is surjective and hence a unitary.

Define a spatial isomorphism $\Gamma^* \colon A \to B(L^2(\sigma(A_0),\mu))$ via $\Gamma^*(x) = UxU^*$. For $x \in A_0$ and $g \in C(\sigma(A_0))$ we have

$$\Gamma^*(x)g = UxU^*g = Ux\left(\Gamma^{-1}(g)\xi_0\right) = U\Gamma^{-1}(\Gamma(x)g)\xi_0 = \Gamma(x)g.$$

By the density of $C(\sigma(A_0)) \subset L^2(\sigma(A_0),\mu)$, it follows that $\Gamma^*(x) = \Gamma(x)$ (where we are viewing $\Gamma(x \in B(L^2(\sigma(A_0),\mu))$ as a pointwise multiplication operator). Thus Γ^* extends the Gelfand transform.

Finally, towards proving $\Gamma^*(A) = L^\infty(\sigma(A_0),\mu)$ we first observe $L^\infty(\sigma(A_0),\mu) = \overline{\Gamma^*(A_0)}^{WOT}$. Indeed,

$$\Gamma^*(A_0) = \Gamma(A_0) = C(\sigma(A_0)) \subset L^\infty(\sigma(A_0),\mu),$$

so that $\overline{\Gamma^*(A_0)}^{WOT} = \overline{C(\sigma(A_0)}^{WOT}$. Recall that by Exercise 1.1.4, the WOT on $L^\infty(\sigma(A_0),\mu)$ corresponds to the weak* topology induced by $L^1(\sigma(A_0),\mu)^* = L^\infty(\sigma(A_0),\mu)$, and $C(\sigma(A_0))$ is dense in this topology by Exercise 2.2.2. Thus

$$\overline{\Gamma^*(A_0)}^{WOT} = \overline{C(\sigma(A_0)}^{WOT} = L^\infty(\sigma(A_0),\mu).$$

Hence, to finish the proof it suffices to prove the following inclusions:

$$\overline{\Gamma^*(A_0)}^{WOT} \subset \Gamma^*(A) \subset \overline{\Gamma^*(A_0)}^{WOT}.$$

To see the first inclusion, suppose $(\Gamma^*(x_i))_{i\in I} \subset \Gamma^*(A_0)$ WOT-converges to some $T \in B(L^2(\sigma(A_0),\mu))$. Then for all $\xi, \eta \in \mathcal{H}$ we have

$$\langle U^*TU\xi, \eta\rangle = \langle TU\xi, U\eta\rangle = \lim_{i\to\infty} \langle Ux_iU^*U\xi, U\eta\rangle = \lim_{i\to\infty} \langle x_i\xi, \eta\rangle .$$

Thus $(x_i)_{i\in I}$ WOT-converges to $U^*TU \in B(\mathcal{H})$. Since $A = \overline{A_0}^{WOT}$, $x := U^*TU \in A$ and $\Gamma^*(x) = UxU^* = T$. So the first inclusion holds. To see the second inclusion, observe that if $(x_i)_{i\in I} \in A$ is a net WOT-converging to $x \in A$, then for any $f, g \in L^2(\sigma(A_0),\mu)$ we have

$$\langle(\Gamma^*(x) - \Gamma^*(x_i))f, g\rangle_{L^2(\sigma(A_0),\mu)} = \langle U(x - x_i)U^*f, g\rangle_{L^2(\sigma(A_0),\mu)} = \langle(x - x_i)U^*f, U^*g\rangle \to 0.$$

Since $\overline{A_0}^{WOT} = A$ (by the Bicommutant Theorem), this implies $\Gamma^*(A) = \Gamma^*(\overline{A_0}^{WOT}) \subset \overline{\Gamma^*(A_0)}^{WOT}$. $\square$

Remark 2.2.7. Observe that if we take $A_0 = A$ in the proof of the previous theorem, then it follows that

$$L^\infty(\sigma(A),\mu) = \Gamma^*(A) = \Gamma(A) = C(\sigma(A)).$$

That is, the μ-measurable essentially bounded functions coincide with the continuous functions on $\sigma(A)$. This should be taken as an indication that the spectrum of a commutative C^*-algebra A is strange when A is also a von Neumann algebra. Indeed, these are *Stonean spaces* and are examples of extremally disconnected spaces.

Let us explore Theorem 2.2.6 when $A = W^*(x)$ for $x \in B(\mathcal{H})$ a normal operator and relate it to the Borel functional calculus. A natural choice for A_0 is $C^*(x)$ (the *unital* C^*-algebra generated by x), which is SOT dense in $W^*(x)$ because $\mathbb{C}[x,x^*] \subset C^*(x)$ is SOT dense. Recall that in this case, $\sigma(C^*(x)) = \sigma(x)$. Suppose $\xi_0 \in \mathcal{H}$ is a cyclic vector for $W^*(x)$, and let $\mu \in M(\sigma(x))$ be as in Theorem 2.2.6. Note that since

$$\int_{\sigma(x)} f\ d\mu = \langle f(x)\xi_0, \xi_0\rangle \qquad \forall f \in C(\sigma(x)),$$

we have $\mu = \mu_{\xi_0,\xi_0}$ where μ_{ξ_0,ξ_0} is defined as in Proposition 2.1.1. Now, since μ is a Borel measure, $L^\infty(\sigma(x),\mu) = B(\sigma(x))/\sim$ where the equivalence relation is μ-almost everywhere equivalence. We claim that for $f \in B(\sigma(x))$ with $[f] \in L^\infty(X,\mu)$, the operator $f(x)$ defined by the Borel functional calculus equals $(\Gamma^*)^{-1}([f])$ where Γ^* is as in Theorem 2.2.6. Indeed, for all $g \in C(\sigma(x))$ we have

$$\left\langle (f(x) - (\Gamma^*)^{-1}([f]))\xi_0, g(x)\xi_0 \right\rangle = \int_{\sigma(x)} f\bar{g}\ d\mu_{\xi_0,\xi_0} - \int_{\sigma(x)} [f]\bar{g}\ d\mu = \int_{\sigma(x)} (f - [f])\bar{g}\ d\mu = 0.$$

The above computation implies $(f(x) - (\Gamma^*)^{-1}([f]))\xi_0 = 0$ because ξ_0 is cyclic for $C^*(x) = \Gamma^{-1}(C(\sigma(x))$ (by virtue of $C^*(x)$ being SOT dense in $W^*(x)$). But ξ_0 is separating for $W^*(x)$ by Corollary 2.2.5, so we have $f(x) = (\Gamma^*)^{-s}([f])$ as claimed. All of which is to say, when $A = W^*(x)$ and $A_0 = C^*(x)$ the $*$-isomorphism in Theorem 2.2.6 respects the Borel functional calculus.

Theorem 2.2.6 also allows us to better understand group von Neumann algebras for commutative groups. We consider a few examples below.

Example 2.2.8. For $n \in \mathbb{N}$ with $n \geq 2$, let

$$\Gamma := \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}.$$

Since Γ is an abelian group, $L(\Gamma)$ is an abelian von Neumann algebra and from Example 2.2.2 we know that δ_0 is a cyclic vector for $L(\Gamma)$. Let $x \in L(\Gamma)$ be the unitary operator corresponding to the group generator $1 \in \mathbb{Z}/n\mathbb{Z}$, so that $L(\Gamma) = W^*(x)$. Since $xx^* = 1 = x^*x$ (i.e. x is normal), from the above discussion we know

$$L(\Gamma) \cong L^\infty(\sigma(x), \mu)$$

for a regular Borel measure $\mu \in M(\sigma(x))$. Observe that the matrix representation of x with respect to the basis $\delta_0, \delta_1, \ldots, \delta_{n-1}$ is the permutation matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

So [Example 3.15.(1), GOALS Prerequisite Notes] implies $\sigma(x)$ is the set of eigenvalues of the above matrix: $\{\exp(\frac{2\pi i k}{n}) \colon k = 0, 1, \ldots, n-1\}$ (**Exercise:** confirm this). Denote $\zeta_k = \exp(\frac{2\pi i k}{n})$ for $k = 0, 1, \ldots, n-1$, then

$$e_k := \frac{1}{\sqrt{n}}\left(\delta_0 + \zeta_k^{-1}\delta_1 + \cdots + \zeta_k^{-(n-1)}\delta_{n-1}\right)$$

is a unit eigenvector of x with eigenvalue ζ_k. Since $z1_{\{\zeta_k\}}(z) = \zeta_k 1_{\{\zeta_k\}}(z)$ for $z \in \mathbb{C}$, the Borel functional calculus implies $x1_{\{\zeta_k\}}(x) = \zeta_k 1_{\{\zeta_k\}}(x)$. That is, $1_{\{\zeta_k\}}(x)$ is the projection onto the ζ_k eigenspace. As this space is spanned by the unit vector e_k, we have $1_{\{\zeta_k\}}(x) = e_k \otimes \overline{e_k}$. Thus we have

$$\mu(\{\zeta_k\}) = \int_{\sigma(x)} 1_{\{\zeta_k\}}\ d\mu = \left\langle 1_{\{\zeta_k\}}(x)\delta_0, \delta_0\right\rangle = \langle e_k \otimes \overline{e_k}\delta_0, \delta_0\rangle = \langle \langle \delta_0, e_k\rangle e_k, \delta_0\rangle = |\langle \delta_0, e_k\rangle|^2 = \frac{1}{n}.$$

Hence μ is the uniform probability distribution on $\{\zeta_k \colon k = 0, 1, \ldots, n-1\}$. ∎

Example 2.2.9. Consider the abelian von Neumann algebra $L(\mathbb{Z})$. As in the previous example, $\delta_0 \in \ell^2(\mathbb{Z})$ is a cyclic vector for $L(\mathbb{Z})$. Let $x \in L(\mathbb{Z})$ be the unitary operator corresponding to $1 \in \mathbb{Z}$, so that $L(\mathbb{Z}) = W^*(x)$. Let

$$\mathbb{T} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\},$$

then $\mathbb{Z}$ and $\mathbb{T}$ are Pontryagin duals to each other via

$$\mathbb{Z} \times \mathbb{T} \ni (n, \zeta) \mapsto \zeta^n.$$

This duality allows us to define a unitary $U\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}, m)$ (where m is the normalized Lebesgue measure on $\mathbb{T}$) via

$$[U(\xi)](\zeta) = \sum_{n\in\mathbb{Z}} \xi(n)\zeta^n \qquad \xi \in \ell^2(\mathbb{Z}),\ \zeta \in \mathbb{T}.$$

If $f\colon L^\infty(\mathbb{T}, m)$ is the identity function $f(\zeta) = \zeta$, we have

$$[Ux\xi](\zeta) = \sum_{n\in\mathbb{Z}} [x\xi](n)\zeta^n = \sum_{n\in\mathbb{Z}} \xi(n-1)\zeta^n = \zeta \sum_{n\in\mathbb{Z}} \xi(n-1)\zeta^{n-1} f(z)[U\xi](\zeta).$$

Hence $UxU^* = f$. Using an argument similar to the on Theorem 2.2.6, one then obtains $L(Z) \cong L^\infty(\mathbb{T}, m)$. We leave the details for you to check in Exercise 2.2.3. ■

Exercises

2.2.1. Let $\mathcal{H}$ be a Hilbert space and let $p \in B(\mathcal{H})$ be a non-trivial projection: $p \neq 0$ and $p \neq 1$. Show that the algebra $A := pB(\mathcal{H})p$ has no cyclic vectors.

2.2.2. Let X be a compact Hausdorff space and let $\mu \in M(X)$ be a positive regular Borel measure. Show that $C(X)$ is weak* dense in $L^\infty(X, \mu)$ by showing that if $f \in L^1(X, \mu)$ satisfies

$$\int_X fg\ d\mu = 0 \qquad \forall g \in C(X)$$

then $f = 0$.

2.2.3. Fill in the remaining details of Example 2.2.9: first show that $UL(\mathbb{Z})U^* = \overline{\mathbb{C}[f, \bar{f}]}^{WOT} = \overline{\mathbb{C}[f, \bar{f}]}^{wk*}$, then argue that $\mathbb{C}[f, \bar{f}]$ (i.e. the set of polynomials) is weak* dense in $L^\infty(\mathbb{T}, m) = L^1(\mathbb{T}, m)^*$.

2.2.4. Let Γ be a discrete abelian group and let $\widehat{\Gamma}$ be its Pontryagin dual group, which is a compact abelian group and hence has a finite Haar measure μ. Show that $L(\Gamma) \cong L^\infty(\widehat{\Gamma}, \mu)$.